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ON COMPUTING THE INSTABILITY INDEX OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR ASSOCIATED WITH COATING AND RIMMING FLOWS*

ALMUT BURCHARD† AND MARINA CHUGUNOVA†

Abstract. We study the problem of finding the instability index of certain non-self-adjoint fourth order differential operators that appear as linearizations of coating and rimming flows, where a thin layer of fluid coats a horizontal rotating cylinder. Our main result shows that the instability index of such operators is determined by its restriction to a finite-dimensional space of trigonometric polynomials. The proof uses Lyapunov's method to associate the differential operator with a quadratic form, whose maximal positive subspace has dimension equal to the instability index. The quadratic form is given by a solution of Lyapunov's equation, which here takes the form of a fourth order linear PDE in two variables. Elliptic estimates for the solution of this PDE play a key role. We include a numerical example.

Key words. non-self-adjoint operator, Pontryagin space, Lyapunov's method, instability index, rimming flows, coating flows

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1. Introduction.

The stability of steady states is a basic problem regarding the dynamics of a partial differential equation that models the evolution of a physical system. Frequently, the first step is to linearize the system about a given equilibrium, and to determine the spectrum of the resulting differential operator A. If the spectrum of A is discrete, an important quantity is the *instability index*, $\kappa(A)$, which counts the number of eigenvalues in the right half plane (with multiplicity).

In order to numerically evaluate the instability index of a given differential operator, its computation should be reduced to a problem of linear algebra. For periodic boundary conditions, it seems natural to restrict A to a finite-dimensional space of trigonometric polynomials. Under what conditions can $\kappa(A)$ be computed from the matrix of this restriction? It is believed that the spectrum of these finite-dimensional matrices should approach the spectrum of A as the degree of the trigonometric polynomials is taken to infinity. However, it is not rigorously known if the entire spectrum of non-self-adjoint operators is recovered in the limit [13]. One difficulty is that the entries of the infinite matrix corresponding to the differential operator A grow with the row and column index, so that truncation is not a small perturbation.

If A is a self-adjoint semibounded differential operator of even order, then the computation of its instability index is well understood through the classical work of Morse [20], who solved this problem completely in the space of vector-valued functions in one independent variable. The instability index is invariant under congruence transformations and agrees with the dimension of the positive cone of the corresponding quadratic form. It can be estimated by variational methods or computed directly from

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†Department of Mathematics, University of Toronto, Toronto, ON, M5S 2E4, Canada (almut@math.utoronto.ca, chungunom@math.utoronto.ca). The work of the first author was supported in part by an NSERC discovery grant. The work of the second author was supported by an NSERC postdoctoral fellowship.

the zeros of the Evans function. An interesting qualitative result about approximating the spectrum of certain self-adjoint operators by nonsymmetric matrices, possibly having complex eigenvalues, was recently obtained by Volkmer [29].

Understanding the spectrum of a non-self-adjoint operator is a much harder problem [17]. It is not at all obvious how to restrict the computation of its instability index to a finite-dimensional subspace. Furthermore, the numerical calculation of eigenvalues can be extremely ill-conditioned even in finite dimensions. One impressive example is the matrix

$$A = \begin{pmatrix} 10^4+1 & 10^6 & 10^4 \\ 10^6 & 2 & 10^6 \\ -(10^4) & -(10^6) & -(10^4-1) \end{pmatrix}.$$

The MATLAB function $\texttt{eig}(A)$ gives for the eigenvalues the numerical results $\lambda_1 = -0.8$, $\lambda_{2/3} = 2.4 \pm 1.7\,i$, which suggests an instability index of $\kappa(A) = 2$. However, the accuracy of the computation is poor. Denoting by V the matrix that contains the (numerically computed) eigenvectors in its columns, and by E the diagonal matrix that contains the (numerically computed) eigenvalues, then

$$\text{Norm}\left(A - VEV^{-1}\right) = 7.6\,.$$

On the other hand, A is similar to an upper triangular matrix

$$A = T \begin{pmatrix} 1 & 10^6 & 10^4 \\ 0 & 2 & 10^6 \\ 0 & 0 & 1 \end{pmatrix} T^{-1}, \quad \text{where } T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and we see that actually $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 2$, and $\kappa(A) = 3$. In contrast, the eigenvalues of the symmetric matrix

$$B = \begin{pmatrix} 10^4+1 & 10^6 & 10^4 \\ 10^6 & 2 & -(10^6) \\ 10^4 & -(10^6) & -(10^4-1) \end{pmatrix}$$

can be determined with the much better computational accuracy:

$$\text{Norm}\left(B - VEV^{-1}\right) = 3.6 * 10^{-10}\,.$$

Note that B differs from A only in the signs of two off-diagonal entries. The chance of encountering a matrix with moderately sized entries and a badly conditioned eigenvalue problem increases rapidly with the dimension of the matrix (see [16, 28]). Such examples demonstrate that the stability problem for a non-self-adjoint operator cannot be easily solved by direct computations of the spectrum.

In this paper, we examine the computation of the instability index for differential operators of the form

$$A[f] = -f'''' - \big(a(x)f\big)'' + \big(b(x)f\big)' - c(x)f\,, \tag{1.1}$$

acting on 2π-periodic square integrable functions. Such operators appear as linearizations of models for thin liquid films moving on the surface of a horizontal rotating cylinder, when the dependence on the longitudinal variable is neglected. A thin film on a rotating cylinder is called a *coating* flow if the fluid is on the outside of the cylinder, and a *rimming* flow if the fluid is on the inside of a hollow cylinder. Coating and rimming flows appear in many applications, including the production of fluorescent light bulbs where a coating solvent is applied inside a spinning glass tube, and different types of molding processes and paper production. Physical experiments show that the flow becomes unstable if the fluid film is thick enough so that drops of fluid can form on the bottom of the cylinder (in the case of a coating flow) or on its ceiling (in the case of a rimming flow). In both cases, surface tension and higher rotation speeds should help to stabilize the fluid, but may also allow for more complicated steady states and interesting dynamics (see, for example, [27, 31, 15]).

Benilov, O'Brien, and Sazonov [7] studied the convection-diffusion equation $\frac{d}{dt}f = Af$, where the operator A is given by

$$A[f] = \big(f + \beta \sin x f'\big)'$$

with periodic boundary conditions on $[0, 2\pi]$. This corresponds to a singular limit of a rimming flow where surface tension is neglected. This operator has remarkable properties: For $|\beta| < 2$, all its eigenvalues are purely imaginary, suggesting neutral stability, but the Cauchy problem

$$\frac{d}{dt}f = A[f], \quad f(0) = f_0$$

is ill-posed in every Sobolev and Hölder space of 2π-periodic functions. The underlying cause is the sign change of the diffusion coefficient as $x \to x+\pi$. This phenomenon of explosive instability of a system with purely imaginary spectrum was studied analytically by Chugunova, Karabash, and Pyatkov [9], who explained it in terms of the absence of the Riesz basis property of the set of eigenfunctions. The spectral and asymptotic properties of A are of interest in operator theory and were analyzed in [12, 30, 8, 10].

One should expect the explosive instability to disappear in complete models that include the smoothing effect of surface tension. Such models have been proposed, for example, by Pukhnachov [23, 24]. In [5, 6], Benilov and coauthors linearized this model about some approximation of a positive steady state solution to obtain

$$A[f] = -\mathcal{X}(f'''' + f'') + ((1 - \alpha \cos x)f + \beta \sin x f')' \tag{1.2}$$

with periodic boundary conditions. After rescaling to $\mathcal{X} = 1$, this is a special case of (1.1). Here, the coefficient $\mathcal{X}$ is related to the surface tension, α is related to the gravitational drainage, and β is a small parameter related to the hydrostatic pressure. They showed numerically that surface tension can stabilize the film if the other coefficients are not too small. For smaller values of α and β, capillary effects destabilize the film. The instability index of A grows if $\mathcal{X}$ is decreased.

We will consider operators on $L^2 = L^2[0, 2\pi]$ with periodic boundary conditions, given by (1.1). Our assumptions on the coefficients are that the distributional derivatives a'', b', and c are bounded measurable functions, and that their Fourier series satisfy

$$M = \sum_{p=-\infty}^{\infty} \Big\{ |\hat{a}(p)| + |\hat{b}(p)| + |\widehat{c-1}(p)| \Big\} < \infty\,. \tag{1.3}$$

We will show that the instability index of A is determined by its projection to a finite-dimensional space of trigonometric polynomials. The dimension of the space depends on a suitable norm of the distributional solution U of the partial differential equation

$$\big(\mathcal{A}_x^* + \mathcal{A}_y^*\big) U(x, y) = \delta_{y-x} \tag{1.4}$$

with periodic boundary conditions on $[0, 2\pi] \times [0, 2\pi]$. Here, the differential operators $\mathcal{A}_x^*$ and $\mathcal{A}_y^*$ are defined by applying the adjoint of the single-variable differential operator A to the x and y variables, respectively (see section 3). Equation (1.4) has a unique solution $U(x, y)$ if the spectra of A and $-A^*$ are disjoint [2, 3]. This solution defines a self-adjoint integral operator.

