Cases of Equality in the Riesz Rearrangement Inequality

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Abstract

We determine the cases of equality in the Riesz rearrangement inequality

$$\iint f(y)g(x-y)h(x)\,dydx \leq \iint f^*(y)g^*(x-y)h^*(x)\,dydx$$

where f^*, g^*, and h^* are the spherically decreasing rearrangements of the functions f, g, and h on $\mathbf{R}^n$. We apply our results to the weak Young inequality.

1 Statement of the results

The Riesz rearrangement inequality states that the functional

$$\mathcal{I}(f,g,h) := \int f * gh\,dx = \iint f(y)g(x-y)h(x)\,dydx \tag{1.1}$$

never decreases under spherical rearrangement, that is,

$$\mathcal{I}(f,g,h) \leq \mathcal{I}(f^*,g^*,h^*) \tag{1.2}$$

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for any triple (f, g, h) of nonnegative measurable functions on $\mathbf{R}^n$ for which the right hand side is defined. The **spherically decreasing rearrangement**, f^*, of a nonnegative measurable function f is the spherically decreasing function equimeasurable to f. We will define it by

$$f^*(x) \;=\; \sup\Big\{ s > 0 \;\mid \mu(\mathcal{N}_s(f)) \geq \omega_n\, |x|^n \Big\} \;,$$

where

$$\mathcal{N}_s(f) := \Big\{ x \in \mathbf{R}^n \;\mid\; f(x) > s \Big\}$$

is the level set of f at height s, and ω_n denotes the measure of the unit ball in $\mathbf{R}^n$. That is, the level sets of f^* are the centered balls of equal measure as the corresponding level sets of f. This definition makes sense if all level sets corresponding to positive values of f have finite measure, for example, if f is in L^p for some $p < \infty$.

In this paper, we determine the cases of equality in (1.2). A triple of functions that satisfies (1.2) with equality will be called an optimizing triple, or **optimizer**, of the inequality. There are many optimizers of (1.2). One reason is that $\mathcal{I}$ is invariant under a large group of affine transformations: For any linear map, $\mathcal{L}$, of determinant ± 1, and vectors a, b, and $c = a + b$ in $\mathbf{R}^n$, we have

$$\mathcal{I}(f, g, h) = \iint f(\mathcal{L}^{-1}y - a) g(\mathcal{L}^{-1}(x - y) - b) h(\mathcal{L}^{-1}x - c)\, dy dx \;. \tag{1.3}$$

Additionally there is the permutation symmetry

$$\mathcal{I}(f, g, h) = \mathcal{I}(g, f, h) = \mathcal{I}(h, g^-, f) \;, \tag{1.4}$$

where g^- denotes the function defined by $g^-(x) := g(-x)$. Clearly, any triple of functions that is equivalent to a triple of spherically decreasing functions under these symmetries is an optimizer.

There is a second reason to expect many optimizers. Consider the case when f and g have compact support. Then also the convolution $f * g$ has compact support. If h is the characteristic function of a set that contains the support of $f * g$, then f, g, h produce equality in (1.2) regardless of the shape of that set.

We will show that the non-uniqueness of the optimizers is due only to these two reasons. In particular, we will give conditions on the three functions that guarantee that any optimizing triple either consists of spherically decreasing functions, or is equivalent to such a triple under the symmetries given in (1.3).

The main result goes back to a conjecture by Lieb and Loss [20]. It describes all cases of equality in (1.2) where f, g, and h are characteristic functions of measurable sets. The spherical rearrangement, A^*, of a measurable set A of finite measure, is defined to be the open centered ball of the same measure as A. With this definition, $\mathcal{X}_{A^*} = (\mathcal{X}_A)^*$, and all level sets of a measurable function f and its spherical rearrangement f^* are related by $\mathcal{N}_s(f^*) = (\mathcal{N}_s(f))^*$. Define the functional

$$\mathcal{J}(A,B,C) := \mathcal{I}(\mathcal{X}_A, \mathcal{X}_B, \mathcal{X}_C) = \iint \mathcal{X}_A(y)\mathcal{X}_B(x-y)\mathcal{X}_C(x)\, dydx \ .$$

Naturally, $\mathcal{J}$ inherits the symmetries of $\mathcal{I}$.

Theorem 1 (Cases of equality, characteristic functions) *Let A, B and C be measurable sets of finite positive measure in $\mathbf{R}^n$. Denote by A^*, B^*, C^* the centered balls of the same measure as A, B, and C, respectively, and by α, β, and γ the radii of these balls.*

If $|\alpha-\beta| < \gamma < \alpha+\beta$, then there is equality in

$$\mathcal{J}(A,B,C) \ \leq \ \mathcal{J}(A^*,B^*,C^*) \tag{1.5}$$

if and only if, up to sets of measure zero,

$$A = a + \alpha E \ , \quad B = b + \beta E \ , \quad C = c + \gamma E \ , \tag{1.6}$$

where E is a centered ellipsoid, and a, b, and $c = a+b$ are vectors in $\mathbf{R}^n$.

Otherwise, permute the three sets so that $\gamma \geq \alpha + \beta$, using (1.4). Then there is equality in (1.5) if and only if A, B, and C can be changed by sets of measure zero so that

$$C \supset A + B \ , \tag{1.7}$$

In particular, for $\gamma = \alpha+\beta$, there is equality in (1.5) if and only if, up to sets of measure zero,

$$A = a + \alpha M \ , \quad B = b + \beta M \ , \quad C = c + \gamma M \ , \tag{1.8}$$

where $M \subset \mathbf{R}^n$ is convex and open, and a, b, and $c = a+b$ are vectors in $\mathbf{R}^n$.

The set $A+B$ that appears in conclusion (1.7) of Theorem 1 is called the pointwise, or **Minkowski sum** of A and B. It is defined by

$$A+B \ := \ \{a+b \ \mid \ a \in A, b \in B\} \ .$$

We will say that three positive numbers α, β, γ satisfy the **strict triangle inequality** if they could form the lengths of the sides of a nondegenerate triangle, that is, if

$$|\alpha-\beta| < \gamma < \alpha+\beta \ .$$

Note that this formula is symmetric under permutation of α, β, and γ. We will often say that the strict triangle inequality holds between the sizes of three sets A, B, and C in $\mathbf{R}^n$, if it holds between the radii of the balls A^*, B^*, and C^*.

In other words, equation (1.6) says that any optimizing triple (A,B,C) such that the radii α, β, γ satisfy the strict triangle inequality is (up to sets of measure zero) equivalent under the symmetries of the functional $\mathcal{J}$ to the triple of balls with these radii, centered at the origin. Equation (1.8) says that any optimizing triple such that α, β, γ are in the critical size relation $\gamma=\alpha+\beta$ is (up to sets of measure zero) of the form $(A,B,A+B)$, where A and B produce equality in the Brunn-Minkowski inequality.

Theorem 1 is the basis for our other results. In principle, it determines all cases of equality in the full Riesz rearrangement inequality, in the following way: Three functions f, g, and h produce equality in (1.2) if and only if almost all triples of level sets of the three functions produce equality in inequality (1.5). Unfortunately, this condition is hard to check in general. However, it is easy to recover the result by Lieb [18] that if the middle function, g, is required to be strictly spherically decreasing, then equality in (1.5) occurs only if f and h are common translates of spherically decreasing functions. The next theorem is the corresponding result in the case where at least two of the three functions are required to have no flat spots.

Theorem 2 (Cases of equality, two strictly decreasing rearrangements) *Let f, g, and h be three nonnegative measurable functions with spherically decreasing rearrangements f^*, g^*, and h^*. Assume that $\mathcal{I}(f,g,h)$ is finite, and that f, g, and h are not everywhere zero.*

If at least two of the three rearrangements f^*, g^*, and h^*, are strictly spherically decreasing, then there is equality in inequality (1.2) if and only if the triple (f,g,h) is equivalent to (f^*,g^*,h^*) under the symmetries (1.3).

The following example shows that the hypothesis that the rearrangements of **two** of the functions are **strictly decreasing** is essential. Let

$$\begin{aligned} f(x) &= \mathcal{X}_{(-2,2)}, \\ g(x) &= \mathcal{X}_{(-1,1)}, \\ h(x) &= \begin{cases} 0 & \text{if } x < -3, \\ 4+x & \text{if } -3 \le x < -1, \\ 4-x & \text{if } -1 \le x < 3, \\ e^{3-x} & \text{if } x \ge 3. \end{cases} \end{aligned}$$

Then f and g are already spherically decreasing, and h coincides with its rearrangement

$$h^*(x) = \begin{cases} 5-2\,|x| & \text{if } |x| < 1, \\ 4-|x| & \text{if } -1 \le |x| < 3, \\ e^{6-2|x|} & \text{if } |x| \ge 3, \end{cases}$$

on the union of the two intervals $[-3,-1]$ and $[1,3]$. Since the convolution $f*g$ is locally constant on the complement of these intervals, it follows that

$$\mathcal{I}(f,g,h) = \mathcal{I}(f^*,g^*,h^*) \ .$$

Note that h^* is strictly decreasing, but h and h^* are not related by a linear transformation.

As an application of Theorem 2, consider the weak Young inequality

$$\left| \iint f(y) g(x-y) h(x)\, dy dx \right| \ \le \ C(p,\lambda,n)\, \|f\|_p\, \|g\|_{w,q}\, \|h\|_r \ , \tag{1.9}$$

where $1 < p,q,r < \infty$, $1/p + 1/q + 1/r = 2$, and $\lambda = n/q$. The weak q-norm is defined by

$$\|g\|_{w,q} := \sup_{s>0}\ s \left(\frac{\mu(\mathcal{N}_s(|g|))}{\omega_n} \right)^{1/q} ,$$

and $C(p,\lambda,n)$ is the best constant in the Hardy-Littlewood-Sobolev inequality

$$\left| \iint f(y)\, |x-y|^{-\lambda}\, h(x)\, dy dx \right| \ \le \ C(p,\lambda,n)\, \|f\|_p\, \|h\|_r \ .$$

With Lieb's proof of this inequality in [19], we obtain the following statement.

