APM 351: Differential Equations in Mathematical Physics Assignment 3, due Oct. 6, 2011)

Summary

The **wave equation**

$$u_{tt} = c^2 \Delta u$$

is the prototype of a **hyperbolic equation**. It is used to describe the propagation of vibrations in an elastic medium such as a string or a membrane, as well as the propagation of electromagnetic waves in vacuum. In many cases, it is an approximation to a nonlinear wave equation that is valid for small amplitudes. The parameter c is called the **wave speed**. The wave equation is invariant under **time reversal**: If $u(x,t)$ solves the wave equation, then so does then $u(x,-t)$.

In one dimension, the general solution of the wave equation has the form

$$u(x,t) = F(x-ct) + G(x+ct)\,,$$

where F and G are arbitrary twice continuously differentiable functions. It can be expressed in terms of the initial amplitude $\phi(x) = u(x,0)$ and initial velocity $\psi(x) = u_t(x,0)$ by **d'Alembert's formula**

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy\,.$$

The lines $x - ct = const.$ and $x + ct = const.$ are called the **characteristics** of the equation. The region between the characteristics that emanate from a point (x_0, t_0) with $t < t_0$ is called the **domain of dependence**, and the corresponding region with $t > t_0$ is called the **domain of influence**; together, they form the **solid light cone**. D'Alembert's formula implies that waves have **finite speed of propagation**, i.e., no signal can travel faster than at speed c. This is closely related with the idea of causality.

An important feature of the wave is equation is that **energy is conserved**:

$$\frac{d}{dt}\int \frac{1}{2}\Big\{|u_t(x,t)|^2 + c^2|u_x|^2\Big\}\, dx = 0$$

(assuming that the integral is finite). Here, the first term in the integrand represents kinetic energy, and the second term represents the potential energy of the wave. Conservation of energy is useful for proving that the initial-value problem is well-posed in a suitable space of square integrable functions.

Solutions of the wave equation can be oscillatory ($u(x,t) = \cos(ct)\cos x$) or traveling waves $u(x,t) = f(x \pm ct)$); in higher dimensions, we will see examples of focusing (wave packets that are initially far apart collide in a small area) and dispersion (a wave packet separates into pieces that run off in different directions).

Assignments:

In Chapter 2 of Strauss, read about the wave and diffusion equation.

1. Consider the wave equation with initial amplitude $\phi(x) = 0$ and initial velocity $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Define $u(x,t)$ by d'Alembert's formula.

   (a) Show that
$$u(x,t) = \frac{1}{2c} \cdot \text{length of } (x - ct, x + ct) \cap (-a, a)\,.$$

   (b) Sketch u vs. x at the times $t = \frac{ka}{2c}$ for $k = 0, 1, 2, 3, 4, 5$.

   (c) Find the greatest amplitude, $\max_x |u(x,t)|$, as a function of t.

2. If $u(x,t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity
$$u(x+h, t+k) + u(x-h, t-k) = u(x+k, t+h) + u(x-k, t-h)$$
   for all x, t, h, and k. Sketch the quadrilateral Q in the x-t-plane whose vertices appear in the identity.

3. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic (complex-analytic) function. Write $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$, and interpret f as a function from $\mathbb{R}^2$ to itself.

   (a) The Cauchy-Riemann differential equations say that
$$u_x = v_y \qquad u_y = -v_x\,.$$
   Show that u and v satisfy Laplace's equation.

   (b) Conversely, assume that $u : \mathbb{R}^2 \to \mathbb{R}$ satisfies Laplace's equation. (We say that u is a **harmonic function**). Show that there exists a function v such that the Cauchy-Riemann differential equations hold. (v is called the **conjugate harmonic function** to u.) The function $u + iv$ is holomorphic.