Let A_0 be given by (1.1) with $a(x) = b(x) = 0$ and $c(x) = 1$, and let $U_0(x, y)$ be the corresponding solution of (1.4). We will see in sections 3 and 4 that U_0 is piecewise smooth, with a jump in the third derivative across the line $x = y$, and that $U(x, y) - U_0(x, y) \in \mathcal{H}^4$. Denote by P_N the standard projection onto the space of trigonometric polynomials of order N,

$$P_N[f](x) = \sum_{|p|<N} \hat{f}(x)e^{ipx} . \tag{1.5}$$

Our first result shows that the instability index of A is determined for sufficiently large N by the restriction of U to the nullspace of $(I - P_N)U$, which has dimension $2N - 1$. Proposition 7.1 implies that

$$\kappa(A) = \kappa\big(P_N U^{-1} P_N\big) \tag{1.6}$$

if $N^2 > M\big(1 + ||U(x, y) - U_0(x, y)||_{\mathcal{H}^4}\big)$. The constant M is given by (1.3), and $\mathcal{H}^4$ is the Sobolev space of doubly periodic functions with four square integrable derivatives; see (4.1). This result has the weakness that (1.6) involves the unknown function U, which is defined as the solution of (1.4). This solution depends sensitively on the spectrum of A, which is exactly the unknown quantity we are concerned with.

Our main result shows that we can instead truncate A itself at sufficiently high Fourier modes without changing the instability index. Proposition 7.4 says that

$$\kappa(A) = \kappa\big(P_N A P_N\big) \tag{1.7}$$

if $N^2 > 4 \max\{M, M^2\} \cdot \big(1 + ||U(x, y) - U_0(x, y)||_{\mathcal{H}^4}\big)$. Note that (1.7) does not involve U at all, and only the norm of the unknown function $U(x, y)$ enters into the condition on N. The selection of N and the problem of numerically estimating $U - U_0$ will be discussed at the end. It would be interesting to extend our results to the case of a more general fourth order differential operator with a third order derivative term, which is absent from (1.1).

Let us add a few words about the proof. Given a distributional solution $U(x, y)$ of the partial differential equation (1.4), we define the integral operator U on L^2 by the operation

$$U[f](x) = \int_0^{2\pi} U(x, y) f(y)\, dy . \tag{1.8}$$

Then

$$\int\int (\mathcal{A}_x^* + \mathcal{A}_y^*) U(x, y)\, \Phi(x, y)\, dxdy = \int \Phi(x, x)\, dx$$

for all smooth doubly periodic test functions. If we choose $\Phi(x, y) = \phi(x)\psi(y)$ and integrate the first summand by parts, we see that U satisfies the operator equation $A^*U + UA = I$. This is an example of *Lyapunov's equation*,

$$A^*U + UA = V . \tag{1.9}$$

It turns out that U is always self-adjoint in L^2 and maps L^2 to the Sobolev space H^4. Classical results, which will be discussed section 2, state that $\kappa(A) = \kappa(U)$, and that the positive and negative cones of U contain the invariant subspaces associated with the spectrum of A in the right and left half planes, respectively.

In section 3, we show that the operator A in (1.1) is sectorial, and we prove some basic bounds. These are used in section 4 to prove that the partial differential equation in (1.4) has a unique solution if the spectra of A and $-A^*$ are disjoint.

Section 5 is dedicated to tail estimates for the operator in (1.8). The estimates required for the proof of (1.6) are straightforward, but the proof of (1.7) requires more subtle off-diagonal estimates. There are two natural topologies on the space of integral operators on L^2, the operator norm $||F||_{L^2\to L^2}$ and the norm of the corresponding integral kernel $F(x, y)$ as a doubly periodic function in $\mathcal{L}^2$. A simple application of Schwarz' inequality shows that

$$||F||_{L^2\to L^2} \le ||F(x, y)||_{\mathcal{L}^2}\,, \tag{1.10}$$

and corresponding inequalities hold for higher order Sobolev spaces. Neither of these norms appears to be particularly useful here. The difficulty is that U defines a bounded linear operator from L^2 to H^4, but its kernel $U(x, y)$ does not lie in the Sobolev space $\mathcal{H}^4$. In Lemma 5.1, we define a new norm $|||\cdot|||$ that depends only on the modulus of the Fourier coefficients of the kernel, while being generally much smaller than the $\mathcal{H}^4$-norm. In particular, $|||U|||$ is finite.

Our analysis of the instability index relies on the indefinite quadratic form associated with U. As part of the argument, we derive an addition formula for the instability index of a self-adjoint operator in terms of its restriction to suitable subspaces; see section 6. The key to the proof of (1.6) in section 7 is that the quadratic form is negative on high Fourier modes, because the fourth order term in A dominates the lower order derivatives. The proof of (1.7) combines (1.6) with the off-diagonal estimate for U from (5.2). The paper concludes with a numerical example in section 8.

2. Lyapunov's equation. Equation (1.9) was first considered by Lyapunov in the case where A is an $n \times n$ matrix, and V is a positive definite symmetric matrix. Assuming that a symmetric matrix U solves (1.9), Lyapunov proved that all eigenvalues of A have negative real part if and only if U is negative definite. The fundamental result on Lyapunov's equation in finite dimensions is due to Taussky [25].

THEOREM 2.1 (Taussky). *Let A be an $n \times n$ complex matrix with characteristic roots α_i, where $\alpha_i + \bar\alpha_k \neq 0$ for $i, k = 1, \ldots, n$. If V is a positive definite Hermitian $n \times n$ matrix, then Lyapunov's equation* (1.9) *has a unique solution U, which is nonsingular and satisfies $\kappa(U) = \kappa(A)$.*

Taussky originally stated the theorem for $V = I$; the more general statement follows with a congruence transformation. An equivalent result was proved by Ostrowski and Schneider [21]. The problem of obtaining information about the sign of eigenvalues of A in situations where both V and U may be indefinite and have nontrivial kernels remains an area of active research.

Lyapunov's equation has many applications in stability theory and optimal control. In typical applications, $\kappa(A) = 0$, so that the system is asymptotically stable, and U is used to study the rate of convergence. Equation (1.9) is a special case of *Sylvester's equation*,

$$AX - XB = C\,,$$

which has been studied extensively in linear algebra, operator theory, and numerical analysis. It is known to be uniquely solvable if and only if the matrices A and B have no eigenvalues in common. In particular, (1.9) has a unique solution if the spectra of A and $-A^*$ are disjoint. Since V is self-adjoint, a unique solution U is automatically self-adjoint as well. These results were extended to bounded operators on infinite-dimensional Hilbert spaces by Daleckĭ and Kreĭn [11] and to unbounded operators by Belonosov [2, 3].

We next state a special case of Belonosov's results. Recall that a closed densely defined operator A on a Banach space is *sectorial* if the spectrum of A is contained in an open sector,

$$S = \{z \in \mathrm{C} \ \big|\, |\arg(\lambda_0 - \mathrm{z})| < \theta\},$$

with vertex at $\lambda_0 \in \mathrm{R}$ and opening angle $\theta < \pi/2$, and the resolvent $R_\lambda(A) = (A - \lambda I)^{-1}$ is uniformly bounded for λ outside S. Sectorial operators are precisely the generators of analytic semigroups. The sector S is invariant under similarity transformations and does not change if the norm on the space is replaced by a equivalent norm.

Theorem 2.2 (Belonosov). *Let A be a sectorial operator on a separable Hilbert space H. Assume that the spectra of A and $-A^*$ are disjoint. If V is a bounded self-adjoint positive definite operator on H, then Lyapunov's equation* (1.9) *has a unique solution U in the class of bounded operators on H. Moreover, U has a densely defined but possibly unbounded generalized inverse U^{-1}, and*

$$\kappa(A) = \kappa(U) = \kappa(U^{-1}) .$$

To explain the geometric meaning of Lyapunov's equation, we introduce on the Hilbert space H the indefinite inner product

$$[f, g] = \langle Uf, g\rangle . \tag{2.1}$$

If U has trivial nullspace and $\kappa(U) < \infty$, then H equipped with $[\cdot, \cdot]$ is called a *Pontryagin space* and will be denoted by Π. The concepts of orthogonality and adjointness are defined in the natural way with respect to the indefinite inner product [22]. A subspace $X \subset \Pi$ is called *positive* if $[f, f] > 0$ for every nonzero vector $f \in X$, and *negative* if $[f, f] < 0$ for every nonzero $f \in X$. Maximal positive subspaces have dimension $\kappa(U)$, while maximal negative subspaces have codimension $\kappa(U)$.