Corollary 1 *There is equality in the weak Young inequality (1.9), if and only if* (f, g, h) *is equivalent to* (f^*, g^*, h^*) *under the symmetries (1.3), and moreover*

$$g^*(x) = const. \, |x|^{-\lambda} \ ,$$

and f^* *and* h^* *produce equality in the Hardy-Littlewood-Sobolev inequality.*

The structure of the paper is as follows. The second section contains an overview over the literature. We are particularly interested in the questions, which of the proofs of inequality (1.2) can be used to identify the cases of equality. The major part of the paper, Sections 4–9, is dedicated to the proof of Theorem 1. Finally, in Section 10, we prove Theorem 2 and Corollary 1.

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2 The history of the problem

The history of the inequality seems to begin with Poincaré's work on the problem of the possible shapes of a fluid body in equilibrium [11, 12]. If the total angular momentum of the body vanishes, then the energy functional has the form (1.1), where $f = h$ is the characteristic function of the body, and $g(x-y) = |x-y|^{-1}$ is (up to a minus sign) the potential of the gravitational attraction between two points located at x and y. Poincaré showed that, under some smoothness assumptions, the potential energy is minimized if and only if the body assumes the shape of a ball. Although he quoted Steiner's work on the isoperimetric inequality, his proof contains no concepts of rearrangement.

Poincaré referred to an earlier proof by Liapunoff [16], which apparently did not use the isoperimetric inequality. He pointed out several shortcomings of Liapunoff's proof, which apparently did not show that minima of the energy functional existed.

All later proofs of inequality (1.2) (with or without requiring the middle function to be spherically decreasing) are based on rearrangement ideas developed by Steiner for the isoperimetric inequality. Using **Steiner symmetrization**, the proof can be split into two parts. First, it is proven in one dimension. Applying the inequality to lower-dimensional cross sections shows that $\mathcal{J}(f, g, h)$ never decreases under Steiner symmetrization, while f^*, g^*, and h^* stay the same. Then the spherical rearrangement is approximated with repeated Steiner symmetrizations. It is a well known problem with Steiner's proof of the isoperimetric inequality that he did not show that such an approximation procedure converges, thus leaving the possibility open that the functional in question may not attain its extremum. The same problem occurs in some of the proofs of the Riesz rearrangement inequality discussed below.

The second universal tool is the **'layer-cake principle'**. Any nonnegative measurable function can be represented as a sum of the characteristic functions of its level sets. Then inequality (1.2) follows easily from the corresponding inequality (1.5) for the level sets (see [23, 15, 18, 6]). We will use this technique to prove Theorem 2.

Blaschke seems to have been the first to consider inequality (1.2) as a geometric inequality, and to use Steiner symmetrization to construct a proof of a special case [4]. With essentially the same methods Carleman showed that the inequality holds for any symmetrically decreasing middle function [8]. Both Blaschke and Carleman used techniques developed by Groß [13] for the isoperimetric inequality to extend the inequality to non-convex sets. The discussion of the cases of equality in [4, 8] was not complete, as it was only shown that convex optimizers have to be balls, but not that all optimizers have to be convex sets. Finally, Lichtenstein [17] extended the inequality with the 'layer-cake principle' to non-homogeneous fluids.

Riesz first stated the full inequality, where all three functions may vary, in one dimension [23] (see also [15]). It is easy to read off the cases of equality from a modification of his proof. Another proof of the inequality in one dimension can be found in Hardy, Littlewood, and Pólya [15]. Since the inequality (1.5) is approximated with a discrete analog, it does not show the cases of equality.

Riesz claimed that the inequality can easily be generalized to several dimensions [23]. He may have been thinking of Steiner's methods discussed above. Following work by Lusternik on the Brunn-Minkowski inequality [21], Sobolev took this approach for the Riesz inequality [24, 25]. Both proofs are incomplete, since it is not shown that the constructed sequence of Steiner symmetrizations converges to the spherical rearrangement in some suitable metric (Hausdorff metric for the Brunn-Minkowski inequality, and symmetric difference for the Riesz inequality). The cases of equality are not discussed. Sobolev was interested in inequality (1.2) because of an application to an inequality of the type of the Hardy-Littlewood-Sobolev inequality.

Inequality (1.5) is closely related to the Brunn-Minkowski inequality. This connection is lost, if the middle function is restricted to be strictly spherically decreasing. Incidentally, Hadwiger and Ohmann proved the Brunn-Minkowski inequality and discussed the cases of equality for measurable sets with more direct geometric methods that do not involve Schwarz or Steiner symmetrization, and thus do not require such a convergence result [14].

The first complete proof of inequality (1.2) in higher dimensions is due to Brascamp, Lieb, and Luttinger [6]. They also give a new proof of the inequality in one dimension, which calls upon the Brunn-Minkowski inequality in a surprising way. This proof gives the more general inequality

$$\int \cdots \int \prod_{i=1}^{m} f_i \left(\sum_{j=1}^{k} a_{ij} x_j \right) dx_1 \dots dx_k \ \leq \ \int \cdots \int \prod_{i=1}^{m} f_i^* \left(\sum_{j=1}^{k} a_{ij} x_j \right) dx_1 \dots dx_k \ . \tag{2.1}$$

The variables x_i are vectors in $\mathbf{R}^n$, and (a_{ij}) is a $m \times k$ matrix of scalars. Inequality (1.2) corresponds to the special case

$$m = 3 \ , \quad k = 2 \ , \quad (a_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$$

The full inequality (1.2), where all three functions are allowed to vary, is needed for applications to variants of Young's inequality, which relate the value of the functional $\mathcal{I}$ to the norms of f, g, and h in some function spaces. Beckner [3] applied the analog of (1.2) for multiple convolutions,

$$\int f_1 * \cdots * f_k f_0 \, dx \leq \int f_1^* * \cdots * f_k^* f_0^* \, dx \ , \tag{2.2}$$

(which is the special case of (2.1) with $m = k + 1$) to a version of Young's inequality. Brascamp and Lieb [5] proved a more general version of Young's inequality and determined the best constants, based on (2.1). Since sharp versions of inequalities (1.2), (2.1) and (2.2) were not available, they used other methods to identify the cases of equality.

A similar inequality on spheres was proven by Luttinger [22] in one dimension, and by Baernstein and Taylor in connection with subharmonic maps [2]. Here, g is a fixed symmetrically decreasing integral kernel. The proof relies on a compression procedure that involves reflections at hyperplanes. Although it is not mentioned in [2], it is easy to identify the cases of equality from this proof. The same proof applies to inequality (1.2) in $\mathbf{R}^n$ in case the middle function is spherically decreasing.

In [18], Lieb used inequality (1.2) to find solutions of a variational problem with rotational symmetry. He proved and used the sharp version with the fixed middle function mentioned above. The proof in one dimension is interesting, because it uses no approximation arguments, but direct geometric considerations for measurable sets. In particular, no regularity is assumed for the optimizers *a priori*. He also showed that the rearrangement inequality for the L^2-norm of the gradient can be obtained with a limiting argument from (1.2) with the heat kernel as the middle function.

Returning to the problem studied by Poincaré, Auchmuty [1] used the sharp form of the inequality with a fixed middle function to show that there are uniquely determined axisymmetric equilibrium shapes for rotating fluid bodies. This inequality also plays a role in Lieb's work on sharp constants in the Hardy-Littlewood-Sobolev inequalities [19]. Similar arguments are needed to identify the optimizers when applying the *'Competing Symmetries'* method of Carlen and Loss [9, 10] to functionals of the form (1.1).

We are convinced that Theorems 1 and 2 can be generalized to include inequality (2.1). The strict triangle inequality in Theorem 1 has to be replaced by a different size condition which depends on the matrix (a_{ij}). However, Riesz' proof of inequality (1.5) does not work for inequality (2.1), and the proof by Brascamp, Lieb, and Luttinger uses approximation arguments in such a way that it seems impossible to read off the cases of equality. Inequality (2.2) is a simpler special case. Theorem 1 holds for this inequality, with the triangle inequality replaced by a condition that the radii form the sides of a polygon with interior. The analogue of Lieb's result holds, if one of the functions $f_0, \ldots, f_k$ is strictly spherically decreasing, and the analogue of Theorem 2 holds, if at least two of them are strictly spherically decreasing (see[7]).

3 Outline of the proof of Theorem 1

The proof of Theorem 1 falls into several parts. In case one of the three sets is large enough compared to the other two, we deduce Theorem 1 from the result by Hadwiger and Ohmann [14] on the Brunn-Minkowski inequality and its cases of equality for measurable sets in $\mathbf{R}^n$ that was mentioned in the previous section.

In the most interesting case, when the strict triangle inequality holds between the sizes of the three sets, we will prove Theorem 1 by induction over the dimension. For the base case, dimension $n = 1$, we will modify Riesz' proof to identify the cases of equality.

The inductive step is based on the observation that the functional $\mathcal{J}$ in $\mathbf{R}^{n+1}$ is just the functional in a lower-dimensional space integrated over the cross sections. It is easy to see that almost all triples of n-dimensional cross sections of the three sets (at heights satisfying a certain relation) are optimizers of the inequality in $\mathbf{R}^n$. Since $\mathcal{J}$ is invariant under rotations, Theorem 1 can be applied to intersections of the three sets with hyperplanes in arbitrary directions.

