How to achieve radial symmetry through simple rearrangements

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We like to think that symmetric problems have symmetric solutions, but symmetry-breaking can be observed everywhere in physics and geometry. Still, shouldn't at least optimal solutions be highly symmetric? Faced with a symmetric optimization problem in several variables, we commonly look for radial solutions, betting that the true optimizer might be among them. After all, balls minimize surface area and electrostatic capacity among bodies of a given volume, and the sharp constants in many integral inequalities, such as the Sobolev inequality, are assumed by particular symmetric decreasing functions. However, it can be difficult to compare a competitor for optimality with a radial solution. This is where symmetrization techniques come in.

In 1838, Steiner gave a "simple proof" of the isoperimetric inequality by comparing a given body not directly with a ball, but rather with a related body of the same volume that has just one hyperplane of symmetry (see Figure 1(a)). For a planar convex body (the case considered by Steiner), the perimeter decreases *strictly* under Steiner symmetrization unless the body is already reflection symmetric. Since the perimeter of a minimizing set cannot be reduced by symmetrization in any direction, the minimizer must be a disk. Steiner's contemporaries objected that this argument does not establish existence, and that the minimizer is identified only within the restricted class of convex bodies. These issues were finally resolved in 1958 by De Giorgi, using a technical notion of perimeter due to Caccioppoli. In the 1970s, symmetrization was recognized by analysts including Talenti, Lieb, and Baernstein as a tool for proving sharp functional inequalities.

A more direct geometric approach is to *construct* a sequence of symmetrizations that converges to a minimizer. This ultimately leads to the "competing symmetries principle" of Carlen and Loss, which has been used to find optimizers for many conformally invariant problems. The basic idea goes back to 1909, when Carathéodory proved the isoperimetric inequality, using a greedy sequence that reduces the moment of inertia by (almost) as much as possible in each step. How can convergence fail? Of course, the set of directions may simply be too small to generate full rotational symmetry. Yet convergence can be achieved by iterating Steiner symmetrization in finitely many well-chosen directions. Symmetrization along a sequence of directions chosen uniformly at random from the unit sphere converges almost surely to the ball.

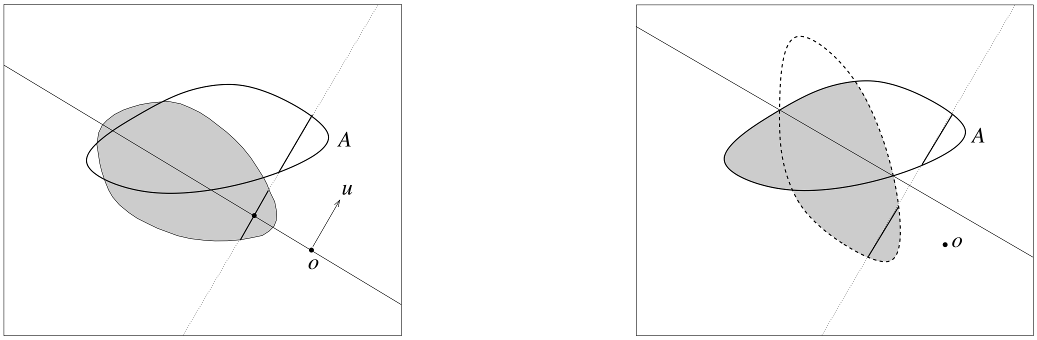

Figure 1: *(a) Left panel:* The Steiner symmetrization of a compact set in the direction of a unit vector u. The set is enclosed by a solid curve, and the symmetrized set is shaded. Steiner symmetrization creates a hyperplane of symmetry by replacing the intersection with each line parallel to u with a symmetric interval. Volume is preserved, and perimeter is reduced. *(b) Right panel:* The polarization of the same compact set with respect to a reflection. The solid line indicates the hyperplane of reflection, the set is enclosed by a solid curve, its mirror image by a dashed curve, and the polarized set is shaded. Polarization pushes mass towards the origin by replacing the portion of the set that lies in the half-space across from the origin whose reflection is not in the set with its mirror image. Both volume and perimeter are preserved. Note that convexity, smoothness, and non-trivial symmetries can be lost.

