APM 351: Differential Equations in Mathematical Physics
Last Handout, April 3, 2012

Summary:

We've spend most of the year studying **linear PDE** that can be written as $Lu = f$, where L is a differential operator, the right hand side f a given function, and u the unknown solution. Laplace's equation, the diffusion equation, the wave equation, and Schrödinger's equation are all examples of second order linear equations. We've constantly relied on the **superposition principle** to construct integral representations of solutions (such as (Poisson's formula for Laplace's equation, and Duhamel's formula for the inhomogeneous diffusion equation). We've used that all these equations have constant coefficients (not depending on x) to introduce methods the Fourier transform and other methods related to the spectrum of L.

We now turn (very briefly) to nonlinear equations, a vast area. Here, we will discuss only first order equations. A first order PDE is

- **linear**, if it has the form $\sum_{i=1}^{n} a_i(x)u_{x_i} = f(x)$

and nonlinear otherwise. The method of characteristics works beautifully for linear equations, although well-posedness can fail for various reasons. For instance, the initial data may fail the **transversality condition**, or the characteristic curves may fal to cover the space in a nice way (e.g., there could be critical points, closed loops, or other global obstacles).

It is useful to further classify nonlinear equations, according to how far away they are from linear equations: A first order PDE is

- **semilinear**, if we can write it as $\sum_{i=1}^{n} a_i(x)u_{x_i} = f(x, u)$,

i.e., the right hand may depend on the unknown function (and its lower-order derivatives), but the coefficients must not. Linearization and linear methods, such as the method of characteristics and (for equations involving time) the Duhamel formula work well here. Higher-order semilinear equations include reaction-diffusion equations, nonlinear Schrödinger equations, and the Korteveg-de Vries equation. Distributions and Fourier transform remain an important tools.

A first order PDE is

- **quasilinear**, if we can write it as $\sum_{i=1}^{n} a_i(x, u)u_{x_i} = f(x, u)$,

i.e., the coefficients and the right hand side may both depend on the unknown function. Quasilinear equations are much harder to analyze — the method of characteristics works in principle, but will break down if characteristics cross. In that case, the solution develops **shocks**, and it becomes necessary to consider **weak solutions**. An important example class of examples are

- **scalar conservation laws** $u_t + a(u)u_x = 0$.

The special case with $a(u) = u$ is called **Burger's equation**. Weak solutions are determined uniquely by two principles:

1. The **Rankine-Hugoniot condition** determines the speed of the shock as $s'(t) = \frac{A(u_L)-A(u_R)}{u_L-u_R}$, where A is the anti-derivative of a. For Burger's eequation, $s'(t) = \frac{1}{2}(u_L + u_R)$;

2. **Lax' entropy condition** says that a shock forms only where characteristics collide, i.e., if $a(u_L) > a(u_R)$. For Burger's equation, the condition is that $u_L > u_R$. Otherwise, a **rarefaction wave** will form. Note that the entropy condition distinguishes the future from the past!

Higher-order quasilinear PDE include the porous-medium and thin-film equations, and many examples from the Calculus of Variations, such as the minimal surface equation, harmonics maps, and wave maps.

All bets are off, if the equation is

- **fully nonlinear** , $F(x, u, \nabla u) = 0$.

The method of characteristics (together with the Implicit Function Theorem) can be adapted to solve first order fully nonlinear equations locally, i.e., in a neighborhood of the initial data (provided a transversality condition holds). Examples of fully nonlinear first order equations are the eikonal equation $|\nabla u| = 1$ (which describes characteristic surfaces for the wave equation), and the Hamilton-Jacoby equations $u_t + H(u, \nabla u) = 0$ (which appear in Classical Mechanics). The most famous fully nonlinear equation is the Monge-Ampère equation $\det D^2 u = f$. Fully nonlinear equations cannot be analyzed by linearization; instead each requires special methods adapted to its geometric and physical meaning.

Assignments:

Read the part of Strauss' Chapter 14 that pertains to Burger's equation. Prepare for the final exam !

Some Problems (no hand-in):

1. Compute explicitly the solution of **Burger's equation**

$$u_t + uu_x = 0\,, \quad u(x, 0) = \phi(x) \tag{1}$$

with initial values

$$\phi(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$

that satisfies both the Rankine-Hugoniot jump condition and the entropy condition. Please draw a sketch of the characteristics and the shocks.

2. For $f \in C_c^\infty(\mathbb{R}^n)$, prove that its Fourier transform $\hat{f}$ is also C^∞. (In fact, $\hat{f}$ is analytic, but you are not asked to show this.) Show also that $|\,|k|^a \hat{f}(k)|$ is a bounded function for each integer $a > 0$.

   Remark: Your calculation relates the smoothness of f to the decay of $\hat{f}$, and vice versa. The **Paley-Wiener theorem** describes this relationship precisely.