Let $f(t) = e^{tA} f_0$ be the solution of the evolution equation

$$\frac{d}{dt} f(t) = Af(t) , \quad f(0) = f_0 .$$

Lyapunov's equation guarantees that the value of the quadratic form $Q(f) = [f, f]$ strictly increases with t,

$$\frac{d}{dt} Q(f(t)) = \big\langle (A^*U + UA) f(t), f(t)\big\rangle = \big\langle Vf(t), f(t)\big\rangle > 0 .$$

Denote by $M_+(A)$ the invariant subspace associated with the part of the spectrum of A located in the right half plane. If $f \in M_+(A)$, then

$$Q(f) > \lim_{t\to-\infty} Q\big(e^{tA} f\big) = 0 ,$$

which shows that $M_+(A)$ is a positive subspace of Π. A similar argument with $t \to \infty$ shows that the complementary subspace $M_-(A)$, which corresponds to the spectrum of A in the left half plane, is a negative subspace of Π. Since (1.9) excludes purely imaginary eigenvalues, these subspaces are maximal, and consequently $\kappa(A) = \kappa(U)$.

One can also interpret Lyapunov's equation as a dissipativity condition on A on the Pontryagin space Π. In general, a densely defined linear operator A on Π is called *dissipative* if $\mathbf{Re}\,[Af, f] \le 0$ for all $f \in \mathrm{Dom}\,(A)$. It is *maximally dissipative* if it has no proper dissipative extension in Π. Assuming Lyapunov's equation, we compute for $f \neq 0$

$$\mathbf{Re}\,[Af, f] = \frac{1}{2}\big\langle (A^*U + UA)f, f\big\rangle = \frac{1}{2}\langle Vf, f\rangle > 0\,,$$

i.e., $-A$ is dissipative. The following result was proved by Azizov and Iokhvidov [1] (but note that they work with $\mathbf{Im}$ rather than $\mathbf{Re}$).

THEOREM 2.3 (Azizov and Iokhvidov). *Let Π be a Pontryagin space with inner product $[\cdot,\cdot]$. If A is an operator on Π such that $-A$ is maximally dissipative, then there exist a maximal nonnegative subspace Π_+ and a maximal nonpositive subspace Π_- of Π such that*

$$\mathbf{Re}\,\sigma(A|_{\Pi_+}) \ge 0\,, \quad \mathbf{Re}\,\sigma(A|_{\Pi_-}) \le 0\,.$$

Moreover, we can choose Π_+ and Π_- to be invariant subspaces for A, and

$$\Pi_+ \supset M_+(A)\,, \quad \Pi_- \supset M_-(A)\,.$$

If, additionally, $\mathbf{Re}\,[Af, f] > 0$ for all nonzero $f \in \mathrm{Dom}\,(A)$, then $M_+(A)$ and $M_-(A)$ are themselves maximal positive and negative subspaces for Π, respectively, and

$$M_+(A) \dot{+} M_-(A) = \Pi\,.$$

Let U be a self-adjoint operator on a Hilbert space H with trivial nullspace and finite instability index, and consider the Pontryagin space Π with inner product $[\cdot,\cdot]$. The second part of Azizov's theorem implies that $\kappa(A) = \kappa(U)$, provided that V in (1.9) is positive definite. This agrees with the conclusion of Theorem 2.2, but note the difference in the hypotheses: Belonosov's assumptions on the spectrum of A provide resolvent estimates that allow one to represent U as a contour integral, thereby proving the existence of a solution to (1.9). The analytic semigroup e^{tA} appears in the proof that $\kappa(A) = \kappa(U)$, as sketched above. In contrast, Azizov and Iokhvidov's theorem makes no assumptions on the spectrum of A, but starts instead from a given solution to (1.9). In the special case where $\kappa(U) = 0$, Theorem 2.3 reduces to a theorem of Phillips that characterizes maximal dissipative operators as generators of strongly continuous contraction semigroups. In particular, the spectrum of A lies in the closed left half plane (see [32, Corollary 1 in section IX.4]).

In the case where A is a sectorial differential operator of even order on an interval, Belonosov proved that the solution of Lyapunov's equation with $V = I$ is given by a self-adjoint bounded operator [4]. His results are formulated for "split" boundary conditions that do not couple the values at the two endpoints. Belonosov's results were extended to second order sectorial differential operators with nonsplit boundary conditions by Tersenov [26]. The operators we consider here are of fourth order with periodic boundary conditions, which are not covered by Belonosov's results.

An interesting open question is how to take advantage of the freedom to choose an arbitrary positive definite self-adjoint bounded operator V for the right-hand side of (1.9). For instance, if A is a sectorial non-self-adjoint differential operator, can V be chosen in such a way that the solution U is the inverse of a differential operator?

3. Basic estimates for A. We start with some simple bounds for the differential operator in (1.1). We will work in the Hilbert space $L^2 = L^2[0, 2\pi]$ and use periodic boundary conditions throughout. The inner product and norm are denoted by

$$\langle f, g\rangle = \int_0^{2\pi} f(x)\bar{g}(x)\, dx\,, \quad ||f||_{L^2} = \left(\int_0^{2\pi} |f(x)|^2\, dx\right)^{1/2}.$$

For the Fourier coefficients we use the conventions

$$\hat{f}(p) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ipx}\, dx\,, \quad f(x) = \sum_{p=-\infty}^{\infty} \hat{f}(p)e^{ipx}\,.$$

On the Sobolev spaces $H^s = H^s[0, 2\pi]$ with periodic boundary conditions, we use the norms

$$||f||^2_{H^s} = 2\pi \sum_{p=-\infty}^{\infty} (1 + p^4)^{s/2}\, |\hat{f}(p)|^2\,. \tag{3.1}$$

The domain of the operator A in (1.1) consists of periodic functions in H^4, and its adjoint is given by the operation

$$A^*[f] = -f'''' - a(x)f'' - b(x)f' - c(x)f\,. \tag{3.2}$$

In particular, A is self-adjoint if $b(x) = a'(x)$.

Lemma 3.1. *Let A be the operator in* (1.1)*, and denote by A_0 the special case with $a(x) = b(x) = 0$ and $c(x) = 1$. If the coefficients of A satisfy* (1.3)*, then*

$$||A^* - A_0||_{H^2\to L^2} \le M\,.$$

Proof. By definition,

$$(A^* - A_0)[f] = -a(x)f'' - b(x)f' - \big(c(x) - 1\big)f\,.$$

Using that $||a||_\infty \le \sum_{p=-\infty}^{\infty} |\hat{a}(p)|$, and correspondingly for b and $c - 1$, we estimate

$$\begin{aligned} ||(A^* - A_0)f||_{L^2} &= ||a(x)f'' + b(x)f' + (c(x) - 1)f||^2_{L^2} \\ &\le ||a||_{L^\infty}||f||_{H^2} + ||b||_{L^\infty}||f||_{H^1} + ||c - 1||_{L^\infty}||f||_{L^2} \\ &\le M||f||_{H^2}\,. \qquad \square \end{aligned}$$

Lemma 3.2. *Let A be as in the previous lemma, and assume that a'', b', and c are bounded measurable functions. Then A^* is sectorial.*

Proof. It suffices to show that the *Hausdorff set* $\big\{\langle A^*f, f\rangle \mid f \in \mathrm{Dom}\,(A^*), ||f||_{L^2} = 1\big\}$ is contained in a closed sector

$$S = \{\lambda_0\} \cup \{\, z \in C : |\arg(\lambda_0 - z)| \le \theta\}$$

with some vertex λ_0 and opening angle $\theta < \pi$, and that $A^* - \lambda_0 I$ is invertible (see p. 280 of [19]). By straightforward computations, we have, for $f \in H^2$,

$$\mathbf{Re}\,\langle f'''' + c(x)f, f\rangle = \int |f''|^2 + c(x)|f|^2\,dx\,, \tag{3.3}$$

$$\mathbf{Re}\,\langle a(x)f'', f\rangle = \int \frac{1}{2}a''(x)|f|^2 - a(x)|f'|^2\,dx\,, \tag{3.4}$$

$$\mathbf{Re}\,\langle b(x)f', f\rangle = -\frac{1}{2}\int b'(x)|f|^2\,dx\,. \tag{3.5}$$

Choose

$$\lambda_0 = \frac{1}{2}\big(1 + \sup_x\{-a''(x) + b'(x) - c(x)\}\big) + \sup_x [a(x)]_+^2,$$
$$\theta = \tan^{-1}\big(\sup_x |a'(x) - b(x)|\big)\,.$$

