APM 351: Differential Equations in Mathematical Physics
Assignment 7, due Nov. 10, 2011

Summary

Separation of variables is a method for finding special solutions of a partial differential equation. Sometimes we can generate all solutions of the PDE from these special solutions, using the

- **Superposition principle:** If the PDE is linear and homogeneous, then any linear combination of solutions is again a solution.

If, moreover, its coefficients are constant, then translates and derivatives of solutions are again solutions; if the equation has additional symmetries (such as rotations and dilations), they can be used to generate yet more solutions.

To explain the method, consider the wave equation on an interval

$$u_{tt} = c^2 u_{xx}\,, \quad (0 < x < \ell)\,,$$

with Dirichlet boundary conditions $u(0,t) = u(\ell,t) = 0$. We seek solutions that can be written as a product

$$u(x,t) = X(x)T(t)$$

for some unknown functions X and T. Inserting this into the PDE and collecting terms, we see that

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} =: -\lambda\,,$$

and deduce that λ can depend neither on x nor on t. We obtain a system of two ODE:

$$-X''(x) = \lambda X(x)\,, \quad -T''(t) = c^2\lambda T(t)\,.$$

Enforcing the Dirichlet boundary conditions $X(0) = X(\ell) = 0$, we conclude that

$$X(x) = \sin\beta x\,, \quad \lambda = \beta^2\,, \quad T(t) = A\cos(c\beta t) + B\sin(c\beta t)\,,$$

where $\beta = n\pi/\ell$ for some integer $n \geq 1$, and A and B are constants. We have found the special solutions

$$u_n(x,t) = \cos\Big(\frac{N\pi ct}{\ell}\Big)\sin\Big(\frac{n\pi x}{\ell}\Big)\,, \quad v_n(x,t) = \sin\Big(\frac{N\pi ct}{\ell}\Big)\sin\Big(\frac{n\pi x}{\ell}\Big)\,.$$

Two questions remain:

- Can we construct the **general solution** of the wave equation from the u_n and v_n by superposition?

- How can we determine the coefficients in the superposition from **initial values**?

Assignments:

Read Chapter 4 and the first three sections of Chapter 5.

1. A metal rod $(0 < x < \ell)$ insulated along its sides but not its ends is initially at a constant temperature u_0. Suddenly both ends are plunged into a bath of temperature zero.
   (a) Write the initial-value problem for the temperature.
   (b) Use the formula

$$1 = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k+1} \sin\Big(\frac{2\pi(k+1)x}{\ell}\Big) , \quad (0 < x < \ell)$$

   to represent the solution $u(x,t)$ as a series.

2. Apply separation of variables to the Schrödinger equation $iu_t = u_{xx}$ with Dirichlet boundary conditions on $0 < x < 1$. Here, i is the imaginary unit, satisfying $i^2 = -1$. You may find Euler's formula useful: $e^{a+ib} = e^a(\cos b + i \sin b)$.

3. Consider the PDE

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} , \quad (r > 0, \theta \in \mathbb{R})$$

   with periodic boundary conditions in θ, i.e., $u(r\theta + 2\pi) = u(r, \theta)$ for all θ.
   (a) Set $u(r, \theta) = f(r)g(\theta)$ and separate variables to obtain a pair of ODE for f and g.
   (b) Solve these ODE to obtain special solutions for the PDE. (*Hint:* Try $f(r) = r^\alpha$.)
   Remark: We will see later this year that this is Laplace's equation in polar coordinates.

Please remember:

Our first midterm test is scheduled for **Friday November 4, 5-7pm, BG 304 (Galbraith Building)**. There will be no lectures on Friday October 27; on the morning of November 4, the lecture will be replaced by a question hour.