APM 351: Differential Equations in Mathematical Physics
Assignment 15, due February 16, 2012

Summary:

For the wave equation $u_{tt} = c^2\Delta u$, the heat equation $u_t = k\Delta u$, and the Schrödinger equation $iu_t = -\Delta u$, separation of variables leads to the same eigenvalue problem

$$-\Delta u = \lambda u\,.$$

It turns out that this eigenvalue problem has no solutions on $\mathbb{R}^n$ that decay at infinity or are even square integrable. (For every vector k, the function $u(x) = e^{-ik\cdot x}$ is a bounded solution with $\lambda = |k|^2$ but these don't lie in L^2.) So we have to investigate other methods of solutions.

- The solutions of the **wave equation** in one, two, and three spatial dimensions are given by the formulas of D'Alembert, Poisson and Kirchhoff. Similar formulas can be derived in higher dimensions.

- The solution of the **heat equation** with $u(x, u) = \phi(x)$ is given by

  $$u(x,t) = (4\pi kt)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4kt}}\phi(y)\,dy\,.$$

  The positivity of the heat kernel $(4\pi kt)^{-n/2}e^{-\frac{|x|^2}{4kt}}$ is a manifestation of the maximum principle.

  This formula remains valid, if k is a complex number with positive real part, provided that we take the square root $\sqrt{k}$ to have positive real part. The integral converges and defines a smooth function, so long as ϕ is bounded and integrable.

- By analytic continuation to $k = i$, we obtain for the **Schrödinger equation** the solution formula

  $$u(x,t) = (4\pi it)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4it}}\phi(y)\,dy\,.$$

  Here, the square root in the first factor should be chosen as $\sqrt{i} = \frac{1+i}{\sqrt{2}}$. Note that the integral is now oscillatory, and will diverge unless ϕ itself decays at infinity. This is related to the wave-like properties of Schrödinger's equation.

  The fact that the kernel $(2\pi it)^{-n/2}e^{-\frac{|x-y|^2}{4it}}$ never vanishes indicates infinite speed of propagation.

Assignments:

Complete Chapter 9 of Strauss and move on into Chapter 10.

1. (a) Verify that the solution formula for the heat equation in $\mathbb{R}^n$ is valid for every product of continuous functions $\phi(x) = \prod_{i=1}^n \phi_i(x_i)$, and hence for all finite linear combinations of such products.

   (b) Use an approximation argument to show that the formula holds more generally for every continuous function ϕ with compact support.

2. (a) Derive the conservation of energy for the wave equation on a domain D with Neumann boundary conditions.

   (b) Solve the wave equation in the square $(0, \pi) \times (0, \pi)$ with homogeneous Neumann conditions on the boundary, and initial conditions $u(x, y, 0) = \sin^2 x$, $u_t(x, y, 0) = 0$.

   (c) Verify that conservation of energy is indeed valid for your solution.

3. (a) Starting from the zeroth Hermite polynomial $H_0(x) = 1$, derive the first four Hermite polynomials from the recursion formula for the coefficients.

   (b) Show that all Hermite polynomials are given by $H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$.

4. (a) Verify that the Hermite polynomials have the orthogonality property

$$\int H_k(x) H_\ell(x)\, e^{-|x|^2}\, dx = 0\,, \quad k \neq \ell\,.$$

   Hint: Use that $v = e^{-\frac{x^2}{2}} H_k(x)$ satisfies the eigenvalue equation $v'' + (\lambda_k - x^2)v = 0$.

   (b) Explain how to use the Gram-Schmidt method to obtain another recursion formula for the Hermite polynomials. (The resulting integrals can be computed explicitly, but you're not asked to do that here.)