There are two major difficulties. First, since Theorem 1 only makes a strong statement if the strict triangle inequality holds, it is crucial to find sufficiently many triples of cross sections whose sizes satisfy this inequality. Since not much can be said about the measures of cross sections of general measurable sets, some regularity is needed.

Second, the information about the cross sections obtained from the inductive assumption has to be pieced together to draw conclusions about the entire sets. In other words, the three sets have to be identified as ellipsoids by local properties. Eventually, we will derive and solve a differential equation for the boundaries of the three sets of an optimizing triple. Additional regularity will be needed to do this.

We will combine a pair of symmetrizations along subspaces with a linear transformation to yield a basic symmetrization operation. The symmetrization procedure (which can also be used to prove the inequality [24, 25, 7]) has the following properties.

(R1) It transforms optimizers of (1.5) into optimizers of (1.5).

(R2) It is rather simple, so that we can show that optimizers that satisfy the conclusions of Theorem 1 after regularization must have satisfied them to begin with.

(R3) It transforms general measurable sets into sets that are sufficiently regular.

Thus we need to prove conclusion (1.6) of Theorem 1 only for optimizing triples consisting of regularized sets. For such triples, we find cross sections whose radii satisfy the strict triangle inequality. We apply the inductive hypothesis and derive some properties of these cross sections. From these properties, we identify the sets as ellipsoids that differ only by scale factors. This will complete the proof of Theorem 1.

4 Some properties of optimizers of inequality (1.9)

In this section, we do those parts of the proof of Theorem 1 that are not related to the induction over the dimension.

Let A, B, C be measurable sets of finite positive measure in $\mathbf{R}^n$. Clearly, neither the functional $\mathcal{J}$, nor the spherical rearrangements A^*, B^*, and C^*, nor the radii α, β, and γ change, if A, B, and C are changed by sets of measure zero. We will often replace the three sets by the sets of their points of density one, or **Lebesgue points**. It is well known that any measurable set differs by a set of measure zero from the set of its Lebesgue points.

We begin with a simple property of optimizers which plays a central role in the inductive step: It will be used to find at least some cross sections of an optimizing triple for whose sizes the strict triangle inequality holds, provided the strict triangle inequality holds for the sizes of the three sets.

Fix two sets A and B in $\mathbf{R}^n$. The convolution $\mathcal{X}_A * \mathcal{X}_B$ of their characteristic functions is continuous, even if A and B are only measurable. Fix the measure of the third set, C. It is easy to see that the functional $\mathcal{J}$ is maximized, if C is a level set of $\mathcal{X}_A * \mathcal{X}_B$ (this is know as the **'bathtub principle'**). It is the purpose of this section to describe this level set more precisely. The number s will be called a **degenerate** value of $\mathcal{X}_A * \mathcal{X}_B$, if the level 'surface'

$$\Big\{x \;\mid\; \mathcal{X}_A * \mathcal{X}_B = s\Big\}$$

has positive n-dimensional Lebesgue measure, and **non-degenerate**, if it has measure zero.

By definition, the symmetrized sets A^* and B^* are the open balls of radius α and β. The convolution

$$\mathcal{X}_{A^*} * \mathcal{X}_{B^*}(x) = \mu\Big(B_\alpha(0) \cap B_\beta(x)\Big)$$

is a nonincreasing function of $|x|$. It is positive on the open ball of radius $\alpha + \beta$, and it achieves its maximum on all points of the closed ball of radius $|\alpha - \beta|$. Its maximal value is either $\mu(A)$ or $\mu(B)$, whichever is smaller. It is strictly spherically decreasing for radii between $|\alpha - \beta|$ and $\alpha + \beta$. Hence, if α, β, γ satisfy the strict triangle inequality, then C^* is a level set corresponding to a nondegenerate value of $\mathcal{X}_{A^*} * \mathcal{X}_{B^*}$. If $\gamma \geq \alpha + \beta$, then C^* contains the support of the convolution. If $\gamma \leq |\alpha - \beta|$, then C^* is a subset of the set where the convolution takes its maximal value.

The following two lemmas show that similar statements hold for general optimizing triples. The first lemma can be seen as a regularity result – it implies that we can assume that the boundary of the three sets have measure zero. In the second lemma, we characterize the support of $\mathcal{X}_A * \mathcal{X}_B$.

Lemma 1 (Optimizers and level sets) *Let (A, B, C) be an optimizing triple of inequality (1.5) in $\mathbf{R}^n$. If α, β, and γ satisfy the strict triangle inequality, then C differs by a set of measure zero from the level set*

$$\Big\{x \;\mid\; \mathcal{X}_A * \mathcal{X}_B > s\Big\}\ , \tag{4.1}$$

where

$$s := \inf_{x \in C^*} \mathcal{X}_{A^*} * \mathcal{X}_{B^*}\,(x)$$

is a nondegenerate value of $\mathcal{X}_A * \mathcal{X}_B$. If $\gamma \geq \alpha + \beta$, then C contains (except for a set of measure zero) the level set

$$\Big\{x \;\mid\; \mathcal{X}_A * \mathcal{X}_B\,(x) > 0\Big\}\ ,$$

if $\gamma = \alpha + \beta$, then the two sets differ by a set of measure zero. If $0 < \gamma \le |\alpha - \beta|$, then C is contained (up to a set of measure zero) in

$$\left\{ x \;\mid\; \mathcal{X}_A * \mathcal{X}_B\,(x) = \min(\mu(A), \mu(B)) \right\} ,$$

if $0 < \gamma = |\alpha - \beta|$, the two sets differ by a set of measure zero.

Proof Assume without loss of generality that $\mu(A) \ge \mu(B)$, that is, $\alpha \ge \beta$. The strict triangle inequality between α, β, and γ guarantees that

$$0 < s < \max \mathcal{X}_{A^*} * \mathcal{X}_{B^*} \ .$$

Clearly, s is a nondegenerate value of $\mathcal{X}_{A^*} * \mathcal{X}_{B^*}$. Define the sets

$$\begin{aligned} C_+ &:= C \cap \left\{ x \mid \mathcal{X}_A * \mathcal{X}_B(x) > s \right\} , \\ C_- &:= C \cup \left\{ x \mid \mathcal{X}_A * \mathcal{X}_B(x) \ge s \right\} . \end{aligned}$$

Then

$$C_+ \subset C \subset C_- \ , \tag{4.2}$$

and

$$C_+ \ \subset \ \left\{ x \;\mid\; \mathcal{X}_A * \mathcal{X}_B(x) > s \right\} \ \subset \ \left\{ x \;\mid\; \mathcal{X}_A * \mathcal{X}_B(x) \ge s \right\} \ \subset \ C_- \ .$$

If C differs by a set of positive measure from the level set (4.1), or if s is a degenerate value of $\mathcal{X}_A * \mathcal{X}_B$, then at least one of the sets $C \setminus C_+$, $C_- \setminus C$ has positive measure. If $\mu(C \setminus C_+) > 0$ then

$$\begin{aligned} \mathcal{J}(A,B,C) &\le \mathcal{J}(A,B,C_+) + s\mu(C \setminus C_+) \qquad \text{because } \mathcal{X}_A * \mathcal{X}_B \le s \text{ on } C \setminus C_+ \\ &\le \mathcal{J}(A^*,B^*,C_+^*) - s\mu(C^* \setminus C_+^*) \qquad \text{by (1.5) and (4.2)} \\ &< \mathcal{J}(A^*,B^*,C^*) \qquad \text{because } \mathcal{X}_{A^*} * \mathcal{X}_{B^*} > s \text{ on } C^* \ . \end{aligned}$$

Similarly, if $\mu(C_- \setminus C) > 0$ then

$$\begin{aligned} \mathcal{J}(A,B,C) &\le \mathcal{J}(A,B,C_-) - s\mu(C_- \setminus C) \qquad \text{because } \mathcal{X}_A * \mathcal{X}_B \ge s \text{ on } C_- \setminus C \\ &\le \mathcal{J}(A^*,B^*,C_-^*) - s\mu(C_-^* \setminus C^*) \qquad \text{by (1.5) and (4.2)} \end{aligned}$$

$$< \quad \mathcal{J}(A^*, B^*, C^*) \qquad \text{because } \mathcal{X}_{A^*} * \mathcal{X}_{B^*} < s \text{ outside the closure of } C^* \ .$$

In either case, it follows that (A, B, C) produces strict inequality in (1.5).

The proofs in the cases $\gamma \geq \alpha + \beta$ and $\gamma \leq \alpha - \beta$ are similar. ■

Lemma 2 (The support of the convolution) *Consider two measurable sets, A and B, in $\mathbf{R}^n$ with characteristic functions $\mathcal{X}_A$ and $\mathcal{X}_B$. Assume that A and B consist exactly of their Lebesgue points. Then*

$$\Big\{x \ \mid \ \mathcal{X}_A * \mathcal{X}_B(x) > 0\Big\} = A + B \ .$$

Proof "$\supset$": Suppose that the convolution takes a positive value at the point x. Then

$$\mu((x - A) \cap B) = \int \mathcal{X}_A(x-y)\mathcal{X}_B(y)\, dy = \mathcal{X}_A * \mathcal{X}_B(x) > 0 \ ,$$

that is, the intersection of $x - A$ with B has positive measure. Any Lebesgue point of this intersection is of the form $b = x - a$ for some Lebesgue points $a \in A$, $b \in B$.

