APM 351: Differential Equations in Mathematical Physics
Assignment 8, due Nov. 17, 2011

Summary

We have seen that separation of variables can lead to linear eigenvalue problems of the form

$$\big(p(x)u'\big)' + q(x)u + \lambda r(x)u = 0 \qquad x \in (a,b)\,, \tag{1}$$

where p, q, r are given functions. Such problems are called **Sturm-Liouville** eigenvalue problems. The goal is to determine the unknown function u and the eigenvalue λ. These depend crucially on the **boundary conditions** that we impose on u. We assume that the boundary conditions are **symmetric**, so that for all u, v that satisfy the boundary conditions, integration by parts yields

$$-\int_a^b \big(p(x)u'\big)'v\,dx = \int_a^b u'v'p(x)\,dx = -\int_a^b u\big(p(x)v'\big)'\,dx\,.$$

Dirichlet, Neumann, Robin, and periodic boundary conditions are all symmetric.

We focus on **regular** Sturm-Liouville problems, where p and r are strictly positive. But some interesting problems are **singular:** The functions p or r have zeroes, p, q, or r may be unbounded, or the interval may be infinite.

We will consider this as a linear algebra problem in infinite dimensions. We introduce the **inner product**

$$\langle u, v\rangle_r = \int_a^b u(x)\bar{v}(x)\,r(x)\,dx\,.$$

If r is strictly positive, this is a positive definite, Hermitian quadratic form, and we can use it to define the following geometric notions.

- **norm:** $||u||=\sqrt{\langle u,u\rangle_r}$;
- **orthogonality:** $u \perp v \Leftrightarrow \langle u, v\rangle = 0$;
- **angle:** $\cos\alpha = \frac{\langle u,v\rangle}{||u||_r\,||v||_r}$.

A key tool is **Schwarz' inequality:** $|\langle u,v\rangle_r| \le ||u||_r\,||v||_r$. It implies in particular that the norm satisfies the **triangle inequality**. The completion of the space of continuous functions with this norm is an example of a **Hilbert space** and denoted by $L^2\big((a,b), r(x)dx\big)$.

We will prove next semester that regular Sturm-Liouville eigenvalue problems have countably many independent solutions. The eigenvalues are real, and the eigenfunctions $\{X_n\}_{n\ge1}$ are orthogonal, in perfect analogy with the diagonalizaton problem for Hermitian $n \times n$ matrices. The expansion of a function f in these eigenfunctions, given by

$$\sum_{n=1}^{\infty} A_n X_n\,, \quad \text{where } A_n = \frac{\langle f, X_n\rangle_r}{||X_n||_r^2}$$

is called a (generalized) **Fourier series** of f. An important question is when, and in which sense, such a Fourier series converges to f.

The most important example of a Sturm-Liouville problem is the equation

$$u'' + \lambda u = 0 \quad x \in (0, \ell) \tag{2}$$

(corresponding to $p, r = 1$ and $q = 0$). In this case, the eigenvalues and eigenfunctions are given by

- **Dirichlet** boundary conditions $u(a) = u(\ell) = 0$:

$$X_n = \sin \tfrac{\pi n x}{\ell}, \quad \lambda_n = \left(\frac{\pi n}{\ell}\right)^2, \qquad n = 1, 2, \dots ;$$

- **Neumann** boundary conditions $u'(0) = y'(\ell) = 0$:

$$X_n = \cos \tfrac{\pi n x}{\ell}, \quad \lambda_n = \left(\frac{\pi n}{\ell}\right)^2, \qquad n = 0, 1, 2, \dots ;$$

- **periodic** boundary conditions $u(x) = u(x + \ell)$:

$$X_n = e^{\frac{2\pi i n x}{\ell}}, \quad \lambda_n = \left(\frac{2\pi n}{\ell}\right)^2, \qquad n = 0, \pm 1, \pm 2, \dots .$$

If $\ell = 2\pi$, the **Fourier coefficients** of a function f are given by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx, \qquad n = 0, \pm 1, \pm 2, \dots ,$$

and the corresponding expansion is called **'the' Fourier series** of f.

Assignments:

Read Chapter 5 of Strauss.

1. (a) On the interval $[-1, 1]$, show that the function x is orthogonal to the constant functions.
(b) Find a quadratic polynomial that is orthogonal to both 1 and x.
(c) Find a cubic polynomial that is orthogonal to all quadratics.
(These are the first three Legendre polynomials.)

2. Let ϕ be a 2π-periodic function with Fourier series $\phi(x) = \sum_n A_n e^{inx}$.
(a) If ϕ is real-valued, show that $A_{-n} = \bar{A}_n$.
(b) If, additionally, ϕ is even, what can you say about the Fourier coefficients? Use this to represent ϕ as a cosine series.
(c) What if ϕ is odd?

3. Let f be real-valued function on the real line. Assume that f is continuously differentiable, and that

$$\left(\int_{\mathbb{R}} |f'(x)|^2\, dx\right)^{\frac{1}{2}} = M < \infty .$$

Use Schwarz' inequality to prove that $|f(y) - f(x)| \leq M\sqrt{|y - x|}$. In particular, f is uniformly continuous.