APM 351: Differential Equations in Mathematical Physics
Assignment 9, due November 24, 2011

Summary

A **Hilbert space** is a vector space $\mathcal{H}$ over $\mathbb{C}$ with an **inner product** $\langle f, g\rangle$ that is

- **linear** in the first slot: $\langle a_1 f_1 + a_2 f_2, g\rangle = a_1\langle f_1, g\rangle + a_2\langle f_2, g\rangle$ for $a_1, a_2 \in \mathbb{C}$
- **Hermitian:** $\langle f, g\rangle = \overline{\langle g, f\rangle}$
- **positive definite:** $\langle f, f\rangle \geq 0$, with equality only for $f = 0$

such that $\mathcal{H}$ is **complete** under the norm $||f|| = \big(\langle f, f\rangle\big)^{\frac{1}{2}}$, in the sense that every Cauchy sequence in $\mathcal{H}$ converges to a limit in $\mathcal{H}$.

Hilbert spaces share many geometric properties of Euclidean space, such as the **Schwarz inequality** $|\langle f, g\rangle| \leq ||f||\,||g||$ and the **parallelogram identity** $||f+g||^2 + |f-g||^2 = 2\big(||f||^2 + ||g||^2\big)$. The most important examples are the finite-dimensional complex vector spaces $\mathbb{C}^m$ with inner product $u \cdot v$, and the function space $L^2(a, b)$ with inner product $\int_a^b f(x)\bar{g}(x)\,dx$.

Two vectors $f, g \in \mathcal{H}$ are **orthogonal**, if $\langle f, g\rangle = 0$. In that case, we write $f \perp g$. We have

- **Pythagoras**: If $f \perp g$, then $||f + g||^2 = ||f||^2 + ||g||^2$,

just as in $\mathbb{R}^m$. If $X_1, X_2, \dots$ is a (finite or countable) sequence of orthogonal vectors in $\mathcal{H}$,

- **Bessel's inequality** $||f||^2 \geq \sum_n |a_n|^2\,||X_n||^2$, where $a_n = \frac{\langle f, X_n\rangle}{||X_n||^2}$

follows from the fact that $f - \sum_n a_n X_n$ is orthogonal to $\sum_n a_n X_n$. For the partial sums

$$S_N = \sum_{n=1}^{N} a_n X_n\,, \quad ||S_N||^2 = \sum_{n=1}^{N} |a_n|^2\,||X_n||^2\,,$$

Bessel's inequality implies that $||S_N||^2$ converges to $\sum_{n=1}^{\infty} |a_n|^2\,||X_n||^2$, and by the Cauchy criterion S_N converges to $S = \sum_{n=1}^{\infty} a_n X_n$.

We say that the $\{X_n\}$ is an **orthogonal basis** for $\mathcal{H}$, if every $f \in \mathcal{H}$ can be written as a convergent series

$$f = \sum_n a_n X_n\,.$$

In that case, we also say that the orthogonal sequence is **complete**. We will prove in Chapter 11 that $\{\sin nx\}_{n\geq 1}$ and $\{\cos nx\}_{n\geq 0}$ are orthogonal bases for $L^2(0, \pi)$, and that $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthogonal bases for $L^2(0, 2\pi)$. In particular, each of the classical Fourier series of an L^2-function f converges in L^2 to f. (We say that the Fourier series **represents** the function.) A more subtle question is under what conditions a Fourier series converges pointwise or even uniformly to f. There are examples of continuous 2π-periodic functions whose Fourier series diverges for every x!

Theorem *Let $X_1, X_2, \dots$ be a sequence of orthogonal vectors. The following are equivalent:*

(1) Finite linear combinations $\sum_{n=1}^{N} b_n X_n$ are **dense** *in $\mathcal{H}$;*

(2) If $\langle f, X_n \rangle = 0$ for all n then $f = 0$;

(3) **Parseval's identity:** *For each $f \in \mathcal{H}$, $||f||^2 = \sum_n |a_n|^2 \, ||X_n||^2$, where $a_n = \frac{\langle f, X_n \rangle}{||X_n||^2}$;*

(4) $\{X_n\}_{n \geq 1}$ is an orthogonal basis.

Proof. $(1) \Rightarrow (2)$: Assume that f satisfies $\langle f, X_n \rangle = 0$ for all n. By (1), we can find a sequence $\{g_k\}$ converging to f, where each g_k is a finite linear combination of the X_n's. By our assumption on f, we have $\langle f, g_k \rangle = 0$ for all k, and therefore

$$||f||^2 = \langle f, f - g_k \rangle \leq ||f|| \, ||f - g_k|| \to 0 \quad (k \to \infty).$$

We conclude that $f = 0$.

$(2) \Rightarrow (3)$: Let $S = \sum_n a_n X_n$ and $S_N = \sum_{n=1}^{N} a_n X_N$. Since

$$\langle f - S, X_n \rangle = \lim_{N \to \infty} \langle f - S_N, X_n \rangle = a_n - a_n = 0$$

for each n, we conclude from (2) that $f = S$. Using Pythagoras,

$$||f||^2 = ||S_N||^2 + ||f - S_n||^2 \to \sum_{n=1}^{\infty} |a_n|^2 \, ||X_n||^2 + ||f - S|| \,,$$

proving Parseval's identity.

$(3) \Rightarrow (4)$: Assuming Parseval's identity, we have by Pythagoras

$$||f - S_N||^2 = ||f||^2 - ||S_N||^2 = ||f||^2 - \sum_{n=1}^{N} |a_n|^2 \, ||X_n||^2 \to 0 \quad (N \to \infty),$$

proving that $f = \sum a_n X_n$.

$(4) \Rightarrow (1)$: Let $f \in \mathcal{H}$. Since $\{X_n\}$ is an orthogonal basis, $f = \lim S_N$.

Assignments:

Read Chapter 5 of Strauss (again).

1. (a) Find the Fourier sine series of the function $f(x) = x$ on $[0, \pi]$.
   (b) Apply Parseval's identity to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.
   (c) Integrate the sine series term by term to obtain a Fourier cosine series for the function $\frac{1}{2}x^2$. Note that the constant of integration appears as the $n = 0$ term in the series.
   (d) Then by setting $x = 0$, find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

2. Let γ_n be a sequence of constants with $\lim_{n \to \infty} \gamma_n = \infty$. Define a sequence of functions on $[0, 1]$ by $f_n(x) = \gamma_n \sin(n\pi x)$ for $0 \leq x \leq \frac{1}{n}$, and $f(x) = 0$ otherwise.
   (a) Show that $f_n \to 0$ pointwise, but not uniformly.
   (b) If $\gamma_n = n^{1/3}$, prove that $f_n \to 0$ in L^2.
   (c) If $\gamma_n = n^{2/3}$, show that f_n does not converge in L^2.

3. Let f be a smooth 2π-periodic function with $\int_0^{2\pi} f(x)\, dx = 0$. Use the Fourier series representation and Parseval's identity to show that $||f|| \leq ||f'||$.