APM 351: Differential Equations in Mathematical Physics
Assignment 14, due February 2, 2012

Summary:

We have studied the wave equation $u_{tt} = c^2 \Delta u$ in two and three spatial dimensions. The initial conditions are given by $u(x,0) = \phi(x)$ and $u_y(x,u) = \psi(x)$. Two important properties of the wave equation in any dimension are:

- The **energy** $E(t) = \frac{1}{2} \int u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx$ is **conserved** (constant in time);
- **Causality:** The solution $u(x,t)$ depends on the initial condition only inside the solid **light cone**

$$\left\{(y,s) \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; |y-x|^2 \leq |t-s|^2\right\}.$$

For the solution of the wave equation $u_{tt} = c^2 \Delta u$ in three dimensions, we have derived **Kirchhoff's formula**

$$u(x,t) = \frac{\partial}{\partial t} \left\{ \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \phi(y) \, dS(y) \right\} + \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \psi(y) \, dS(y) \, .$$

Remarkably, the solution depends on the initial data only on the (surface of the) light cone, i.e., waves travel exactly at the speed of light. This is called **Huygens principle**. It is typical for solutions of the wave equation in all odd dimensions $n = 2k+1 \geq 3$.

In two dimensions, we have **Poisson's formula**

$$u(x,t) = \frac{\partial}{\partial t} \left\{ \frac{1}{2\pi c} \int_{|y-x|<ct} \frac{\phi(y)}{(c^2t^2 - |y-x|^2)^{1/2}} \, dy \right\} + \frac{1}{2\pi c} \int_{|y-x|<ct} \frac{\psi(y)}{(c^2t^2 - |y-x|^2)^{1/2}} \, dy \, .$$

Note that Huygens' principle fails in two dimensions (and generally in even dimensions.)

The key to the proof of Kirchhoff's formula is the observation that the **spherical mean** of a solution, given by

$$\bar{u}(r,t;x) = \frac{1}{n\omega_n r^{n-1}} \int_{|y-x|=r} u(y,t) \, ds(y)$$

satisfies **Darboux' equation**

$$u_{tt} = c^2 \left(u_{rr} + \tfrac{n-1}{r} u_r \right) \, .$$

(Here, the denominator $n\omega_n$ is the $n-1$-dimensional surface area of the n-ball. In $n = 3$ dimensions, its value is 4π.) Darboux's equation can be solved explicitly when n is odd, and Kirchhof's formula follows by setting $u(x,t) = \bar{u}(0,t;x)$. From there, we obtain the solution in even dimensions by using **Hadamard's method of descent**.

Assignments:

Read Chapter 9 of Strauss.

1. (a) Verify that $u(x,t) = (c^2t^2 - |x|^2)^{-1}$ satisfies the three-dimensional wave equation except on the light cone.

   (b) Use Kirchhoff's formula to find the solution of the three-dimensional wave equation with initial data $u(x,u) = 0$,$u_t(x,0) = x_2$.
   Hint: The linear function $\Psi(x) = x_2$ has the mean value property.

   (c) Use the Darboux equation (for radial solutions of the wave equation) to solve the three-dimensional wave equation with initial data $u(x,0) = 0$, $u_t(x,u) = |x|^2$.

2. Consider the **Klein-Gordon equation** $u_{tt} - c^2\Delta u + m^2 u$, where $m > 0$.

   (a) What is the energy? Show that it is conserved.

   (b) Prove the causality principle for it.

3. (a) For any solution of the two-dimensional wave equation with initial data vanishing outside some circle, prove that $u(x,t) = O(t^{-1})$ as $t \to \infty$ for each fixed $x \in \mathbb{R}^2$, i.e., $tu(x,t)$ is bounded in t for each fixed x.

   (b) Also show that $\sup_x u(x,t) = O(t^{-1/2})$, i.e., $t^{1/2}u(\cdot,t)$ is bounded *uniformly* in x as $t \to \infty$.