APM 351: Differential Equations in Mathematical Physics
Assignment 20, due April 5, 2012

Summary:

The **Fourier transform** of a smooth complex-valued function f on $\mathbb{R}^n$ is defined by

$$\mathcal{F}(f)(k) = \hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k\cdot x} f(x)\, dx\,.$$

This integral makes sense, provided that f is at least integrable. In that case, $\hat{f}$ turns out to be bounded and continuous. The most important properties of the Fourier transform are its relationship with the natural symmetries of $\mathbb{R}^n$:

- **Translation:** For $v \in \mathbb{R}^n$, define $T_v f(x) = f(x - v)$. Then $\widehat{T_v f}(k) = e^{-2\pi i k\cdot v}\hat{f}(k)\,.$
- **Rotation:** If $R^t R = I$, define $Rf(x) = f(R^{-1}x)$. Then $\widehat{Rf}(k) = \hat{f}(R^{-1}k)\,.$
- **Scaling:** For $\lambda > 0$, define $S_\lambda f(x) = f(\frac{x}{\lambda})$. Then $\widehat{S_\lambda f}(k) = \lambda^n \hat{f}(\lambda k)\,.$

In other words, the Fourier transform diagonalizes translations (in the sense that the translation is represented as a multiplication operator), commutes with rotations, and has a simple commutation relation with scaling.

In the theory of PDE, the Fourier transform appears as a fundamental tool for solving linear, constant-coefficient equations. The reason is that as a consequence of the translation invariance,

$$\widehat{\frac{\partial}{\partial x_j} f}(k) = 2\pi i k_j \hat{f}(k)\,, \quad \widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)\,.$$

More subtle are the applications of the Fourier transform to nonlinear dispersive equation, such as the nonlinear Schrödinger and the KdV equation.

By **Plancherel's theorem**, the Fourier transform can be extended as a unitary transformation from L^2 onto itself, i.e., $||f||_{L^2} = ||\hat{f}||_{L^2}$. More generally, we have

- **Parseval's identity**: for all $f, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \bar{f}(x) g(x)\, dx = \int_{\mathbb{R}^n} \bar{\hat{f}}(k)\hat{g}(k)\, dk\,.$$

- **Fourier inversion formula**: $\mathcal{F}^{-1}(g)(x) = \check{g}(x) = \hat{g}(-x)\,.$

Our proof of Parseval's identity and the inversion formula was based on the fact that $G(x) = e^{-\pi |x|^2}$ is unchanged under the Fourier transform. The Fourier transform can be extended to even larger spaces of functions and distributions, most notably the **Schwarz space** $\mathcal{S}$ of rapidly decaying functions, and its dual $\mathcal{S}'$ consisting of tempered distributions. In the sense of distributions, $\hat{\delta}(k) = 1$.

Assignments:

Read Section 12.3 of Strauss, and look at the part of Chapter 14 that pertains to Burger's equation. Supplementary reading on the Fourier transform: Chapter 5 of Lieb & Loss.

1. Under what assumptions on f is its Fourier transform $\hat{f}$ (a) real? (b) even?

2. Use the Fourier transform to solve the ODE $-u_{xx} + a^2 u = \delta$, where δ is the delta distribution.

3. Let f be a continuous function on $\mathbb{R}$ such that its Fourier transform satisfies $\hat{f}(k) = 0$ for $|k| > \frac{1}{2}$. Such a function is called **band-limited**.

   (a) Show that
   $$f(x) = \sum_{\ell=-\infty}^{\infty} f(\ell) \frac{\sin[\pi(x-\ell)]}{\pi(x-\ell)} .$$
   That is, f is completely determined by its values at the integers.

   (b) If $\hat{f}(k) = 1$ for $|x| \leq \frac{1}{2}$ and $\hat{f}(k) = 0$ for $|k| > \frac{1}{2}$, calculate both sides of (a) directly to verify that they are equal.

4. "Solve" the wave equation using the Fourier transform, as follows:

   (a) If u solves the initial-value problem
   $$\begin{aligned} &u_{tt} = \Delta u \,, && x \in \mathbb{R}^n, t > 0 \\ &u(x,0) = \phi(x), u_t(x,0) = \psi(x) \,, && x \in \mathbb{R}^n, t = 0 \end{aligned}$$
   write down a formula for $\hat{u}(k,t)$ in terms of the initial conditions $\hat{\phi}(k)$ and $\hat{\psi}(k)$.

   (b) Use Fourier inversion to write a "formula" for $u(x,t)$ in terms of $\phi(x)$ and $\psi(x)$.

   (c) Suppose that $n = 3$ and $\phi = 0$. Can you see any relation between your formula and Huygens' principle? Kirchhoff's formula? Anything at all? Why not?