# APM 351: Differential Equations in Mathematical Physics
# Assignment 6, due Oct. 27, 2011

## Summary

We turn to inhomogeneous linear equations, such as diffusion equations with sources, and the wave equations with forcing. We will formally write such equations as

$$u_t = Lu + f(\cdot, t)\,, \quad u(\cdot, 0) = \phi\,, \tag{1}$$

where $L$ is a linear differential operator that involves only the $x$ variable, $f$ is the inhomogeneity, and $\phi$ is the initial condition.

- To ground ourselves, let us first consider a linear ODE

$$\frac{dy}{dt} = Ay + f(t) \quad y(0) = y_0\,, \tag{2}$$

where $A$ is a $n \times n$ matrix $f$ is a given function, $y_0 \in \mathbb{R}^n$ a given vector, and the unknown function $y(t)$ takes values in $\mathbb{R}^n$. **Duhamel's principle** says that the solution is given by

$$y(t) = e^{tA} y_0 + \int_0^t e^{A(t-s)} f(s)\, ds\,.$$

(This can be proved by **Variation of Constants**.) Here, $e^{tA}$ denotes the fundamental solution of the homogeneous equation $\frac{d}{dt}y = Ay$. By definition, $y(t) = e^{tA}y_0$ solves the initial-value problem

$$\frac{dy}{dt} = Ay\,, \quad y(0) = y_0\,.$$

There are many equivalent ways to compute the matrix-valued function $e^{tA}$, by its power series, by diagonalizing $A$, or by special techniques such as contour integrals.

- For the **heat equation with sources**

$$u_t = ku_{xx} + f(x,t)\,, \quad u(x,0) = \phi(x)\,, \tag{3}$$

the Duhamel principle yields

$$u(\cdot, t) = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s) f(\cdot, s)\, ds\,.$$

Here, $\mathcal{S}(t)$ is the fundamental solution of the heat equation. Explicitly,

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy + \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4k(t-s)}} f(y,s)\, dyds\,.$$

- For the linear transport equation (Assignment 2 Problem 2),

$$u_t + bu_x = f(x,t)\,, \quad u(x,0) = \phi(x) \tag{4}$$

the solution of the homogeneous problem is $e^{-bt\partial_x}u_0 = u_0(x - bt)$. By Duhamel's principle,

$$u(x,t) = \phi(x - bt) + \int_0^t f(x - b(t-s), s)\, ds\,.$$

- Consider finally the initial-value problem for the wave equation with forcing,

$$u_{tt} = c^2 u_{xx} + f(x,t)\,, \quad u(x,u) = \phi(x), u_t(x,0) = \psi(x)\,. \tag{5}$$

  We rewrite this equation as a system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ c^2\partial_x^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix}, \quad \begin{pmatrix} u(\cdot,0 \\ v(\cdot,0) \end{pmatrix} = \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

  and then apply Duhamel's principle

$$\begin{pmatrix} u(\cdot,t) \\ v(\cdot,t) \end{pmatrix} = \mathcal{T}(t)\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \int_0^t \mathcal{T}(t-s)\begin{pmatrix} 0 \\ f(\cdot,s) \end{pmatrix} ds\,.$$

  Here, $\mathcal{T}(s)$ is the fundamental solution of the above system. Using D'Alembert's formula, we obtain for the first component

$$u(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds\,.$$

We've given a recipe how to construct solutions of the inhomogeneous heat, transport, and wave equations. Our heuristic derivation does of course not constitute a proof. But once we guess the correct formulas, we can check by direct computation that they indeed satisfies Eqs. (3)-(5).

## Assignments:

Read the second half of Chapter 3, specifically the last sections that contains the proof of the solution formula for the heat equation, and start with Chapter 4.

1. Verify that $u(x,t) = \frac{1}{\sqrt{1-t}} e^{\frac{x^2}{4(1-t)}}$ satisfies the heat equation on the real line for $0 < t < 1$. Note that $u_t > 0$ for each $t > 0$, i.e., $u(x,t)$ is increases with $t$. How do you reconcile this with the maximum principle?

2. (a) Use the method of reflections to write the solution of the boundary-value problem

$$\begin{aligned} &u_t = u_{xx}\,, && 0 < x < 1, t > 0\,, \\ &u_x(0,t) = u_x(1,t) = 0\,, && t > 0\,, u(x,0) = \phi(x) \end{aligned}$$

   as a series. Here, $\phi$ is a continuously differentiable function with $\phi'(0) = \phi'(1) = 0$.
   (b) Does the series converge? In what sense?

3. (a) Find a pair a pair of ODE for $f$ and $g$ such that $u(x,y) = f(x)g(y)$ solves the PDE

$$u\Delta u = |\nabla u|^2\,, \quad (x,y \in \mathbb{R})\,.$$

   (Do not try to solve these equations).
   (b) Is the general solution a superposition of such product form solutions? Why not?