"$\subset$": Any point of the form $x = a + b \in A + B$ is a Lebesgue point of both $a + B$ and $b + A$, so it is a Lebesgue point of the intersection. Consequently, $b = x - a$ is a Lebesgue point of $(x - A) \cap B$, and the convolution takes a positive value at x. ■

Combining Lemmas 1 and 2 shows that, if (A, B, C) is an optimizer of inequality (1.5) and the strict triangle inequality holds between α, β, and γ, then A, B, and C differ by sets of measure zero from open sets satisfying

$$\bar{C} \ \subset \ A + B \quad , \tag{4.3}$$

where $\bar{C}$ is the closure of C.

The two lemmas also establish a connection between Theorem 1 in case the strict triangle inequality does not hold between the sizes of the three sets, and the Brunn-Minkowski inequality. The Brunn-Minkowski inequality for measurable sets in $\mathbf{R}^n$ as proved by Hadwiger and Ohmann in [14] states that the measure of the pointwise sum of any two nonempty measurable sets A and B in $\mathbf{R}^n$ satisfies the inequality

$$\mu(A + B)^{1/n} \geq \mu(A)^{1/n} + \mu(B)^{1/n} \ . \tag{4.4}$$

There is equality if and only if either A is a point, or B is a point, or A and B are of the form

$$A = \bar{A} \setminus N_A, \quad B = \bar{B} \setminus N_B \tag{4.5}$$

where N_A and N_B are sets of measure zero, and $\bar{A}$ and $\bar{B}$ are convex sets that can be mapped to each other by scaling and translation.

Before we turn to the proof of Theorem 1 in case one of the three sets is large compared to the other two, we would like to remark that conversely, the Riesz rearrangement inequality implies the Brunn-Minkowski inequality for measurable sets, and Theorem 1 implies characterization (4.5) of the cases of equality (see [7]).

Proof of Theorem 1, $\boldsymbol{\gamma \geq \alpha + \beta}$ The assertion (1.7) follows immediately from Lemmas 1 and 2. If $\gamma = \alpha + \beta$, the Brunn-Minkowski inequality (4.4) together with (1.7) implies that

$$\mu(C)^{1/n} \geq \mu(A + B)^{1/n} \geq \mu(A)^{1/n} + \mu(B)^{1/n} \ ,$$

that is, $\gamma \geq \alpha + \beta$. Hence A and B produce equality in (4.5), and C differs by a set of measure zero from its subset $A + B$. By (4.5), A, B, and consequently $A + B$, are convex sets differing only by scaling and translation. This proves assertion (1.8). ∎

5 Proof of Theorem 1, $n = 1$, $|\alpha - \beta| < \gamma < \alpha + \beta$

In this section, we adapt Riesz' proof of inequality (1.5) in one dimension [23] (see also [15]) to general measurable sets, and use it to determine the cases of equality when the strict triangle inequality holds between the sizes of the three sets.

The idea of the proof is to replace two of the three sets by smaller sets, so that the three sets are in the critical size relation $\gamma = \alpha + \beta$, in which case we have just proved the theorem.

Proof of Theorem 1, $\boldsymbol{n = 1}$, $\boldsymbol{|\alpha - \beta| < \gamma < \alpha + \beta}$ For $\delta \geq 0$, define

$$\begin{aligned} A_\delta &= \left\{ x \in A \mid \int_{-\infty}^{x} \mathcal{X}_A(s)\, ds > \delta/2, \int_{x}^{\infty} \mathcal{X}_A(s)\, ds > \delta/2 \right\} \\ B_\delta &= \left\{ x \in B \mid \int_{-\infty}^{x} \mathcal{X}_B(s)\, ds > \delta/2, \int_{x}^{\infty} \mathcal{X}_B(s)\, ds > \delta/2 \right\} . \end{aligned}$$

The sets A_δ and B_δ are obtained by cutting off subsets of measure $\delta/2$ from both ends of A and B. In general, $\mu(A_\delta) = (\mu(A) - \delta)_+$. Set

$$\delta := \frac{1}{2}(\mu(A) + \mu(B) - \mu(C)) = \alpha + \beta - \gamma.$$

With this choice of δ,

$$\begin{aligned}
\mu(A_\delta) &= \mu(A) - \frac{1}{2}\Big(\mu(A) + \mu(B) - \mu(C)\Big) = \alpha + \gamma - \beta \ , \\
\mu(B_\delta) &= \mu(B) - \frac{1}{2}\Big(\mu(A) + \mu(B) - \mu(C)\Big) = \beta + \gamma - \alpha \ .
\end{aligned}$$

Both expressions are positive because α, β, and γ satisfy the strict triangle inequality. Moreover, the measures of A_δ, B_δ, and C are in the critical size relation

$$\mu(A_\delta) + \mu(B_\delta) = \mu(C) \ .$$

To estimate how the value of the functional $\mathcal{J}$ changes when A and B are replaced by A_δ and B_δ, observe that for any two measurable sets,

$$(A \cap B)_\delta = \Big\{x \in A \cap B \mid \int_{-\infty}^{x} \mathcal{X}_A \mathcal{X}_B > \delta/2, \int_x^\infty \mathcal{X}_A \mathcal{X}_B > \delta/2\Big\} \subset A_\delta \cap B_\delta \ ,$$

and hence

$$\mu(A_\delta \cap B_\delta) \ge \mu((A \cap B)_\delta) \ge \mu(A \cap B) - \delta \ .$$

Applying this inequality to the intersection of A with $x - B$ and integrating gives

$$\begin{aligned}
\mathcal{J}(A, B, C) - \mathcal{J}(A_\delta, B_\delta, C) &= \int_C \mu((A \cap (x - B)) - \mu(A_\delta \cap (x - B_\delta)) \, dx \\
&\le \delta \mu(C) \ .
\end{aligned} \tag{5.1}$$

It is easy to calculate directly, that A^*, B^*, and C^* produce equality in (5.1). It follows that

$$\mathcal{J}(A, B, C) - \mathcal{J}(A_\delta, B_\delta, C) \le \mathcal{J}(A^*, B^*, C^*) - \mathcal{J}(A^*_\delta, B^*_\delta, C^*) \ . \tag{5.2}$$

Adding inequality (1.5) for (A_δ, B_δ, C)

$$\mathcal{J}(A_\delta, B_\delta, C) \le \mathcal{J}(A^*_\delta, B^*_\delta, C^*) \tag{5.3}$$

to inequality (5.2) gives (1.5) for (A, B, C).

It is necessary for equality in (1.5) that A_δ, B_δ, and C produce equality in (5.3). Conclusion (1.8) of Theorem 1, which was proved in the previous section, implies that C differs from an interval by a set of measure zero. By the permutation symmetry (1.4), the other two sets A and B have to be intervals up to sets of measure zero as well. It is easy to calculate directly that the centers must satisfy $a + b = c$. ∎

6 Steiner and Schwarz symmetrization

The main tool for the proof of Theorem 1 in higher dimensions is symmetrization along lower-dimensional subspaces, which we now define. Let A be a measurable set in $\mathbf{R}^{n+1}$. Write points in $\mathbf{R}^{n+1}$ as $\mathbf{x} = (x^0, x)$ with $x^0 \in \mathbf{R}$ and $x \in \mathbf{R}^n$. Denote the n-dimensional cross section of A at $x^0 = z$ by

$$A(z) := \{x \in \mathbf{R}^n \ \mid \ (z, x) \in A\} \ .$$

The **Schwarz symmetrization**, $\mathcal{S}_1 A$, of A is defined by the property that its n-dimensional cross sections perpendicular to the x^0-axis are centered open balls whose measures equal the measures of the corresponding cross sections of A. In short, for all $z \in \mathbf{R}$,

$$(\mathcal{S}_1 A)(z) = (A(z))^* \ , \tag{6.1}$$

where * denotes symmetrization of the cross section in $\mathbf{R}^n$. If a cross section is not measurable or does not have finite measure, the corresponding cross section of $\mathcal{S}_1 A$ is defined to be $\mathbf{R}^n$.

Similarly, the **Steiner symmetrization**, $\mathcal{S}_2 A$, of A is defined by the property that its intersections with lines parallel to the x^0-axis are intervals of the same lengths as the measures of the corresponding intersections with A. In short, for all $x \in \mathbf{R}^n$,

$$(\mathcal{S}_2 A)(x) = (A(x))^* \ , \tag{6.2}$$

where * denotes symmetrization in $\mathbf{R}$ of the linear cross section at x. If a cross section is not measurable, or does not have finite measure, the corresponding cross section of $\mathcal{S}_2 A$ is defined to be $\mathbf{R}$.

Schwarz and Steiner symmetrization are uniquely determined by properties (6.1) and (6.2). They preserve the measure of A by Fubini's theorem. Moreover, if A is changed by a set of measure zero, then $\mathcal{S}_1 A$ and $\mathcal{S}_2 A$ change by sets of measure zero. We will write $\mathcal{S}_1(A, B, C)$ and $\mathcal{S}_2(A, B, C)$ for the triples consisting of the Schwarz and Steiner symmetrizations of A, B, and C.

The key to the inductive step is the observation that these partial symmetrizations transform optimizing triples into optimizing triples: The functional $\mathcal{J}$ can be written as

$$\mathcal{J}(A, B, C) = \int_{\mathbf{R}} \int_{\mathbf{R}} \mathcal{J}(A(w), B(z-w), C(z))\, dwdz\ .$$

Inequality (1.5) applied to the n-dimensional cross sections gives

$$\mathcal{J}(A, B, C) \leq \int_{\mathbf{R}} \int_{\mathbf{R}} \mathcal{J}((A(w))^*, (B(z-w))^*, (C(z))^*)\, dwdz = \mathcal{J}(\mathcal{S}_1(A, B, C))\ .$$

Applying inequality (1.5) in $\mathbf{R}^{n+1}$ gives

$$\mathcal{J}(A, B, C) \leq \mathcal{J}(\mathcal{S}_1(A, B, C)) \leq \mathcal{J}(A^*, B^*, C^*)\ .$$

Moreover, almost all n-dimensional cross sections of any optimizing triple (A, B, C) at z_A, z_B, and $z_C = z_A + z_B$ form optimizing triples for the inequality in $\mathbf{R}^n$. Similar arguments show that also Steiner symmetrization transforms optimizers into optimizers.

