APM 351: Differential Equations in Mathematical Physics Assignment 2, due Sept. 27, 2011)

Summary

The general **first order linear PDE** in two variables has the form

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = d(x,y)\,.$$

Initial conditions are given by prescribing a curve $\Gamma(s) = (x_0(s), y_0(s), u_0(s))$. The objective is to find an **integral surface** for the PDE that contains the initial curve. The **method of characteristics** builds the integral surface from curves that emanate from the initial curve by solving a system of ODE, as follows:

1. Determine the **characteristics** in the (x,y)-plane by solving
$$\frac{dx}{dt} = a(x,y)\,, \qquad \frac{dy}{dt} = b(x,y)\,. \tag{1}$$

2. Along the characteristics, solve
$$\frac{d}{dt}z + cz = d\,, \tag{2}$$
where $c = c(x(t), y(t))$, $d = d(x(t), y(t))$. The curves $(x(t), y(t), z(t))$ are called the **characteristic curves** of the PDE.

3. Denote by $(x(t,s), y(t,s), z(t,s))$ the characteristic curve that passes through the point $\Gamma(s)$ on the initial curve at $t = 0$. The solution of the PDE is **implicitly defined** by
$$u(x(t,s), y(t,s)) = z(t,s)\,.$$
This is a parametric representation of the integral surface. The final step is to eliminate the parameters and solve for $u(x,y)$.

Note that the characteristic equations (1) can be nonlinear, even when the PDE is linear, and hence its solutions may not be defined globally. Even if the characteristic equations have global solutions, final step may be problematic. The Inverse Function Theorem guarantees that we can solve for $u(x,y)$ in some neighborhood of the initial curve, provided that $\Gamma(s)$ intersects the characteristics **transversally**, i.e.,

$$\det \begin{pmatrix} a(x_0(s), y_0(s)) & x_0'(s) \\ b(x_0(s), y_0(s)) & y_0'(s) \end{pmatrix} \neq 0\,.$$

Here, the first column is tangent to the characteristic, and the second column is tangent to the initial curve.

The method is easily adapted to the **quasilinear equation**

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)\,.$$

Here, Eqs. (1) and (2) are coupled, and the transversality condition involves also $u_0(s)$.

Assignments:

Read Chapter 1 and Section 2.1 of Strauss.

1. By trial and error, find a solution of the heat equation $u_t = u_{xx}$ with initial condition $u(x,0) = x^2$.

2. Use the method of characteristics to solve the initial-value problem for the transport equation

$$u_t + bu_x = f(x,t)$$

   with initial values $u(x,0) = g(x)$. Is the problem well-posed?

3. If F is a continuous vector field on $\mathbb{R}^3$ and $|F(x)| \leq (1 + |x|^3)^{-1}$, prove that

$$\int_{\mathbb{R}^3} \mathrm{div} F \, dx = 0\,.$$

   Hint: Consider a large ball B_R and then take $R \to \infty$.