Errors and omissions: Rearrangement inequalities

March 2010

- **Definiton of f^*, p. 3:** The equation should read

$$f^*(x) = \int_0^\infty \mathcal{X}_{\{y:f(y)>t\}^*}(x)\,dt$$

  Similarly, Eq. (1.3) on p. 4 should read

$$f(x) = \int_0^\infty \mathcal{X}_{\{y:f(y)>t\}}(x)\,dt$$

- **Exercise 1.7, p.6:** This really needs a hint, and perhaps a sketch. One could add the formula

$$||f-g||_p^p \le \int_0^\infty \mu(\{f>t\} \triangle \{g>t\})\, pt^{p-1}\,dt\,.$$

  Perhaps as in Exercise 1.6 one should also consider increasing functions of f and g.

- **Exercise 2.14, p. 20:** I had intended this as an application of the approximation by polarization.

  One may also prove directly that symmetrization improves the modulus of continuity, by using the Brunn-Minkowski inequality to show that the minimal distance between level sets increases.
  (Thanks to Piotr Hajlasz.)

- **p. 25, after Eq. (3.2):** Missing period after "dilations".

- **Fig. 3.2, p. 27:** In the top row, the pictures on the left and right should show the function concentrated near the south pole (not the north pole), see Fig. 3.1 on p. 26 .
  (Thanks to Marc Fortier.)

- **After Section 3.2, p. 31:** Add new section on random symmetrizations. Discuss results of Mani-Levitska, van Schaftingen, Volčič.

- **Proof of Theorem 4.7, p. 37:** For $p = 1$, the proof by polarization is not complete. We need to show that the weak limit g is in $W^{1,p}$ (not just in BV), i.e., $|\nabla g|$ is an integrable function (not just a measure). We need to use the concentration inequality from Exercise 4.9.
(Thanks to Etienne Sandier.)

  p. 38, top: The fact that the volume of the set of critical points can only decrease under symmetrization requires proof. This can be done as follows: If $f \in W^{1,p}$, then the critical values of f^* form a set of (one-dimensional) measure zero. By Lieb & Loss, Theorem 6.19, the gradient of f vanishes on the inverse image of that set. Since f and f^* are equimeasurable, the volume of the set of critical points of f is at least as large as for f^*. In dimension $n > 1$, the inequality can be strict; Almgren and Lieb characterized the functions for which this can happen in their big paper on the (dis)-continuity of symmetrization. However, Steiner symmetrization preserves the measure of the set of critical points.
(Thanks to Yannick Privat.)

- **Section 5.2, p. 44:** Need to add reference to Gromov, and discuss recent results of Figalli/Maggi/Pratelli.

- **After Section 5.3, p. 47:** Add section on Barthe's proof of Young's inequality.

- **Add chapter:** Domination and weak convergence.

- **Add chapter:** Equality cases and stability results.