We will use the inductive assumption in form of the following lemma. It says that, provided the sizes of the cross sections of an optimizing triple satisfy the strict triangle inequality, they must be homothetic ellipsoids whose midpoints lie on parallel lines.

Lemma 3 (Shape and midpoints of cross sections) *Suppose that Theorem 1 has been proven in some fixed dimension n. Let (A, B, C) be an optimizing triple of inequality (1.5) in $\mathbf{R}^{n+1}$.*

Assume that there exist nonempty intervals I_A, I_B, and I_C such that for almost all $z_0 \in I_C$ there exists $w_0 \in I_A$ with $z_0 - w_0 \in I_B$ so that for almost all (z, w) near (z_0, w_0), the sizes of the n-dimensional cross sections $A(w)$, $B(z-w)$, and $C(z)$ satisfy the strict triangle inequality. Finally, assume that these assumptions also hold for any triple that can be obtained from (A, B, C) by the permutations (1.4).

Then there exists a fixed centered ellipsoid $\hat{E}$ in $\mathbf{R}^n$, and vectors $\hat{a}$, $\hat{b}$, and $\hat{c} = \hat{a} + \hat{b}$ in

$\mathbf{R}^n$, *so that for almost all* z *in* I_A, I_B, *and* I_C, *respectively,*

$$\begin{aligned} A(z) &= \hat{a} + z\hat{v} + \alpha(z)\hat{E} \ , \\ B(z) &= \hat{b} + z\hat{v} + \beta(z)\hat{E} \ , \\ C(z) &= \hat{c} + z\hat{v} + \gamma(z)\hat{E} \ , \end{aligned} \tag{6.3}$$

except for sets of n*-dimensional measure zero.*

Remark It is easy to see that

$$\hat{a} = \hat{b} = \hat{c} = 0 \tag{6.4}$$

if A, B, and C are symmetric about the origin. In other words, the midpoints of the cross sections of the three sets lie all on one line through the origin.

Proof Fix (z_0, w_0) as in the assumptions. Since (A, B, C) is an optimizer, almost all triples of cross sections $A(w)$, $B(z-w)$, $C(z)$ are optimizing for inequality (1.5) in $\mathbf{R}^n$. Conclusion (1.6) of Theorem 1 shows that almost all those cross sections with (z, w) near (z_0, w_0) are given by

$$\begin{aligned} A(w) &= a(w) &+ \quad &\alpha(w)E(z,w) \\ B(z-w) &= b(z-w) &+ \quad &\beta(z-w)E(z,w) \\ C(z) &= c(z) &+ \quad &\gamma(z)E(z,w) \end{aligned} \tag{6.5}$$

(up to sets of n-dimensional measure zero), where for each (z, w), $E(z, w)$ is a centered ellipsoid in $\mathbf{R}^n$, and the vectors $a(w), b(z-w), c(z) \in \mathbf{R}^n$ satisfy

$$c(z) = a(w) + b(z-w) \ . \tag{6.6}$$

Clearly, the ellipsoid $E(z, w)$ in (6.5) cannot depend on z or w. Equation (6.6) implies that for small $|h|$

$$d(h) \ := \ c(z+h) - c(z) = a(w+h) - a(w) \tag{6.7}$$

is a function of h only, and that

$$d(h_1 + h_2) \ = \ d(h_1) + d(h_2) \ . \tag{6.8}$$

The function d is measurable, because $\hat{a}(z)$ is the center of gravity of the cross section $A(z)$,

$$a(z) = \mu(A(z))^{-1} \int_{\mathbf{R}^n} x \mathcal{X}_A(x, z)\, dx \ .$$

Therefore relation (6.8) implies that d coincides with a linear function except on a set of measure zero. By definition (6.7), for almost all values of z near w_0, $z_0 - w_0$, z_0, respectively,

$$a(z) = \hat{a} + z\hat{v} \ , \quad b(z) = \hat{b} + z\hat{v} \ , \quad c(z) = \hat{c} + z\hat{v} \ ,$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{v}$ are vectors in $\mathbf{R}^n$, and $\hat{a} + \hat{b} = \hat{c}$. Since I_C is connected, the formula for $c(z)$ holds for almost all $z \in I_C$. Permute the three sets, using (1.4), to see that the formulas for $a(z)$ and $b(z)$ hold for almost all z in I_A and I_B, respectively. This proves the claim. ■

7 Regularization

We will use the following basic symmetrization operation in the inductive step: Let $\mathcal{R}$ be a rotation by an angle that is not a rational multiple of $\pi/2$ in the x^0-x^1-plane which leaves the other coordinates fixed, and define $\mathcal{S}$ by

$$\mathcal{S}A := \mathcal{S}_2\mathcal{S}_1\mathcal{R}\, A \ . \tag{7.1}$$

It can be shown (see [6, 7]) that for any measurable set A of finite measure, the sequence $\left\{\mathcal{S}^k A\right\}_{k\geq 0}$ converges to A^* with respect to symmetric difference. This suggests that $\mathcal{S}$ should have regularizing properties. We now verify that $\mathcal{S}^2$ indeed has the properties (R1)–(R3) announced in Section 3.

Proof of (R1)

We already showed in Section 6 that Schwarz and Steiner symmetrization transform optimizers into optimizers. By the rotational symmetry of $\mathcal{J}$, the same holds for $\mathcal{S}$ and $\mathcal{S}^2$.

Proof of (R2)

We will show that $\mathcal{S}^2$ has property (R2) in Lemma 5. In the proof we will use Lemma 4 to show that optimizers that can be transformed into ellipsoids by Schwarz symmetrization have sufficiently many cross sections whose sizes satisfy the strict triangle inequality.

Lemma 4 *Let α, β, γ be three positive numbers satisfying the strict triangle inequality. Then for any z with $|z| < \gamma$ there exists a nonempty open interval such that for w in this interval, the numbers*

$$\sqrt{\alpha^2 - w^2}\ , \quad \sqrt{\beta^2 - (z-w)^2}\ , \quad \sqrt{\gamma^2 - z^2} \tag{7.2}$$

satisfy the strict triangle inequality as well.

Proof Fix z with $|z| < \gamma$. We will construct a triangle in the plane with side lengths given by equation (7.2).

Consider a triangle in $\mathbf{R}^3$ with side lengths α, β, and γ, and denote the corners by O, P, and Q. Arrange it so that $O = (0,0,0)$, and $P = (\sqrt{\gamma^2 - z^2}, 0, z)$, and that the distances $|Q - O|$ and $|P - Q|$ are α and β, respectively. Denote the image of P and Q under projection onto the x_1-x_2-plane by $P\prime$ and Q'. The triangle $OP'Q'$ is non-degenerate triangle unless Q happens to lie in the x_1-x_3-plane, directly above or below the line OP.

Let w be the third component of Q, so $z - w$ is the third component of the difference vector $P - Q$. Then (by Pythagoras) the distances of O, P', and Q' are the three numbers of equation (7.2). Varying Q for fixed O, P, α, and β, we see that w can take any value in a certain open interval. ■

Lemma 5 (Optimizers may be regularized) *Assume that Theorem 1 has been proven for some fixed dimension $n \geq 1$. Let (A, B, C) be an optimizing triple $\mathbf{R}^{n+1}$ so that the radii α, β, γ satisfy the strict triangle inequality.*

If $\mathcal{S}^2(A, B, C)$ satisfies conclusion (1.6) of Theorem 1, then (A, B, C) satisfies (1.6) as well.

Proof We will first show the claims

$$A = a + \alpha E\ , \quad B = b + \beta E\ , \quad C = c + \gamma E \tag{7.3}$$

and

$$a + b = c \tag{7.4}$$

under the stronger assumption that

$$\mathcal{S}_1\, A = a' + \alpha E' \ , \quad \mathcal{S}_1\, B = b' + \beta E' \ , \quad \mathcal{S}_1\, C = c' + \gamma E' \ , \tag{7.5}$$

where a', b' and $c' = a' + b'$ are fixed vectors in $\mathbf{R}^{n+1}$. By the translation symmetry of $\mathcal{J}$, we may assume that the centers of gravity of A and B are at the origin, that is,

$$\mu(A)^{-1} \int_A x\, dx \ = \ \mu(B)^{-1} \int_B x\, dx = 0 \ . \tag{7.6}$$

By (7.5), the ellipsoid E' is symmetric under rotation about the x^0-axis, so

$$E' = \left\{ \mathbf{x} = (x^0, \hat{x}) \in \mathbf{R}^{n+1} \ \mid \ |\hat{x}|^2 < c_1^2 - c_2^2 x_0^2 \right\} \ ,$$

where c_1 and c_2 are constants. Since Schwarz symmetrization does not change the measure of the cross sections, we have

$$\left(\frac{\mu(A(z))}{\omega_n} \right)^{1/n} = \left(\frac{\mu(\mathcal{S}_2 A(z))}{\omega_n} \right)^{1/n} = \sqrt{(c_1^2 \alpha^2 - c_2^2 z^2)_+} \ . \tag{7.7}$$

It follows that

$$c_1^{-1} \left(\frac{\mu(A(zc_1/c_2))}{\omega_n} \right)^{1/n} = \sqrt{(\alpha^2 - z^2)_+} \ .$$