We estimate with the help of (3.3)–(3.5)

$$\begin{aligned}\mathbf{Re}\,\langle(\lambda_0 - A^*)f, f\rangle &= \int_0^{2\pi} |f''|^2 - a(x)|f'|^2 + \Big(\lambda_0 + \frac{1}{2}a''(x) - \frac{1}{2}b'(x) + c(x)\Big)|f|^2\,dx \\ &\geq 2\pi \sum_{p=-\infty}^{\infty} \frac{1+p^4}{2}\,|\hat{f}(p)|^2 \\ &= \frac{1}{2}||f||_{H^2}^2\,.\end{aligned} \tag{3.6}$$

This shows that the spectrum of A^* lies in the half plane $\mathbf{Re}\ z < \lambda_0 - \frac{1}{2}$. Similarly,

$$\begin{aligned}|\mathbf{Im}\,\langle(\lambda_0 - A^*)f, f\rangle| &\leq \int_0^{2\pi} |a'(x) - b(x)|\,|\mathbf{Im}\,f'\bar{f}|\,dx \\ &\leq 2\pi \sup_x |a'(x) - b(x)| \sum_{p=-\infty}^{\infty} |p|\,|\hat{f}(p)|^2\,.\end{aligned}$$

For $||f|| = 1$ it follows that

$$\frac{|\mathbf{Im}\ \langle(\lambda_0 - A^*)f, f\rangle\,|}{\mathbf{Re}\ \langle(\lambda_0 - A^*)f, f\rangle} \leq \big(\sup_x |a'(x) - b(x)|\big)\Big(\sup_{p\in\mathrm{Z}} \frac{2|p|}{1+p^4}\Big) = \sup_x |a'(x) - b(x)|\,,$$

which yields the claim. $\square$

The lemma implies that the Cauchy problem for A^* has a unique solution $f(t,\cdot) = e^{tA^*}f_0$ for every initial value $f_0 \in L^2$. This solution is analytic in t for $t > 0$, and for any fixed $t > 0$, the function $f(t,\cdot)$ lies in $\mathrm{Dom}\,(A^*)$. If the coefficients of A^* are analytic, then f is analytic in both variables for $t > 0$. An application of the Lax–Milgram theorem similar to Lemma 4.1 below shows that $(\lambda_0 - A^*)^{-1}$ maps L^2 into H^2. It follows that the resolvent is a compact operator of Hilbert–Schmidt type, and that the spectrum of A^* is discrete.

4. The function $U(x, y)$. We next discuss the differential equation in (1.4), but first let us give some more notation. In the hope of minimizing confusion, we will denote functions on $[0, 2\pi] \times [0, 2\pi]$ by uppercase letters (F, Φ, ...) to distinguish them from functions of a single variable (f, ϕ, ...). The Sobolev spaces of doubly periodic functions will be denoted by $\mathcal{H}^s$, and their norms are defined by

$$||F||_{\mathcal{H}^s}^2 = 4\pi^2 \sum_{p,q=-\infty}^{\infty} (2+p^4+q^4)^{s/2} \, |\hat{F}(p,q)|^2 \,. \tag{4.1}$$

Note that for $s=0$, this agrees with the definition of the $\mathcal{L}^2$-norm as the square integral. The choice of the Fourier multipliers $(2+p^4+q^4)^{s/2}$ in place of the standard $(1+p^2+q^2)^s$ allows for an easier comparison between functions of one and two variables. Abusing notation, we will identify a function $F(x,y) \in \mathcal{L}^2$ with the corresponding integral operator F on L^2. Operators on functions of two variables will be denoted by calligraphic letters $(\mathcal{F}, \mathcal{G}, \dots)$.

Let A be the differential operator from (1.1), and define the differential operator $\mathcal{A}$ acting on doubly periodic functions $F(x,y)$ by

$$\begin{aligned} \mathcal{A}[F] = &-(\partial_x^4+\partial_y^4)F - \partial_x^2\big(a(x)F\big) + \partial_y^2\big(a(y)F\big) \\ &+ \partial_x\big(b(x)F\big) + \partial_y\big(b(y)F\big) - \big(c(x)+c(y)\big)F \,. \end{aligned} \tag{4.2}$$

Note that $\mathcal{A} = \mathcal{A}_x + \mathcal{A}_y$, where $\mathcal{A}_x$ and $\mathcal{A}_y$ denote the operators that act on the x or y variable of F while keeping the other one fixed. Assume that the spectra of A and $-A^*$ are disjoint. We have seen that the integral operator U given by a weak solution $U(x,y)$ of

$$\mathcal{A}^* U(x,y) = \delta_{y-x} \tag{4.3}$$

satisfies Lyapunov's equation (1.9) with $V=I$. This solution is unique by Belonosov's theorem.

Let us solve (4.3) in the special case where $\mathcal{A}_0$ is defined by (4.2) with $a=b=0$ and $c=1$. Since the operator has constant coefficients, the solution can be written as $U_0(x,y) = u_0(x-y)$, where

$$2A_0 u_0 = \delta_0 \,;$$

in other words, $2U_0(x,y)$ is the Green's function of A_0 on $[0,2\pi]$ with periodic boundary conditions. One can compute $u_0(x)$ explicitly as a linear combination,

$$u_0(x) = C_1 \cos\frac{x-\pi}{\sqrt{2}} \cosh\frac{x-\pi}{\sqrt{2}} + C_2 \sin\frac{x-\pi}{\sqrt{2}} \sinh\frac{x-\pi}{\sqrt{2}} \,,$$

where the coefficients are adjusted so that u_0 is periodic and twice differentiable, and its third derivative jumps by $-1/2$ at $x=0$. From this representation, it is clear that U_0 is smooth away from the line $x=y$, that its second derivative is continuous, and that its third derivative is bounded. Alternately, we easily obtain from the Fourier series representation $\widehat{A_0 u}(p) = -(1+p^4)\hat{u}(p)$ that $\hat{u}_0(p) = -\frac{1}{4\pi(1+p^4)}$, and

$$U_0(x,y) = -\frac{1}{4\pi} \sum_{p=-\infty}^{\infty} \frac{1}{1+p^4} e^{ip(x-y)} \,. \tag{4.4}$$

In particular, $U_0(x,y) \in \mathcal{H}^s$ for all $s < \frac{7}{2}$, and $||U_0||_{\mathcal{H}^3} \le 1$.

The difference

$$K(x,y) := U(x,y) - U_0(x,y)$$

solves the partial differential equation

$$\mathcal{A}^* K(x,y) = -\big(\mathcal{A}^* - \mathcal{A}_0\big) U_0(x,y)\,. \tag{4.5}$$

The next lemma provides a weak solution of (4.5). Note that the right-hand side lies in $\mathcal{L}^2$, because

$$||\mathcal{A}^* - \mathcal{A}_0||_{\mathcal{H}^2 \to \mathcal{L}^2} \leq 2M \tag{4.6}$$

by computations analogous to Lemma 3.1.

LEMMA 4.1 (construction of K). *Let $\mathcal{A}$ be the operator on $\mathcal{L}^2$ given by* (4.2)*. If A satisfies* (1.3) *and a'', b', and c are bounded measurable functions, then the resolvent of $\mathcal{A}^*$ is compact and maps $\mathcal{L}^2$ into $\mathcal{H}^2$.*

Proof. Let λ_0 be the vertex of the sector computed in Lemma 3.2, and assume that $F(x,y) \in \mathcal{L}^2$. We verify that the equation

$$\big(2\lambda_0 - \mathcal{A}^*\big) K(x,y) = F(x,y)$$

satisfies the assumptions of the Lax–Milgram theorem, as stated in [14, p. 297].

Define a bilinear form on smooth doubly periodic functions Φ, Ψ by

$$B(\Phi,\Psi) = \langle (2\lambda_0 - \mathcal{A}^*)\Phi, \Psi\rangle_{\mathcal{L}^2}\,.$$

Then B is extended continuously to $\mathcal{H}^2$ by

$$B(\Phi,\Psi) = \langle \Phi,\Psi\rangle_{\mathcal{H}^2} + 2\lambda_0 \langle \Phi,\Psi\rangle_{\mathcal{L}^2} - \big\langle (\mathcal{A}^* - \mathcal{A}_0)\Phi, \Psi\big\rangle_{\mathcal{L}^2}\,.$$

On the other hand, it follows from (3.6) that

$$B(\Phi,\Phi) \geq \frac{1}{2} ||\Phi||^2_{\mathcal{H}^2}\,.$$

Finally, the map $\Phi \mapsto -\langle \Phi, F\rangle_{\mathcal{L}^2}$ defines a continuous linear form on $\mathcal{H}^2$. The Lax–Milgram theorem asserts that there exists a unique function $K(x,y) \in \mathcal{H}^2$ such that

$$B(K,\Psi) = \big\langle F, \Psi\big\rangle_{\mathcal{L}^2}$$

for all $\Psi \in \mathcal{H}^2$. By the resolvent identity, the equation

$$\big(\mathcal{A}^* - \lambda\big) K(x,y) = F(x,y)$$

has a unique weak solution in $\mathcal{H}^2$ for every value of λ that is not an eigenvalue of $\mathcal{A}^*$ and every $F(x,y) \in \mathcal{L}^2$. Since $\mathcal{H}^2$ is compactly embedded in $\mathcal{L}^2$, the resolvent is compact. $\square$

LEMMA 4.2 (regularity). *Assume the hypotheses of Lemma* 4.1*. If $K(x,y) \in \mathcal{H}^2$ solves* (4.5)*, then $K(x,y) \in \mathcal{H}^4$, and*

$$||K(x,y)||_{\mathcal{H}^4} \leq 2M ||U_0(x,y) + K(x,y)||_{\mathcal{H}^2}\,,$$

where the constant is given by (1.3).

