APM 351: Differential Equations in Mathematical Physics Assignment 4, due Oct. 13, 2011)

Summary

The **diffusion equation** (or **heat equation**)

$$u_t = ku_{xx}$$

is the prototype of a **parabolic** equation. It is used to describe the diffusion of a chemical substance by Brownian motion, or the flow of heat in a body. A variant of this equation appears in the Black-Scholes equation for the price of a stock option. The parameter $k > 0$ is called the **diffusion constant** or the **volatility**. The diffusion equation is not time reversible; we will see that the initial-value problem is well-posed forward in time, but the backwards heat equation is ill-posed in most commonly used function spaces.

The most striking property of the heat equation is the **maximum principle**: If we consider a solution on a region $a \leq x \leq b$, $t_0 \leq t \leq T$, then its maximal value is assumed either at at the initial time ($t = t_0$), or at the boundary ($x = a$ or $x = b$). The **strong maximum principle** says that the maximum *cannot* be asssumed at some point (x_1, t_x) with x_1 in the interior of the interval and $t_1 > t_0$ unless u is constant up to time t_1. (We have not proved the strong maximum principle here.) One consequence is that the solution of the heat equation with nonnegative data remains nonnegative. In fact, unless the data are zero, the solution will immediately become positive everywhere — the diffusion equation has **infinite speed of propagation** !

The maximum principle implies that solutions of boundary-value problems on finite intervals are **unique**. Note that this argument fails for solutions that are defined on the entire real line – there we need additional growth conditions (i.e., boundary conditions at infinity) to ensure uniqueness.

Typical solutions of the diffusion equation on the real line spread out and decay over time. One manifestation of this is that **energy decreases**:

$$\frac{d}{dt} \int \frac{1}{2} u^2(x,t)\, dx \leq 0$$

(assuming that the integral is finite). This is useful for understanding well-posedness and analyzing the long-time behavior.

The **fundamental solution** of the diffusion equation is given by the **source function**

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}\ .$$

Physically, this represents the diffusion of a substance on the real line that was initially concentrated at $x = 0$. The total mass is given by

$$\int_{-\infty}^{\infty} S(x,t)\, dt = 1$$

for all $t > 0$. The concentration at time t is given by a Gaussian bell-shaped curve of width proportional to $\sqrt{kt}$. The function

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy$$

solves the diffusion equation with initial values $u(x,y) = \phi(x)$. The formula says that the concentration at position x, time t is a weighted average of the concentrations at time $t = 0$.

Assignments:

Read the sections in Chapter 2 of Strauss about the diffusion equation, and solve the folllowing problems.

1. Consider the wave equation $u_{tt} - c^2 u_{xx} = 0$ with initial values $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$. If both ϕ and ψ are odd in x, prove that u is odd in x.

2. Prove the **comparison principle** for the diffusion equation: If u and v are two solutions of $u_t = ku_{xx}$ for $0 < x < \ell$ and $t > 0$, and $u \leq v$ initially ($t = 0$) and on the boundary ($x = 0, \ell$), then $u(x,t) \leq v(x,y)$ for all t.

3. Let U be a connected open bounded set with smooth boundary, let ∂U be its boundary, and let $\overline{U}$ its closure. Let u be a solution of **Laplace's equation**

   $$u_{xx} + u_{yy} = 0\,, \quad (x, y \in U)$$

   that is continuous on $\overline{U}$.

   (a) Prove that u satisfies the **maximum principle**:

   $$\sup_{(x,y)\in U} u(x,y) = \max_{(x,y)\in\partial U} u(x,y)\,.$$

   (b) Conclude that **Poisson's problem**

   $$\begin{aligned} &u_{xx} + u_{yy} = f(x,y)\,, \quad (x, y \in U)\,, \\ &u(x,y) = g(x,y)\,, \quad (x,y) \in \partial U \end{aligned}$$

   can have at most one solution.