The same formula holds for for B and C with α replaced by β and γ, respectively. By Lemma 4, A, B, and C satisfy the assumptions of Lemma 3 with

$$I_A = (-\alpha c_1/c_2\,,\, \alpha c_1/c_2) \ , \quad I_B = (-\beta c_1/c_2\,,\, \beta c_1/c_2) \ , \quad I_C = (-\gamma c_1/c_2\,,\, \gamma c_1/c_2) \ .$$

Assumption (7.6) gives

$$\hat{a} = \mu(A)^{-1} \int_A \hat{x}\ d\mathbf{x} \ = \ 0 \ , \quad \hat{b} = \mu(B)^{-1} \int_B \hat{x}\ d\mathbf{x} \ = \ 0 \ ,$$

so, by Lemma 3, also

$$\hat{c} = \hat{a} + \hat{b} = 0 \ .$$

To show that A, B, and C are ellipsoids, write the ellipsoid $\hat{E}$ of (6.3) as

$$\hat{E} = \left\{ \hat{x} \in \mathbf{R}^n \ \mid \ \hat{Q}(\hat{x}) < 1 \right\}$$

where $\hat{Q}$ is a positive definite quadratic form on $\mathbf{R}^n$. Then, by equations (6.3) and (7.7), conclusion (1.6) holds with the ellipsoid

$$E = \left\{ \mathbf{x} \in \mathbf{R}^{n+1} \ \mid \ \hat{Q}(\hat{x} - x^0 \hat{v}) < c_1^2 - c_2^2 x_0^2 \right\} ,$$

and $a = b = c = 0$.

In case (A, B, C) satisfies

$$\mathcal{S}_2 \, A = a' + \alpha E' \ , \quad \mathcal{S}_2 \, B = b' + \beta E' \ , \quad \mathcal{S}_2 \, C = c' + \gamma E' \ , \tag{7.8}$$

where a', b' and $c' = a' + b'$ are fixed vectors in $\mathbf{R}^{n+1}$, take $\mathcal{R}_0$ to be the rotation by $\pi/2$ that maps the x^0-axis to the x^1-axis and leaves the other coordinate axes fixed. Since Fubini's theorem implies that Steiner symmetrization does not change the measures of n-dimensional cross sections perpendicular to the x^1-axis, we have

$$\mathcal{S}_1 \mathcal{R}_0 = \mathcal{S}_1 \mathcal{R}_0 \mathcal{S}_2 \ .$$

Assumption (7.8) implies

$$\mathcal{S}_1 \mathcal{R}_0 \, A = \mathcal{S}_1 \mathcal{R}_0 \mathcal{S}_2 A = \alpha E \ , \quad \mathcal{S}_1 \mathcal{R}_0 \, B = \beta E \ , \quad \mathcal{S}_1 \mathcal{R}_0 \, C = \gamma E \ .$$

Applying the first case to $\mathcal{R}_0(A, B, C)$ shows the claim (7.3).

Finally, by definition (7.1), the assertion follows from the rotational invariance of $\mathcal{J}$ and the two results just proved. ∎

Proof of (R3)

We will show that $\mathcal{S}^2$ transforms a general measurable set A into the rotational solid of a nonincreasing function,

$$\mathcal{S}^2 A \ = \ \left\{ \mathbf{x} \in \mathbf{R}^{n+1} \ \mid \ |\hat{x}| < \alpha(x^0) \right\} , \tag{7.9}$$

where α is bounded, even, and nonincreasing for positive arguments. Since a nonincreasing function can have at most countable many discontinuities, we can change $\mathcal{S}^2 A$ by a set of measure zero, so that it becomes open, and α becomes lower semicontinuous. The next lemma deals with the smoothness properties of such sets.

Lemma 6 (Continuity of intersections) *Let A be an open set of finite positive measure, given by*

$$A = \Big\{ \mathbf{x} = (x^0, \hat{x}) \in \mathbf{R}^{n+1} \ \mid \ |\hat{x}| < \alpha(x^0) \Big\} \ ,$$

where α is an even, nonnegative function that is nonincreasing for positive arguments. Consider the intersection of A with hyperplanes of the form

$$x^0 = m x^1 + t \ , \tag{7.10}$$

where m and t are scalars. If $m \neq 0$ then the n-dimensional measure of the intersections is uniformly bounded in t, and jointly continuous in (m,t) at $(m,0)$.

If additionally α is bounded and continuous at 0, then the measure of the intersections is also jointly continuous in (m,t) at $(0,0)$.

Proof Write points in $\mathbf{R}^{n+1}$ as $\mathbf{x} = (x^0, x^1, \hat{x})$, if $n > 1$, and points in $\mathbf{R}^2$ as $\mathbf{x} = (x^0, x^1)$. The intersection of a hyperplane of the form (7.10) with A is given by the equations

$$x^0 = m x^1 + t \ , \quad |\hat{x}|^2 < \alpha^2\Big(x^0\Big) - (x^1)^2 \ ,$$

for $n = 1$ set $|\hat{x}| = 0$. Integrating over $\hat{x}$, the measure of this intersection is $(1+m^2)^{n/2} I(m,t)$, where

$$I(m,t) = \omega_{n-1} \int_{-\infty}^{\infty} \Big(\alpha^2(ms+t) - s^2\Big)_+^{(n-1)/2} \ ds \ ,$$

with the convention that $\omega_0 = 1$, and $0^0 = 0$. The integrand cannot be positive unless $\alpha(ms+t) > 1$ or $|s| < 1$ or both. Thus, for $m \neq 0$, we can decompose

$$I \leq I_1 + I_2 \ ,$$

where I_1 is given by

$$\begin{aligned} I_1 &= \frac{\omega_{n-1}}{m} \int_{-s_0}^{s_0} \Big(\alpha^2(s) - ((s-t)/m)^2\Big)_+^{(n-1)/2} \ ds \\ &\leq \frac{\omega_{n-1}}{m} \int_{-s_0}^{s_0} \alpha^n(s) \, dt \\ &= \frac{\omega_{n-1}}{m \, \omega_n} \mu(A) \end{aligned}$$

with $s_0 = \inf\Big\{ s > 0 \;\;|\;\; \alpha(s) \le 1 \Big\}$. The other part of the integral is

$$I_2 = \omega_{n-1} \int_{-1}^{1} \alpha^{n-1}(ms + t)\, ds \ .$$

It is easy to see from these representations that I_1, I_2, and I are bounded uniformly in t, and that $\lim_{|t|\to\infty} I(m,t) = 0$.

Continuity for $n > 1$ follows from the fact that the integrand in

$$I(m,t) = \omega_{n-1} \int_{-\infty}^{\infty} m^{-1} \Big(\alpha^2(s) - ((s-t)/m)^2 \Big)_+^{(n-1)/2} \, ds$$

depends continuously on (m,t), and vanishes for fixed (m,t) if s is outside a compact set. To show continuity for $n = 1$, note that by the monotonicity properties of α, any line (7.10) with $m \ne 0$, $t = 0$ intersects the boundary of A transversally.

Finally, if α is continuous at 0, then the integrand of I converges pointwise to

$$\Big(\alpha^2(0) - s^2 \Big)_+^{(n-1)/2} \ ,$$

except possibly for $n = 1$ at $s = \pm\alpha(0)$. Dominated convergence shows that I is continuous at $(0,0)$. ∎

By construction, S transforms any measurable set A in $\mathbf{R}^{n+1}$ into a rotational solid,

$$\mathcal{S}A = \Big\{ \mathbf{x} \in \mathbf{R}^{n+1} \;\;|\;\; |\hat{x}| < \alpha(x^0) \Big\} \ .$$

where α is even, nonnegative, and nonincreasing for positive arguments, so, clearly, $\mathcal{S}^2 A$ satisfies (7.9). We only need to show that α is bounded. Recall that by definition (7.1),

$$\mathcal{S} = \mathcal{S}_2 \mathcal{S}_1 \mathcal{R} \ ,$$

where $\mathcal{R}$ rotates the x^0-axis by a non-integer multiple of $\pi/2$. The function α describing the cross sections of $\mathcal{S}_2\mathcal{S}_1\mathcal{R}\mathcal{S}A$ is obtained by symmetric decreasing rearrangement from the function describing the cross sections of $\mathcal{R}\mathcal{S}A$ perpendicular to the x^0-axis, which are uniformly bounded by Lemma 6, because they correspond to intersections of $\mathcal{S}A$ with hyperplanes with $m \ne 0$. So α is bounded.

8 Identifying Ellipsoids

The following two lemmas were proven in collaboration with Michael Loss. They provide the criteria we will use to identify optimizers as triples of ellipsoids.

Lemma 7 (Ellipses, local) *Let A be an open set in $\mathbf{R}^2$ of the form*

$$A = \Big\{ (x, y) \in \mathbf{R}^2 \ \mid \ |y| < \alpha(x) \Big\} \tag{8.1}$$

where α is an even, bounded, nonnegative function which is nonincreasing for $x \geq 0$, and continuous at 0. Consider the intersections of A with the family of lines

$$x = my + t \ . \tag{8.2}$$

Assume that there exists a family of lines

$$y = b(m)x \tag{8.3}$$

with the following property: For all m with $|m| < \varepsilon$, the intersection of A with almost all lines (8.2) with $|t| < \varepsilon$ differs by a set of one-dimensional measure zero from a line segment whose midpoint lies on the line (8.3).