Proof. If $K(x,y)$ solves (4.5), then

$$\mathcal{A}_0 K = -(\mathcal{A}^* - \mathcal{A}_0)(U_0 + K)\,. \tag{4.7}$$

Since $\mathcal{A}_0$ defines an isometry from $\mathcal{H}^4$ to $\mathcal{L}^2$, it follows from (4.6) that

$$||K(x,y)||_{\mathcal{H}^4} \leq ||\mathcal{A}^* - \mathcal{A}_0||_{\mathcal{H}^2 \to \mathcal{L}^2} \cdot ||U_0 + K||_{\mathcal{H}^2} \leq 2M ||U_0 + K||_{\mathcal{H}^2}\,. \qquad \square$$

5. The operator *U*. In this section, we derive bounds for $U = U_0 + K$ as an operator on L^2. Since $K(x,y) \in \mathcal{H}^4$, while $U_0(x,y) \in \mathcal{H}^s$ only for $s < 7/2$, the Fourier coefficients of $K(x,y)$ decay more quickly than the Fourier coefficients of $U_0(x,y)$. This in turn implies that the restriction of U to high Fourier modes is dominated by U_0. We now provide the relevant estimates.

As a consequence of the regularity result in Lemma 4.2 we see that U defines a bounded linear operator from L^2 to H^4, with

$$||U||_{L^2\to H^4} \le ||U_0||_{L^2\to H^4} + ||K||_{L^2\to H^4} \le \frac{1}{2} + ||K(x,y)||_{\mathcal{H}^4}\,.$$

We have used that $A_0U_0 = \frac{1}{2}\delta_{y-x}$ and applied (1.10) to $A_0K(x,y)$.

One attractive property of the $\mathcal{H}^4$-norm is that it depends only on the magnitude of the Fourier coefficients, not on the phases. In contrast, the operator norm

$$||F||_{L^2\to H^4} = \sup_{||\phi||_{L^2}=||\psi||_{L^2}=1} \langle A_0F\phi,\psi\rangle = 4\pi^2 \sum_{p,q=-\infty}^{\infty} (1+p^4)\hat{F}(p,q)\hat{\phi}(q)\hat{\psi}(p)$$

can change drastically if we replace $\hat{F}(p,q)$ by $|\hat{F}(p,q)|$. This sensitivity to cancellations can cause difficulties in estimates: Multiplying the Fourier coefficients of F with factors $\alpha(p,q) \in [0,1]$ may either increase or decrease the operator norm. On the other hand, the $\mathcal{H}^4$-norm provides only a loose bound on the norm of the corresponding integral operator. For instance, the function $U_0(x,y)$ (and consequently $U(x,y)$) does not lie in $\mathcal{H}^4$, even though $||U_0||_{L^2\to H^4} = \frac{1}{2}$.

We find it useful to introduce another norm on integral kernels that lies between the $\mathcal{H}^4$-norm (as a function of two variables) and the operator norm (as a linear transformation from L^2 to H^4). By construction, this norm depends only on the modulus of the Fourier coefficients.

LEMMA 5.1 (auxiliary norm). *Define, for smooth doubly periodic functions F,*

$$|||F||| := 4\pi^2 \sup_{||\phi||_{L^2}=||\psi||_{L^2}=1} \sum_{p,q=-\infty}^{\infty} (2+p^4+q^4)|\hat{F}(p,q)|\cdot|\hat{\phi}(p)|\cdot|\hat{\psi}(q)|\,.$$

Then

$$|||F||| \le ||F(x,y)||_{\mathcal{H}^4}$$

and

$$|||F||| \ge \max\big\{||F||_{L^2\to H^4}, ||F||_{H^{-4}\to L^2}, 2||F||_{H^{-2}\to H^2}\big\}\,.$$

Proof. From the Fourier representation, we see that

$$\begin{aligned}|||F||| &\le \sup_{||\Phi(x,y)||_{\mathcal{L}^2}=1} 4\pi^2 \sum_{p,q=-\infty}^{\infty} (2+p^4+q^4)|\hat{F}(p,q)|\,|\hat{\Phi}(p,q)| \\ &\le \sup_{||\Phi(x,y)||_{\mathcal{L}^2}=1} \langle \mathcal{A}_0F,\Phi\rangle_{\mathcal{L}^2} \\ &= ||F(x,y)||_{\mathcal{H}^4}\,.\end{aligned}$$

On the other hand,

$$||F||_{L^2\to H^4} = \sup_{||\phi||=||\psi||=1} \sum_{p,q=-\infty}^{\infty} (1+p^4)\hat{\phi}(p)\hat{\psi}(q)\hat{F}(p,q) \le |||F|||\,,$$

and similarly

$$||F||_{H^{-4}\to L^2} \le |||F|||\,, \quad ||F||_{H^{-2}\to H^2} \le \frac{1}{2}|||F|||\,. \qquad \square$$

If F has positive Fourier coefficients, then $|||F|||$ agrees with the operator norm of $\mathcal{A}_0 F$ as a linear transformation from L^2 into itself. In general, we can interpret $|||F|||$ as the norm of $\mathcal{A}_0\tilde{F}$, where $\tilde{F}$ is obtained from F by replacing all its Fourier coefficients with their absolute values,

$$\tilde{F}(x,y) = \sum_{p,q=-\infty}^{\infty} |\hat{F}(p,q)| e^{i(px+qy)}\,.$$

LEMMA 5.2 (tail estimate). *Assume that* $K(x,y)$ *solves* (4.5)*, set* $U = U_0 + K$*, and let* M *be given by* (1.3)*. Then*

$$||(I-P_N)K(I-P_N)||_{H^{-2}\to H^2} \le \frac{1}{2} M N^{-2} |||U|||\,. \tag{5.1}$$

Furthermore,

$$||(I-P_N)K||_{H^{-2}\to H^2} = ||K(I-P_N)||_{H^{-2}\to H^2} \le \frac{3}{4} M N^{-2} |||U|||\,. \tag{5.2}$$

Proof. Let $\widetilde{A - A_0}$ be the differential operator obtained from $A - A_0$ by replacing the Fourier coefficients of $a(x)$, $b(x)$, and $c(x)-1$ with their absolute values, and let $\tilde{K}(x,y)$ and $\tilde{U}(x,y)$ be the functions with Fourier coefficients $|\hat{K}(p,q)|$ and $|\hat{U}(p,q)|$, respectively.

By Lemma 5.1, we obtain for the left-hand side of (5.1)

$$\begin{aligned} ||(I-P_N)K(I-P_N)||_{H^{-2}\to H^2} &\le \frac{1}{2}|||(I-P_N)K(I-P_N)||| \\ &= \frac{1}{2}||(I-P_N)\big(\mathcal{A}_0\tilde{K}\big)(I-P_N)||_{L^2\to L^2}\,. \end{aligned}$$

We next use (4.7) to write

$$\mathcal{A}_0 K(x,y) = -(\mathcal{A}^* - \mathcal{A}_0)U(x,y)\,,$$

and recall that $\mathcal{A}^*U(x,y)$ is the integral kernel of the operator $A^*U + UA$ to obtain

$$||(I-P_N)\big(\mathcal{A}_0\tilde{K}\big)(I-P_N)||_{L^2\to L^2} \le 2||\widetilde{A^* - A_0}||_{H^2\to L^2} \cdot ||\tilde{U}||_{H^{-2}\to H^2} \cdot ||I-P_N||_{L^2\to H^{-2}}\,.$$

We can use Lemmas 3.1 and 5.1 to bound the first two factors on the right-hand side by $M|||U|||$. The last factor is bounded by N^{-2}, because the pth Fourier coefficient enters into the H^{-2}-norm with a weight of $(1+p^4)^{-\frac{1}{2}} \le p^{-2}$.