Over the last ten years, subtle geometric properties of Steiner symmetrization have become much better understood through the work of Bianchi, Chlebík, Cianchi, Fusco, Gronchi, Klain, Klartag, Lutwak, V. Milman, Van Schaftingen, Volčič, D. Yang, G. Zhang, and very recently Marc Fortier, Greg Chambers, and myself. Steiner symmetrization of a set of finite perimeter in a random direction almost surely decreases perimeter, unless the set is a ball. For any given convex body in $\mathbb{R}^d$ there exists a sequence of $3d$ Steiner symmetrizations that reduce its ratio of outradius over inradius to an absolute constant. Chambers and I find that Steiner symmetrization along d arbitrary linearly independent directions transforms every compact subset of $\mathbb{R}^d$ into a set of finite perimeter. It has been a nice surprise to discover new properties of such a classical tool.

The understanding of infinite sequences of Steiner symmetrizations has grown similarly. Every sequence that uses only a *finite* set of directions converges to a body that has at least partial symmetry. On the other hand, convergence can fail even for Steiner symmetrizations of a convex body along a *dense* set of directions, see Figure 2. Symmetrizations along a dense set of directions can be made to converge or diverge by simply reordering the sequence; furthermore, any sequence of directions (convergent or not) can be realized as a subsequence of one that fails to converge. In a forthcoming paper, Bianchi, Gronchi, Volčič, and I show that the known examples of non-convergent sequences of symmetrizations do converge "in shape" (i.e., up to a rotation), to limits that need not be ellipsoids or even convex. The proof relies on the fact that the intersection between any pair of sets can only grow while the perimeter of each set shrinks.

I am interested in the corresponding questions for an even simpler rearrangement, known as "polarization", or two-point symmetrization, see Figure 1(b). First introduced by Littlewood and Ahlfors in the 1940s for studying conformal invariants, polarization is particularly useful for proving geometric inequalities on spheres. It also lends itself well to quantitative estimates for path

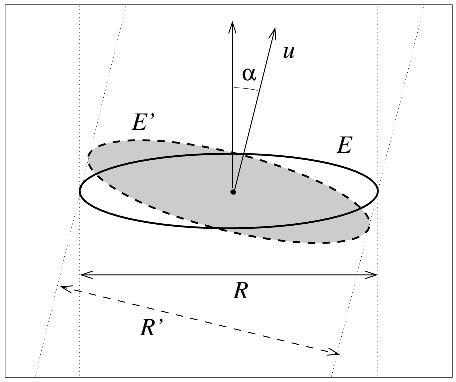

Figure 2: Steiner symmetrization transforms ellipses into ellipses. The unit vector u determines the direction of symmetrization, the original ellipse E is enclosed by a solid curve, and the symmetrized ellipse E' is shaded and enclosed by a dashed curve. Note that u is an axis of symmetry for E'. The diameters of E' and E satisfy $R' > R\cos\alpha$, where α is the angle between u and the minor semiaxis of E. If the angles $\alpha_i \in (0, \pi/2)$ between consecutive directions satisfy $\sum \alpha_i = \infty$, then the sequence of directions is dense in S^1. If, moreover, $\prod \cos\alpha_i > 0$, then the corresponding Steiner symmetrizations of E spin slowly about the origin while their shape converges to an ellipse that is not a ball.

integrals. While Steiner symmetrization (when it applies) reduces geometric inequalities to one-dimensional problems, polarization reduces them to combinatorial identities.

In his 2010 Master's thesis, Fortier found conditions for sequences of random polarizations to converge to the ball, which sharpen results that Van Schaftingen had proved five years earlier. In subsequent joint work, Fortier and I consider random sequences that may be far from uniformly distributed, and may concentrate on certain small sets or converge weakly to zero. If the distribution is sufficiently uniform, we show that the expected symmetric difference to the ball decreases proportionally to n^{-1} in the number of steps. Many questions remain open: Are there (non-random) sequences that converge more quickly? Is a greedy approach the best? We know that random polarizations of a compact set converge in Hausdorff distance, but we have no estimate for the rate. What distinguishes convergent from non-convergent sequences? Is there always convergence in shape?

As corollaries, Fortier and I obtain new convergence results for Steiner symmetrization, including a bound on the rate of convergence that does not require the set to be convex. This bound cannot be sharp, because Steiner symmetrization has much stronger smoothing properties than polarization. For convex bodies, Klartag had shown earlier that the achievable rate of convergence is faster than any power law. Remarkably, his constants grow only polynomially with the dimension. Can this result be extended to non-convex sets? Is the best possible rate of convergence in fact exponential and how does it depend on dimension? Given a configuration of d directions in $\mathbb{R}^d$, what geometric and algebraic conditions determine the ergodic properties of the corresponding Steiner symmetrizations? All these questions await answers.