Then there exists a constant c, and $\delta > 0$ such that for $|x| < \delta$

$$\alpha^2(x) = \alpha^2(0) - c^2 x^2 \ , \tag{8.4}$$

that is, the intersection of A with the vertical strip $|x| < \delta$ coincides with the intersection of an ellipsoid ($c \neq 0$) or a horizontal strip ($c = 0$) with this strip. The shape of the ellipsoid is determined by

$$c^2 = -\frac{b(m)}{m} \tag{8.5}$$

for any nonzero value of m.

Proof We will first show that

$$A_\delta = \Big\{ (x, y) \in A \ \mid \ |x| < \delta \Big\} \ .$$

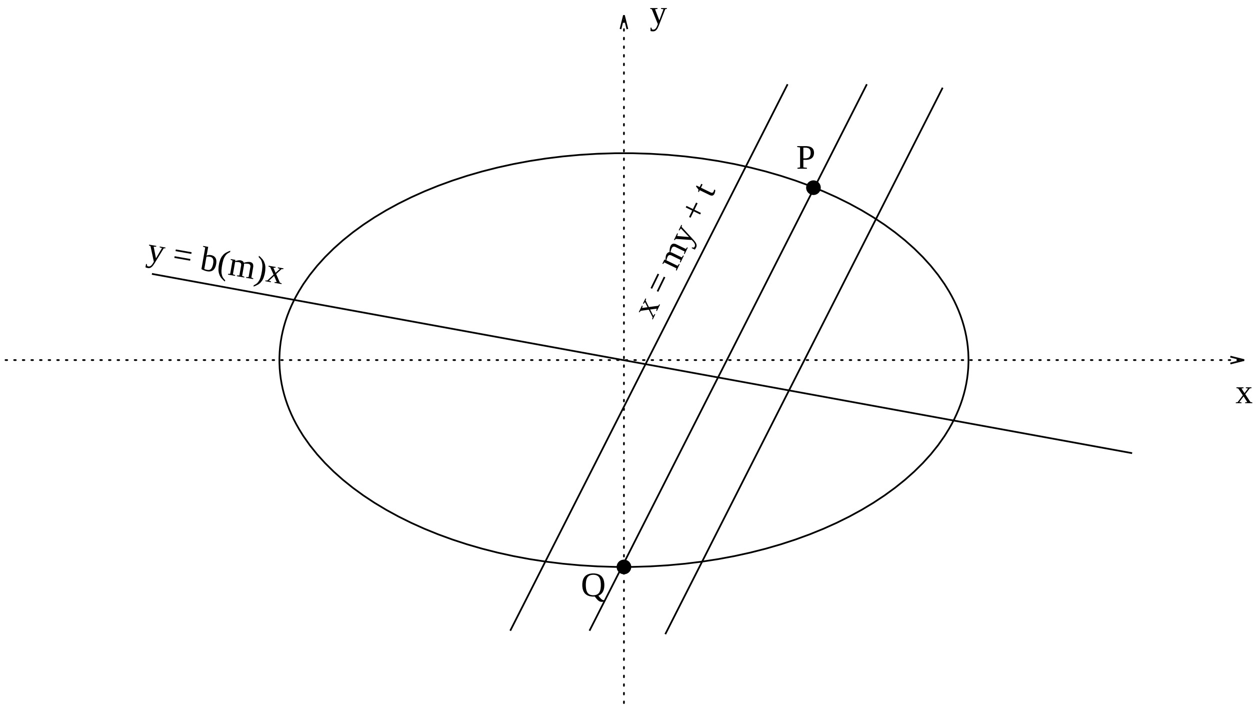

Figure 1: A property characterizing ellipses

is convex if δ is small enough. We need to find through each boundary point of A_δ a line, so that A_δ is contained in one of the closed half-spaces defined by that line. Such a line will be called a **support line** of A at the given boundary point.

Let $Q = (0, -\alpha(0))$ be the point of the boundary of A with the smallest y-coordinate. Clearly, A does not intersect the lower of the two half-spaces defined by the line $y = -\alpha(0)$ through Q. Consider a point of the form $P = (x, \alpha(x))$ in the boundary of A. The line joining P and Q is given by equation (8.2) with

$$m = \frac{x}{\alpha(x) + \alpha(0)} \ , \quad t = -m\alpha(0) \ . \tag{8.6}$$

If x is close enough to zero, then $|t| < \varepsilon$, $|m| < \varepsilon$.

Fix m as in (8.6), and consider the linear transformation $\mathcal{L}$ which fixes the line $y = b(m)x$ pointwise, and maps all points on the line $x = my$ to their negatives. $\mathcal{L}$ is conjugate to a reflection. By assumption, for all t with $|t| < \varepsilon$, the intersection of A with the lines (8.2) differs only by a set of one-dimensional measure zero from a line segment centered on the line $y = b(m)x$. In other words, the intersections of A and $\mathcal{L}A$ with the strip

$$\Big\{(x,y) \ \mid \ |my - x| < \varepsilon\Big\}$$

differ by a set of measure zero. By definition, A, and consequently $\mathcal{L}A$ consist exactly of their Lebesgue points, so the two sets must coincide. In particular, $\mathcal{L}$ maps Q to P, and the half-space below the line $y = -\alpha(0)$ to a half-space containing P in its boundary which does not meet A in a neighborhood of P. It follows that A_δ is convex provided δ is small enough.

By construction, $\mathcal{L}$ maps support lines at P into support lines at Q, so there is a unique support line at P if and only if there is a unique support line at Q. Since P was an arbitrary point on the upper arc of the boundary of A_δ, and since, by convexity, all but countably many points on this arc have unique support lines, every such point has a unique support line. Hence, α is differentiable everywhere on the interval $(-\delta, \delta)$.

We will derive a differential equation for α. Consider a pair of points $P = (x, \alpha(x))$ on the upper arc, and $Q = (z, -\alpha(z))$ on the lower arc of the boundary of A_δ. By assumption (8.3), the slope of the line though the midpoint of the line segment PQ,

$$b = \frac{\alpha(x) - \alpha(z)}{x + z} \ , \tag{8.7}$$

is a function of the slope of the line segment,

$$m = \frac{x - z}{\alpha(x) + \alpha(z)} \ .$$

Both the midpoint and the slope are differentiable functions of (x, z) for $x + z \neq 0$. Since b is a function of m, the gradients of $b(x, z)$ and $m(x, z)$ must be linearly dependent. We calculate these gradients at $(x, 0)$ $(x \neq 0)$

$$\begin{aligned}
\nabla b(x,0) &= x^{-2}\begin{pmatrix} x\alpha'(x) - \alpha(x) + \alpha(0) \\ -\alpha(x) + \alpha(0) \end{pmatrix} \\
\nabla m(x,0) &= (\alpha(x) + \alpha(0))^{-2}\begin{pmatrix} \alpha(x) + \alpha(0) - x\alpha'(x) \\ -\alpha(x) - \alpha(0) \end{pmatrix} \ ,
\end{aligned}$$

where we have used that $\alpha'(0) = 0$ since α is even. They are linearly dependent, if

$$\Big(x\alpha'(x) - \alpha(x) + \alpha(0)\Big)\Big(\alpha(x) + \alpha(0)\Big) = \Big(\alpha(x) - \alpha(0)\Big)\Big(\alpha(x) + \alpha(0) - x\alpha'(x)\Big) \ .$$

Collecting terms, we see that α satisfies the differential equation

$$x(\alpha^2(x))' = -2(\alpha^2(0) - \alpha^2(x)) \ .$$

The general solution of this differential equation with $\alpha(x) \leq \alpha(0)$ is given by (8.4). Inserting (8.4) into (8.7) gives the formula (8.5) for c. ■

Lemma 8 (Ellipses, continuation) *Let $A \subset \mathbf{R}^2$ be as in equation (8.1) of Lemma 7. Assume that there exists a number $\delta > 0$ with $c\delta < \alpha(0)$, so that for $x \in [-\delta, \delta]$, the function α is given by formula (8.4). Let*

$$m_0 = \frac{\delta}{\sqrt{\alpha^2(0) - c^2\delta^2}} \ .$$

Assume moreover that there exist $\varepsilon > 0$ and a function $b(m)$ such for all m with $|m - m_0| < \varepsilon$ and for almost all t with $|t| < \varepsilon$ the intersection of A with the line given by (8.2) differs by a set of one-dimensional measure zero from a line segment whose midpoint lies on the line given by (8.3).

Then equation (8.4) holds on an open neighborhood of $[-\delta, \delta]$.

Proof For $0 < m < m_0$ define

$$\varepsilon' = \delta - m\sqrt{\alpha^2(0) - c^2\delta^2} > 0$$

Choose m with $m_0 - \varepsilon < m < m_0$ close enough to m_0 that $\varepsilon' < \varepsilon$. By assumption, the intersection of A with lines with parameters (m, t), $|t| < \varepsilon$ consists of line segments whose midpoints are given by (8.3). In other words, the intersection of A with the strip

$$\left\{(x, y) \in \mathbf{R}^2 \ \mid \ |x - my| < \varepsilon\right\}$$

differs from its image under the skewed reflection $\mathcal{L}$ constructed in the proof of Lemma 7 by a set of measure zero. Since A consists exactly of its Lebesgue points, it follows that the strip is symmetric under $\mathcal{L}$. The upper arc of the strip with

$$-\varepsilon < x - my < \varepsilon'$$

and the lower arc with

$$-\varepsilon' < x - my < \varepsilon$$

consist of points $x, \alpha(x)$, and $x, -\alpha(x)$, respectively, where α is given by formula (8.4). Since the strip is symmetric under the linear map $\mathcal{L}$, also the image of the lower arc under $\mathcal{L}$ is described by a quadratic equation. The image of the lower arc intersects the upper arc in an open arc since the line $x = my$ intersects both arcs. Consequently, (8.4) holds for the whole part of the boundary of A with $|x - my| < \varepsilon$. Repeating the argument for $m = -m_0$ proves the claim. ∎

9 Proof of Theorem 1 in higher dimensions

We proved Theorem 1 in case the strict triangle inequality does not hold between the radii α, β, and γ in Section 4. Following the outline in Section 3, we will prove the remaining case by induction. The base case $n = 1$ was discussed in Section 5.