For (5.2), we use again (4.7) to write

$$K(x,y) = -\mathcal{A}_0^{-1}\big\{\big((A^*-A_0)U + U(A^*-A_0)\big)(x,y)\big\}\,.$$

In the first summand, we replace the Fourier multiplier $(2+p^4+q^4)^{-1}$ of $\mathcal{A}_0^{-1}$ by $(1+p^4)^{-1}$ to obtain

$$\begin{aligned} &||(I-P_N)\big(\mathcal{A}_0^{-1}(A^*-A_0)U\big)||_{H^{-2}\to H^2} \\ &\qquad \le ||(I-P_N)A_0^{-1}(\widetilde{A^*-A_0})\tilde{U}||_{H^{-2}\to H^2} \\ &\qquad \le ||I-P_N||_{H^4\to H^2} \cdot ||\widetilde{A^*-A_0}||_{H^2\to L^2} \cdot ||\tilde{U}||_{H^{-2}\to H^2} \\ &\qquad \le \frac{1}{2} M N^{-2} |||U|||\,. \end{aligned}$$

For the second summand, we replace $(2+p^4+q^4)^{-1}$ by $\frac{1}{2}(1+p^4)^{-1/2}(1+q^4)^{-1/2}$ and estimate

$$\begin{aligned}
&||(I-P_N)\big(\mathcal{A}_0^{-1}U(A^*-A_0)\big)||_{H^{-2}\to H^2}\\
&\qquad \le \frac{1}{2}||(I-P_N)D^{-2}\tilde{U}(\widetilde{A^*-A_0})D^{-2}||_{H^{-2}\to H^2}\\
&\qquad \le \frac{1}{2}||I-P_N||_{H^4\to H^2}\cdot||\tilde{U}||_{H^{-2}\to H^2}\cdot||\widetilde{A^*-A_0}||_{L^2\to H^{-2}}\\
&\qquad \le \frac{1}{4}MN^{-2}|||U|||\,.
\end{aligned}$$

Adding the two inequalities gives (5.2). ◻

Lemma 5.3. *Under the assumptions and with the notation of Lemma* 5.2*,*

$$1 \le |||U||| \le \frac{1}{1-\frac{3}{2}MN^{-2}}\big(1+|||P_NKP_N|||\big)\,, \tag{5.3}$$

provided that $N^2 > \frac{3}{2}M$*.*

Proof. For the first inequality, we use that U satisfies

$$\begin{aligned}
|||U||| &\ge |||(I-P_N)U_0(I-P_N)||| - |||(I-P_N)K(I-P_N)|||\\
&\ge 1 - MN^{-2}|||U|||
\end{aligned}$$

and take $N\to\infty$. For the second inequality, we write

$$\begin{aligned}
|||U||| &\le 1+|||P_NKP_N|||+|||(I-P_N)KP_N|||+|||K(I-P_N)|||\\
&\le 1+|||P_NKP_N|||+\frac{3}{2}MN^{-2}|||U|||
\end{aligned}$$

and solve for $|||U|||$. ◻

6. Addition rule for the instability index. We return to the Pontryagin space Π introduced in section 2, with the indefinite inner product given by (2.1) on an underlying Hilbert space H. Let Π_1 be a finite-dimensional subspace of Π, and let

$$\Pi_2 = \Pi_1^{\perp_U} = \big\{f\in\Pi\ \big|[f,g]=0 \text{ for all } g\in\Pi_1\big\}$$

be its U-orthogonal complement. By construction, $\dim\Pi_1 = \operatorname{codim}\Pi_2$. The natural question is, can we compute $\kappa(U)$ from the restrictions $\kappa(U|_{\Pi_1})$ and $\kappa(U|_{\Pi_2})$? The difficulty is that Π need not be a direct sum of Π_1 and Π_2, because the two subspaces may intersect nontrivially in a subspace where the quadratic form vanishes.

A subspace $X\subset\Pi$ is called *neutral* if $[f,f]=0$ for all $f\in X$. Two finite-dimensional neutral subspaces X and Y of H are Π-*skewly linked* if

$$\dim X = \dim Y$$

and the inner product $[\cdot,\cdot]$ does not degenerate on the direct sum $X\dot{+}Y$. In particular, no vector of X different from 0 is orthogonal to the skewly linked subspace Y, and vice versa.

Theorem 6.1 (see [18, Theorem 3.4]). *Let* Π *be a Pontryagin space with inner product* $[\cdot,\cdot]$ *given by* (2.1)*. Consider an arbitrary subspace* Π_1 *of* Π*, its* U*-orthogonal* *complement* Π_2*, and their intersection* $X = \Pi_1 \cap \Pi_2$*. There exists a neutral subspace* $Y \subset \Pi$ *that is skewly linked to* X *and provides a* U*-orthogonal decomposition*

$$\Pi = \Pi_1' \oplus (X \dot{+} Y) \oplus \Pi_2' \,, \tag{6.1}$$

where

$$\Pi_1 = \Pi_1' \oplus X, \qquad \Pi_2 = \Pi_2' \oplus X.$$

The theorem was originally formulated for the case of regular Pontryagin spaces, where the quadratic form U is a bounded operator with bounded inverse. Under the assumption that Π_1 is finite-dimensional, the result easily extends to the situation where the inverse of U is unbounded but densely defined. Although the above decomposition is not unique in general, it yields the following addition formula for instability indices.

Proposition 6.2 (addition rule). *Let* Π *be a Pontryagin space with inner product* $[\cdot,\cdot]$ *given by* (2.1). *If* Π_1 *is any finite-dimensional subspace* Π*, and* Π_2 *is its* U*-orthogonal complement, then the instability index of* U *is given by*

$$\kappa(U) = \kappa(U|_{\Pi_1}) + \kappa(U|_{\Pi_2}) + \dim(\Pi_1 \cap \Pi_2)\,.$$

In particular, if $U|_{\Pi_2}$ *is negative definite, then* $\kappa(U) = \kappa(U|_{\Pi_1})$*.*

Proof. Theorem 6.1 provides subspaces Π_1' and Π_2' such that

$$\kappa(U) = \kappa(U|_{\Pi_1'}) + \kappa(U|_{\Pi_2'}) + \kappa(U|_{X\dot{+}Y})\,.$$

By construction, we have $\kappa(U|_{\Pi_1'}) = \kappa(U|_{\Pi_1})$ and $\kappa(U|_{\Pi_2'}) = \kappa(U|_{\Pi_2})$.

Since X and Y are skewly linked and finite-dimensional, there exists for each basis $\phi_1, \phi_2, \ldots, \phi_m$ of X a basis $\psi_1, \psi_2, \ldots, \psi_m$ of Y such that $[\phi_i, \psi_j] = \delta_{ij}$ $(i, j = 1, \ldots, m)$. By expanding an arbitrary element $f \in X \dot{+} Y$ as

$$f = \sum_{i=1}^m \alpha_i \phi_i + \sum_{j=1}^m \beta_j \psi_j \,,$$

the restriction of the indefinite inner product to this subspace can be expressed as

$$[f, f] = 2 \sum_{i=1}^m \alpha_i \beta_i = \frac{1}{2} \left(\sum_{i=1}^m (\alpha_i + \beta_i)^2 - \sum_{i=1}^m (\alpha_i - \beta_i)^2 \right) \,.$$

This explicit representation in terms of positive and negative squares shows that

$$\kappa(U|_{X\dot{+}Y}) = \dim(X)\,.$$

If, moreover, $U|_{\Pi_1}$ is negative definite, then $\kappa(U|_{\Pi_2}) = 0$, $\Pi_1 \cap \Pi_2 = \emptyset$, and $\kappa(U) = \kappa(U|_{\Pi_1})$. ☐

7. Main results. We first consider the claim in (1.6).

Proposition 7.1 (projecting out high Fourier modes). *Let* A *be given by* (1.1)*, where* a''*,* b'*, and* c *are bounded measurable functions that satisfy* (1.3)*. Assume that the spectra of* A *and* $-A^*$ *are disjoint, and let* $U(x, y)$ *be the kernel of the unique solution of Lyapunov's equation constructed in section* 4*. If*

$$N^2 > M |||U||| \,, \tag{7.1}$$

where M is the constant from (1.3)*, then*

$$\kappa(A) = \kappa(P_N U^{-1} P_N)\,.$$

For the proof, we find it useful to introduce the following first order pseudodifferential operator. Let D be the unique positive definite self-adjoint operator on L^2 such that

$$D^4[f] = -A_0[f] = f'''' + f\,. \tag{7.2}$$

Then D provides an isometry from H^{s+1} onto H^s for every value of s. In the Fourier series representation, it is given by the multiplication operator $(1+p^4)^{1/4}$. We will frequently use the fact that $D^4U_0 = D^2U_0D^2 = U_0D^4 = -\frac{1}{2}I$.

