APM 351: Differential Equations in Mathematical Physics
Assignment 5, due Oct. 20, 2011)

Summary

We have derived representation formulas for solving the wave and diffusion equations on the whole real line. These formulas can be used to construct the solutions of certain homogeneous boundary-value problems by the **method of reflections**. To explain the method, consider the wave equation on the positive half-line, with (homogeneous) **Dirichlet** boundary conditions at $x = 0$,

$$\begin{array}{ll} u_{tt} = c^2 u_{xx}\,, & x, t > 0\,, \\ u(0,t) = 0\,, & t > 0\,, \\ u(x,0) = \phi(x), u_t(x,0) = \psi(x)\,, & x > 0\,. \end{array}$$

We proceed in three steps.

- **Extend** the problem to the real line by **odd reflection**: Set

  $$\tilde{\phi}(x) = \begin{cases} \phi(x)\,, & x \geq 0 \\ -\phi(-x)\,, & x < 0\,, \end{cases} \qquad \tilde{\psi}(x) = \begin{cases} \psi(x)\,, & x \geq 0 \\ -\psi(-x)\,, & x < 0\,. \end{cases}$$

  In order to obtain a twice continuously differentiable solution, we assume the **compatibility condition** that $\phi(0) = \psi(0) = 0$, i.e., the initial values themselves satisfy the boundary condition.

- **Solve** the extended problem. By uniqueness, the solution $\tilde{u}$ is odd in x, see last week's assignment. Hence $\tilde{u}(0,t) = 0$ for all t. An explicit representation for u can be obtained from d'Alembert's formula.

- **Restrict** the solution to the positive half-line to obtain a solution of the original problem, $u(x,t) := \tilde{u}(x,t)$ for $x > 0$.

To solve the wave equation on the positive half-line with (homogeneous) **Neumann** boundary conditions $u_x(0,t) = 0$, we continue u by **even reflection**, $\tilde{u}(x) = u(-x)$ for $x < 0$, and proceed as above. Here, the compatibility condition is that $\phi_x(0) = \psi_x(0) = 0$. We can similarly solve homogeneous Dirichlet and Neumann problems on finite intervals by reflecting at both endpoints. In that case, the extended functions $\tilde{\phi}(x)$, $\tilde{\psi}(x)$, and $\tilde{u}(x,t)$ are **periodic** in x.

The same method can be used to solve the heat equation with Dirichlet or Neumann boundary conditions on the half-line and on bounded intervals. Here, we take advantage of the fact that the fundamental solution $S(x,t)$ is even in x.

Assignments:

Finish reading Chapter 2 of Strauss, start on Chapter 3, and solve the following problems.

1. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ with initial values $u(x,0) = x^2$, $u_t(x,0) = e^x$.
   Hint: Factor the differential operator into two first order operators, as we did for the wave equation.

2. Consider the diffusion equation $u_t = u_{xx}$ on $-1 < x < 1$ with **Robin boundary conditions**

   $$u_x(-1,t) - au(-1,t) = 0 = u_x(1,t) + au(1,t)\,.$$

   (a) If $a > 0$, show that the energy $E(t) = \frac{1}{2}\int_{-1}^{1} u^2(x,t)\,dx$ decreases.

   (b) If a is much smaller than zero, show by example that energy may increase or decrease.
   Hint: Try to find solutions of the form $u(x,t) = h(t)\cosh(bx)$.

3. Suppose that ϕ is bounded and continuous everywhere except for a jump discontinuity at the point a, i.e., the right- and left-sided limits $\phi(a^+) = \lim_{x\to a^+}\phi(x)$ and $\phi(a^-) = \lim_{x\to a^-}\phi(x)$ exist. Let S be the fundamental solution of the heat equation $u_t = ku_{xx}$, and set

   $$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy\,.$$

   (a) Explain why u solves the heat equation for $x \in \mathbb{R}$, $t > 0$.

   (b) Show that for every $x \in \mathbb{R}$,

   $$\lim_{t\to 0} u(x,t) = \frac{1}{2}\left\{\lim_{y\to x_-}\phi(y) + \lim_{y\to x_+}\phi(y)\right\}.$$

   Hint: Change variables and prove that

   $$\lim_{t\to 0^+}\frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-p^2/4}\phi(\sqrt{4kt}p)\,dp = \frac{1}{2}\lim_{y\to 0^+}\phi(y)\,.$$