APM 351: Differential Equations in Mathematical Physics
Assignment 17, due March 8, 2012

Summary:

Consider the eigenvalue problem for the Laplacian on a domain $D \subset \mathbb{R}^d$ with Dirichlet boundary conditions

$$-\Delta u = \lambda u \text{ on } D\,, \qquad u = 0 \text{ on } \partial D\,.$$

We assume that D is bounded and that its boundary is smooth (e.g., define as a level set $\{g(x) = 0\}$, where g is a smooth function that satisfies the hypotheses of the Implicit Function Theorem.) Our goal is to prove that there is an infinite sequence of positive eigenvalues $\lambda_1 < \lambda_2 \leq \ldots$, whose growth is governed by **Weyl's law:** $\lambda_n \sim (4\pi^2)\left(\frac{n}{\operatorname{Vol} D}\right)^{\frac{2}{d}}$. Furthermore, we have completeness, i.e., $L^2(\mathbb{R}^d)$ has an orthonormal basis consisting of the corresponding eigenvectors $\{v_n\}$.

The main tool for the proof is the **variational characterization of eigenvalues:**

- **max-min:** $\lambda_n = \max\limits_{w_1,\ldots w_{n-1}} \left\{ \min\limits_{w \perp w_1,\ldots,w_{n-1}} \dfrac{\int_D |\nabla w|^2\,dx}{||w||^2} \right\}$;
- **min-max:** $\lambda_n = \min\limits_{w_1,\ldots,w_n} \left\{ \max\limits_{w \in \text{span}\{w_1,\ldots,w_n\}} \dfrac{\int_D |\nabla w|^2\,dx}{||w||^2} \right\}$.

(The objective functions is called the **Rayleigh quotient**. It is minimized by the lowest eigenvalue

$$\lambda_1 = \min_{||w||=1} \int_D |\nabla w|^2\,dx\,.$$

In these formulas, it is understood that $w_1, \ldots, w_n$ and w should all satisfy the Dirichlet boundary conditions. For both the max-min and the min-max principle, the functions w_i must be linearly independent (but they need not be orthonormal). The min-max principle is widely used to obtain upper bounds on eigenvalues. The max-min principle can provide lower bounds, but it is difficult to apply, since it requires to solve two infinite-dimensional problems. The following finite-dimensional approximation method is surprisingly powerful.

- **Rayleigh-Ritz principle:** Choose n orthonormal *"trial functions"* $w_1, \ldots w_n$ that satisfy the Dirichlet boundary conditions. Define a symmetric matrix A by

  $$A_{ij} = \int_D \nabla w_i \cdot \nabla w_j\,dx\,,$$

  and let $\mu_1 \leq \cdots \leq \mu_n$ be its eigenvalues. Then $\lambda_i \leq \mu_i$ for each $i = 1, \ldots, n$.

(There are more complicated versions of this that do not require orthogonality.)

The proof of Weyl's law proceeds by comparing D with a finite union of rectangles. Once we have Weyl's law, we will obtain completeness of the eigenfunctions from the min-max principle.

Assignments:

Complete Chapter 10 of Strauss and start on Chapter 11.

1. (a) Consider Schrödinger's equation with a radial potential $V(|x|)$ on $\mathbb{R}^2$ in polar coordinates

$$iu_t = -\frac{1}{2}\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right) + V(r)u\,.$$

Separate variables to obtain special solutions of the form $u = T(t)R(r)\Theta(\theta)$.
Remark: You will obtain an ODE for R that depends on the potential V.

(b) Assuming that $V(r) = r^2$, substitute $R(r) = e^{-\frac{r^2}{2}}r^{|m|}P(r)$. Write down a differential equation for P, and explain how to obtain polynomial solutions.
Remark: Only these polynomial solutions yield eigenfunctions that lie in L^2 (why?)

2. Consider the eigenvalue problem for the Neumann Laplacian on the two-dimensional unit disc,

$$\begin{cases} -\Delta u = \lambda u\,, & |x| < 1 \\ \frac{\partial u}{\partial n} = 0\,, & |x| = 1\,. \end{cases}$$

As in the previous problem, this can be split into two eigenvalue problems, and the angular problem is solved explicitly by the functions e^{inx}, where n is an integer.

(a) Write down the radial problem corresponding to a given value of λ and n. Remember to state the boundary conditions !

(b) Given n, express the solutions of the radial in terms of the n-th order Bessel function $J_n(r)$. Please explain your reasoning (a picture may help.)

3. Let $f(x)$ be a function on the interval $[0, 3]$ such that

$$f(0) = f(3) = 0\,, \quad \int_0^3 |f(x)|^2\,dx = 1\,, \quad \int_0^3 |f'(x)|^2\,dx = 1\,.$$

Find such a function if you can. If it cannot be found, explain why not.

4. Estimate the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in the triangle

$$D = \{(x, y) \mid x + y < 1, x > 0, y > 0\}\,,$$

using the Rayleigh quotient with trial function $xy(1 - x - y)$.