Proof of Proposition 7.1. By Theorem 2.2, we have $\kappa(A) = \kappa(U)$. Let $[f,g] = \langle Uf, g\rangle$ be the indefinite inner product associated with U. Choose Π_2 to be the range of $I - P_N$, and let $\Pi_1 = \Pi_2^{\perp_U}$ be its U-orthogonal complement. Then $\dim \Pi_1 = \operatorname{codim} \Pi_2 = 2\mathrm{N} - 1$. We will show that

$$\kappa(U) = \kappa(U|_{\Pi_1})\,. \tag{7.3}$$

This will establish the conclusion, because

$$\Pi_1 = \text{Nullspace}\,(I - P_N)U = U^{-1}(\text{Range}\,(P_N))\,.$$

Since U is self-adjoint, its restriction to Π_1 is determined by the quadratic form

$$[U^{-1}P_Nf, U^{-1}P_Nf] = \langle P_NU^{-1}P_Nf, f\rangle\,.$$

Set $\varepsilon_N = MN^{-2}|||U||| < 1$. Writing $U = U_0 + K$, and using that $D^2U_0D^2 = -\frac{1}{2}I$, we see that

$$[D^2f, D^2f] = \langle (U_0+K)D^2f, D^2f\rangle = -\frac{1}{2}||f||^2_{L^2} + \langle D^2KD^2f, f\rangle$$

for all $f \in \Pi$ for which the left-hand side is finite. For $f \in \Pi_2 = \text{Range}\,(I - P_N)$, this becomes

$$\begin{aligned}
[D^2f, D^2f] &= -\frac{1}{2}||f||^2_{L^2} + \langle (I-P_N)D^2KD^2(I-P_N)f, f\rangle \\
&\le \Big(-\frac{1}{2} + ||(I-P_N)K(I-P_N)||_{H^{-2}\to H^2}\Big)\cdot ||f||^2_{L^2} \\
&\le -\frac{1}{2}\big(1 - MN^{-2}|||U|||\big)\cdot ||f||^2_{L^2}\,.
\end{aligned}$$

In the second line, we have used that D commutes with P_N, and interpreted the largest eigenvalue of the quadratic form $D^2(I-P_N)K(I-P_N)D^2$ as the norm of the corresponding operator. The last line follows from (5.1) of Lemma 5.2. Replacing f with $D^{-2}f$, and recalling that $U_0 = D^{-4}$, we obtain, for $f \in \Pi_2$,

$$\langle Uf, f\rangle \le -\frac{1}{2}(1 - \varepsilon_N)\langle U_0 f, f\rangle\,. \tag{7.4}$$

By Proposition 6.2, the claim in (7.3) follows. $\square$

For the main result, we want to replace Π_1 by the range of P_N. The next two lemmas concern the restriction of U to this space.

LEMMA 7.2 (Lyapunov equation for $P_N A P_N$). *Under the assumptions of Proposition* 7.1*, if*

$$N^2 > \frac{3}{2} M^2 |||U|||\,,$$

then

$$\kappa(P_N A P_N) = \kappa(P_N U P_N)\,.$$

Proof. We will show that

$$(P_N A P_N)^*(P_N U P_N) + (P_N U P_N)(P_N A P_N) \geq \Big(1 - \frac{3}{2} M\varepsilon_N\Big) P_N\,, \tag{7.5}$$

where $\varepsilon_N = MN^{-2}|||U|||$, and then apply Taussky's theorem.

For $f \in L^2$, we write $f_1 = P_N f$, $f_2 = (I - P_N)f$ and decompose

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

In this notation, Proposition 7.1 says that U_{22} is negative definite. From (1.9), we see that U_{11} solves Lyapunov's equation

$$A_{11}^* U_{11} + U_{11} A_{11} = V$$

with $V = I_{11} - A_{12}^* U_{21} - U_{12} A_{21}$. We claim that the right-hand side is positive definite on the range of P_N.

To prove this claim, first observe that we can replace A by $A - A_0$ and U by $K = U - U_0$ in the definition of V, because A_0 and U_0 are diagonal in the Fourier representation. We estimate

$$\begin{aligned}
||A_{12}^* U_{21} &+ U_{12} A_{21}||_{L^2 \to L^2} \\
&\leq 2||P_N(A^* - A_0)(I - P_N)K P_N||_{L^2 \to L^2} \\
&\leq 2||A^* - A_0||_{H^2 \to L^2} \cdot ||(I - P_N)K||_{H^{-2} \to H^2} \cdot ||P_N||_{L^2 \to H^2} \\
&\leq \frac{3}{2} M^2 N^{-2} |||U|||
\end{aligned}$$

by Lemma 5.2. It follows that $V \geq (1 - \frac{3}{2} M\varepsilon_N) P_N > 0$ on the range of P_N. ∎

LEMMA 7.3. *Under the assumptions of Lemma* 7.2*, $P_N U P_N$ is invertible on the range of P_N, and*

$$||\big(D^2 P_N U P_N D^2\big)^{-1}||_{L^2 \to L^2} \leq \frac{2(M+1)}{1 - \frac{3}{2} M\varepsilon_N}\,.$$

Proof. Let us write $A_N = D^{-2} P_N A P_N D^2$ and $U_N = D^2 P_N U P_N D^2$. Lemma 7.2 implies that A_N solves Lyapunov's equation on the range of P_N with a positive definite right-hand side. Explicitly, we proved in (7.5) that

$$A_N^* U_N + U_N A_N \geq \Big(1 - \frac{3}{2} M\varepsilon_N\Big) D^4 P_N$$

as quadratic forms on the range of P_N. Here, $\varepsilon_N = MN^{-2}|||U|||$, as in Lemma 7.2. We apply this inequality to an eigenfunction ϕ_0 of U_N with eigenvalue μ_0,

$$2\mu_0 \mathbf{Re}\,\langle A_N^*\phi_0, \phi_0\rangle \geq \Big(1 - \frac{3}{2}M\varepsilon_N\Big)\langle D^2\phi_0, D^2\phi_0\rangle\,.$$

(We have used that ϕ_0 lies in the range of P_N to simplify the right-hand side.) Let μ_0 be the eigenvalue of U_N with minimal modulus, and let $\psi_0 = D^2\phi_0$. From Lemma 3.1, we conclude that

$$||U_N^{-1}||_{L^2\to L^2} = |\mu_0|^{-1} \leq 2\sup_{\psi\in L^2} \frac{\big|\mathbf{Re}\,\langle A_N^* D^{-2}\psi, D^{-2}\psi\rangle\big|}{(1-\frac{3}{2}M\varepsilon_N)||\psi||_{L^2}^2}\,.$$

Since $A_N = A_0 + (A_N - A_0)$ and $A_0 = D^4$, the numerator is bounded by $1 + ||A^* - A_0^*||_{H^2\to H^{-2}}||\psi||_{L^2}^2$. This yields

$$||U_N^{-1}||_{L^2\to L^2} \leq 2\frac{1 + ||A^* - A_0||_{H^2\to H^{-2}}}{1-\frac{3}{2}M\varepsilon_N} \leq \frac{2(M+1)}{1-\frac{3}{2}M\varepsilon_N}\,. \qquad \square$$

We are finally ready for the main result.

Proposition 7.4 (reduction to trigonometric polynomials). *Let A be the differential operator given by* (1.1)*, where a'', b', and c are bounded measurable functions that satisfy* (1.3)*. Assume that the spectra of A and $-A^*$ are disjoint, and let $U(x,y)$ be the unique weak solution of* (4.3)*. If*

$$N^2 > 4\max\{M, M^2\}\cdot |||U|||\,, \tag{7.6}$$

where M is given by (1.3)*, then*

$$\kappa(A) = \kappa(P_N A P_N)\,.$$

Proof. Since U solves Lyapunov's equation, Theorem 2.2 implies that $\kappa(A) = \kappa(U)$, and we already know from Lemma 7.2 that $\kappa(P_N A P_N) = \kappa(P_N U P_N)$. It remains to prove that $\kappa(U) = \kappa(P_N U P_N)$. We want to apply Proposition 6.2 in the case where Π_1 is the range of P_N. On the complementary space

$$\Pi_2 = \mathrm{Range}\,(P_N)^{\perp_U} = \big\{f\in L^2 \mid U_{11}f_1 + U_{12}f_2 = 0\big\}\,,$$

we compute for the indefinite inner product

$$\begin{aligned}[f,f] &= \langle U_{11}f_1, f_1\rangle + \langle U_{12}f_2, f_1\rangle + \langle U_{21}f_1, f_2\rangle + \langle U_{22}f_2, f_2\rangle \\ &= -\big\langle U_{21}U_{11}^{-1}U_{12}f_2, f_2\big\rangle + \langle U_{22}f_2, f_2\rangle\,.\end{aligned}$$

Set $\varepsilon_N = MN^{-2}|||U|||$. In the first summand above, we bound the middle factor by Lemma 7.3,

$$||\big(D^2U_{11}D^2\big)^{-1}||_{L^2\to L^2} \leq \frac{2(M+1)}{1-\frac{3}{2}M\varepsilon_N}\,.$$

For the two outer factors, we use that U_0 is diagonal in the Fourier representation to replace U with K, and then apply Lemma 5.2,

$$||D^2U_{21}D^2||_{L^2\to L^2} = ||D^2U_{12}D^2||_{L^2\to L^2} \leq \frac{3}{4}\varepsilon_N\,.$$

Since $\varepsilon_N \le \frac{1}{4}$ and $M\varepsilon_N \le \frac{1}{4}$, we obtain for the product

$$||D^2 U_{21} U_{11}^{-1} U_{12} D^2||_{L^2 \to L^2} \le \frac{2(M+1)}{1 - \frac{3}{2} M \varepsilon_N} \cdot \left(\frac{3\varepsilon_N}{4} \right)^2 \le \frac{9}{40} \,.$$

On the other hand, (7.4) of Proposition 7.1 says that the second summand is negative on the nullspace of P_N and satisfies the bound

$$D^2 U_{22} D^2 \le -\frac{1}{2}\big(1 - \varepsilon_N\big)(I - P_N) \le -\frac{3}{8} I$$

as a quadratic form on Π_2. We conclude that

$$D^2 \big\{ -U_{21} U_{11}^{-1} U_{12} + U_{22} \big\} D^2 \le -\frac{3}{20} I$$

as a quadratic form on Π_2. The claim now follows from Proposition 6.2. ☐

In many applications, including the thin film example in (1.2), the operator A in (1.1) results from linearizing an evolution equation that conserves mass, and the zeroth coefficient vanishes ($c = 0$). Proposition 7.4 does not immediately apply to such operators because the constant functions lie in the kernel of A^*, violating the hypothesis that the spectra of A and $-A^*$ are disjoint.

In this case, we restrict A to the space L_0^2 of square integrable periodic functions that have zero mean. The adjoint of A on L_0^2 is given by

$$A^*[f](x) = -f'''' - a(x) f'' - b(x) f' + \frac{1}{2\pi} \int_0^{2\pi} a(y) f''(y) + b(y) f'(y)\, dy \,.$$

We can replace the constant from (1.3) by

$$M_0 = \sum_{p \ne 0} \Big\{ |\hat{a}(p)| + |\hat{b}(p)| \Big\} \,. \tag{7.7}$$

On the Sobolev spaces H_0^s of periodic functions with mean zero, we use the homogeneous norms

$$||f||_{H_0^s} = 2\pi \sum_{p \ne 0} p^{2s} |\hat{f}(p)|^2 \,.$$

On the corresponding Sobolev spaces $\mathcal{H}_0^s$ of doubly periodic functions in $\mathcal{H}^s$ whose integral over each cross section $x = 0$ and $y = 0$ vanishes, we use the norms

$$||F(x,y)||_{\mathcal{H}_0^s} = 4\pi^2 \sum_{p,q \ne 0} (p^4 + q^4)^{s/2} |\hat{F}(p,q)|^2 \,,$$

and in place of the auxiliary norm in Lemma 5.1 we use

$$|||F|||_0 := 4\pi^2 \sup_{||\phi||_{L_0^2} = ||\psi||_{L_0^2} = 1} \sum_{p,q \ne 0} (p^4 + q^4) |\hat{F}(p,q)| \cdot |\hat{\phi}(p)| \cdot |\hat{\psi}(q)| \,.$$

With these adjustments, the following conclusion remains valid.

Corollary 7.5 (reduction to trigonometric polynomials for $c = 0$). *Let A be the operator on L_0^2 given by* (1.1) *with $c = 0$, where a'' and b' are bounded measurable* *functions, and define* M_0 *by* (7.7)*. Assume that the spectra of* A *and* $-A^*$ *are disjoint, and let* $U(x,y)$ *be the unique weak solution of* (4.3) *in* $\mathcal{H}_0^2$*. If*

$$N^2 > 4\max\{M_0, M_0^2\} \cdot |||U|||_0\,,$$

then

$$\kappa(A) = \kappa(P_N A P_N)\,.$$

Proof. Set $A_0[f] = -f''''$, and let D be the unique positive definite operator with

$$D^4[f] = -A_0[f] = f''''\,.$$

Then Lemma 3.1 holds with M reduced to M_0. Clearly, A^* is sectorial L_0^2 and has compact resolvent on L_0^2. The solution of the equation $2A_0 u_0 = \delta_0$ on $\mathcal{L}_0^2$ is given by

$$U_0(x,y) = -\frac{1}{4\pi}\sum_{p\neq 0}\frac{1}{p^4}e^{ip(x-y)}\,;$$

see (4.4). All estimates from sections 4 and 5 carry over without further changes, and the claim follows by the proof of Proposition 7.4. ∎

8. Numerical example. Before we look at examples, let us briefly discuss how to verify the hypotheses on N in (7.1) or (7.6). The conditions involve the solution of the partial differential equation in (4.3). We propose two ways to use these conditions.

(a) *(Using Proposition* 7.1.*)* Solve (4.3) by a Galerkin approximation, and use this solution to compute, approximately, the value of $|||U|||$. If (7.1) is satisfied for some value of N much below the dimension of the Galerkin approximation, we apply Proposition 7.1 and compute the instability index for the restriction of U to $\big(\mathrm{Range}\,(\mathrm{I}-\mathrm{P_N})\big)^{\perp_\mathrm{U}}$. We use the Gram–Schmidt algorithm to find a basis for this subspace. If even (7.6) is satisfied, then we simply compute the instability index of U as $\kappa(P_N U P_N)$.

(b) *(Using Proposition* 7.4.*)* Start with a value of N such that $N^2 > 4\max\{M, M^2\}$. Write the matrix $P_N A P_N$ in the Fourier representation, find its eigenvalues, and bring it into triangular form. Solve Lyapunov's equation

$$P_N A^* P_N X_N + X_N P_N A P_N = I$$

for the finite matrix $X_N(p,q)$. This matrix is our numerical approximation to the Fourier representation of $P_N U P_N$. Compute μ_{max}, the largest eigenvalue of the matrix

$$\big((2+p^4+q^4)|X_N(p,q)-\delta_{pq}|\big)_{|p|,|q|<N}\,.$$

By Lemma 5.3, our best estimate for the norm of the true solution of (4.3) is given by

$$|||U||| \le \frac{1}{1-\frac{3}{2}MN^{-2}}\big(1+|||P_N K P_N|||\big) \approx \frac{1+\mu_{max}}{1-\frac{3}{2}MN^{-2}}\,.$$

If (7.6) is satisfied with the current value of N, accept $\kappa(P_N A P_N)$ as the instability index for A. Otherwise, increase N accordingly and repeat the above steps.

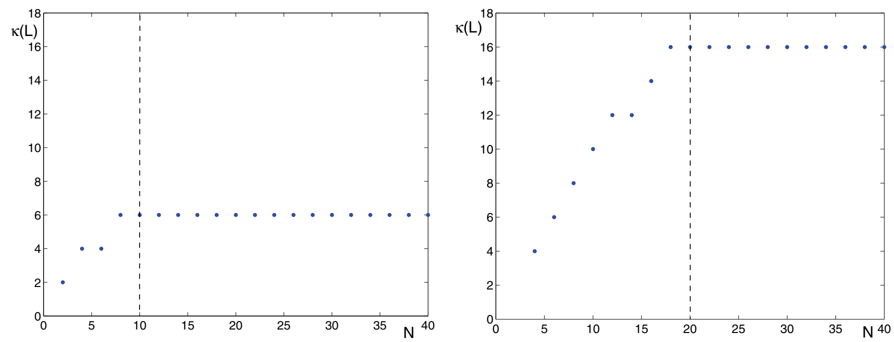

FIG. 1. *Numerical computation of the instability index of the operator A from* (1.2) *from the truncated Fourier matrices $P_N A P_N$, as a function of N. The parameter values are, on the left: $\mathcal{X} = 0.02$, $\alpha = 0$, $\beta = 1$; on the right: $\mathcal{X} = 0.0022$, $\alpha = 0$, $\beta = 1$. The dashed line shows the Fourier mode beyond which the instability index appears insensitive to truncation.*

Consider as an example the operator studied by Benilov in [6] given in (1.2). After rescaling the leading coefficient to 1, the operator has the form in (1.1) with

$$a(x) = 1 + \frac{\beta}{\mathcal{X}} \sin x\,, \quad b(x) = \frac{1-(\alpha+\beta)\cos x}{\mathcal{X}}\,, \quad c(x) = 0\,.$$

Corollary 7.5 reduces the computation of the stability index of A to a finite-dimensional linear algebra problem. This is illustrated in Figure 1. We see that if the parameter $\mathcal{X}$ is small, then the surface tension is not strong enough to overcome the effect of gravity. The model is unstable, and its instability index grows as the surface tension decreases.

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