Suppose that Theorem 1 has been proven for dimensions up to n. Let (A, B, C) be an optimizing triple of inequality (1.5) in $\mathbf{R}^{n+1}$, so that the radii α, β, γ satisfy the strict triangle inequality. By properties (R1)–(R3) of the regularization discussed in Section 7, we may assume without loss of generality that A, B, and C are 'nice', that is

$$\begin{aligned} A &= \left\{\mathbf{x} \in \mathbf{R}^{n+1} \mid |\hat{x}| < \alpha(x^0)\right\} \\ B &= \left\{\mathbf{x} \in \mathbf{R}^{n+1} \mid |\hat{x}| < \beta(x^0)\right\} \\ C &= \left\{\mathbf{x} \in \mathbf{R}^{n+1} \mid |\hat{x}| < \gamma(x^0)\right\} \end{aligned}$$

The functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ are even, nonnegative, bounded, lower semicontinuous, and nonincreasing for positive arguments. Lemmas 1 and 2 imply that

$$\bar{C} \subset A + B$$

(see equation (4.3)). Hence,

$$\gamma(0) < \sup_{w \in \mathbf{R}} \left\{\alpha(w) + \beta(-w)\right\} \leq \alpha(0) + \beta(0) .$$

By the symmetry (1.4), α, β, and γ may be permuted in this inequality to see that $\alpha(0)$, $\beta(0)$, $\gamma(0)$ satisfy the strict triangle inequality. By Lemma 6, the measures of the intersections of

A, B and C with hyperplanes of the form

$$x^0 = mx^1 + t \tag{9.1}$$

are jointly continuous in (m, t) at $(0, 0)$. That is, there exists $\varepsilon > 0$ so that the sizes of these intersections satisfy the strict triangle inequality for (m, t) with $|m| < \varepsilon$, and $|t| < \varepsilon$.

To apply the inductive hypothesis, fix m with $|m| < \varepsilon$. Since we have formulated Lemma 3 only for cross sections perpendicular to the x^0-axis, we rotate the three sets simultaneously in the x^0-x^1-plane (fixing the other coordinates), so that the cross sections defined by (9.1) are mapped to cross sections of the rotated sets perpendicular to the x^0-axis. The rotated sets satisfy the assumptions of Lemma 3 with $I_A = I_B = I_C = (-\varepsilon', \varepsilon')$, where ε' can be chosen independently of m for m small. By Lemma 3, almost all intersections of A, B and C with hyperplanes of the form (9.1) with $|m| < \varepsilon'$ and $|t| < \varepsilon'$ are ellipsoids. Since A, B, and C are symmetric about the origin, for every fixed m, the midpoints of these ellipsoids all lie on one line through the origin by equation (6.4). Since the three sets are in symmetric under rotation about the x^0-axis, this line lies in the x^0-x^1-plane, and can be written as

$$x^1 = b(m)x^0 \ , \quad x^i = 0 \ (i > 1) \ .$$

By Lemma 7, the parts of the three sets near the equatorial hyperplane $x^0 = 0$ are pieces of similar ellipsoids. That is, there exists $\delta > 0$ so that

$$\begin{aligned} \alpha^2(x) &= \alpha^2(0) - c^2x^2 \ , && \text{if } |x| < \alpha(0)\delta \\ \beta^2(x) &= \beta^2(0) - c^2x^2 \ , && \text{if } |x| < \beta(0)\delta \\ \gamma^2(x) &= \gamma^2(0) - c^2x^2 \ , && \text{if } |x| < \gamma(0)\delta \end{aligned} \tag{9.2}$$

where the constant c is the same for the three sets by (8.5). Note that the three quadratic equations are related by scale factors $\alpha(0) : \beta(0) : \gamma(0)$.

To show that equations (9.2) hold globally, we will make a continuation argument. Set

$$\delta_0 = \sup\Big\{\delta \ \ | \ \ (9.2) \text{ holds for } \delta\Big\} \ . \tag{9.3}$$

We have just shown that $\delta_0 > 0$. Assume that $c\delta_0 < 1$. Then $\sqrt{1 - c^2\delta_0^2} > 0$. Define as in Lemma 8

$$m_0 = \frac{\delta_0}{\sqrt{1 - c^2\delta_0^2}}$$

to be the slope of the hyperplane passing through the end points of the parts of the boundaries of the three sets that are described by the quadratic equations (9.2). Since hyperplanes with $|m| < m_0$ meet only those parts of the three sets that are described by (9.2), the intersections of A, B, and C with these hyperplanes are in the fixed scaling proportions $\alpha(0) : \beta(0) : \gamma(0)$. By the continuity results from Lemma 6, also the intersections with the hyperplane $x^0 = mx^1$ are in this proportion. Applying Lemma 6 again we see that the sizes of cross sections with hyperplanes with m near m_0 and $|t|$ small enough satisfy the strict triangle inequality.

To apply Lemma 3 again, fix m near m_0, and rotate the three sets simultaneously in the x^0-x^1-plane so that the intersections of A, B, and C with hyperplanes (9.1) are mapped to cross sections of the rotated sets perpendicular to the x^0-axis. It follows from Lemma 3 applied to the rotated triple that all three sets satisfy the assumptions of Lemma 8. So (9.2) holds on a neighborhood of $[-\delta_0, \delta_0]$, which contradicts definition (9.3).

It follows that either $c\delta_0 = 1$, or $c = 0$ and $\delta_0 = +\infty$. In the first case, A, B, and C are ellipsoids that are related by scale factors $\alpha(0) : \beta(0) : \gamma(0)$. Clearly, these factors must be in proportion $\alpha : \beta : \gamma$, which proves conclusion (1.6) in this case. In the second case, the three sets would have to be infinite strips, which contradicts the assumption that they have finite measure. This completes the proof of Theorem 1. ∎

10 Proof of Theorem 2 and Corollary 1

Proof of Theorem 2 Let (f, g, h) be an optimizer for inequality (1.2) satisfying the assumptions of Theorem 2. We have to show that there exist a linear map, $\mathcal{L}$, and vectors a, b, and $c = a + b$ such that

$$f(x) = f^*\left(\mathcal{L}^{-1}x - a\right), \quad g(x) = g^*\left(\mathcal{L}^{-1}x - b\right), \quad h(x) = h^*\left(\mathcal{L}^{-1}x - c\right) \quad \text{a.e.} \qquad (10.1)$$

We use the 'layer-cake' representation

$$f(x) = \int_0^\infty \mathcal{X}_{f(x)>\alpha}\, d\alpha$$

to decompose f, g, and h, and their spherically decreasing rearrangements into the characteristic functions of their level sets. It is easy to see that the integrand is jointly measurable in x and α. By Fubini's theorem, inequality (1.2) is equivalent to

$$\iiint \mathcal{J}\left(\mathcal{N}_r(f), \mathcal{N}_s(g), \mathcal{N}_t(h)\right)\, drdsdt \leq \iiint \mathcal{J}\left(\mathcal{N}_r(f)^*, \mathcal{N}_s(g)^*, \mathcal{N}_t(h)^*\right)\, drdsdt \ .$$

For equality we need that almost all triples of level sets are optimizers of inequality (1.5).

Define $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ to be the radii of the level sets of f^*, g^*, and h^* at height s, respectively. That is, these functions are constant multiples of the n-th roots of the distribution functions. As in the inductive step of the proof of Theorem 1, we want to find triples (r, s, t) such that the strict triangle inequality holds between $\alpha(r)$, $\beta(s)$, and $\gamma(t)$.

Fix r_0 and s_0 such that $\alpha(r_0)$ and $\beta(s_0)$ are positive. The assumption that f^* and h^* are strictly spherically decreasing implies that α and γ are continuous, nonincreasing, and assume all positive real values. It follows that there exists t_0 such that $\alpha(r_0)$, $\beta(s_0)$, $\gamma(t_0)$ satisfy the strict triangle inequality. By continuity, the strict triangle inequality holds for $\alpha(r), \beta(s_0), \gamma(t)$ with r, t in an open neighborhood of r_0, t_0. By conclusion (1.6) of Theorem 1, we can write the corresponding level sets as

$$\begin{aligned}
\mathcal{N}_r(f) &= a(r) &+& \ \alpha(r)E(r, s_0, t) \ , \\
\mathcal{N}_{s_0}(g) &= b(s_0) &+& \ \beta(s_0)E(r, s_0, t) \ , \\
\mathcal{N}_t(h) &= c(t) &+& \ \gamma(t)E(r, s_0, t) \ ,
\end{aligned}$$

where for each value of r and t, E is an centered ellipsoid in $\mathbf{R}^n$, and a and c are vectors in $\mathbf{R}^n$ with

$$a(r) + b(s_0) = c(t) \ .$$

Clearly, E cannot depend on r and t near r_0 and t_0, and a and c are locally constant. But the set

$$\Big\{(r, t) \ \mid \ \alpha(r), \beta(s_0), \gamma(t) \ \text{ satisfy the strict triangle inequality}\Big\}$$

is connected, so a, c must be constant, and E is independent of r and t. Since s_0 was arbitrary, E and b cannot depend on s, either. The claim (10.1) follows by choosing $\mathcal{L}$ to be a linear transformation that maps the unit ball to E. ∎

Proof of Corollary 1 We may assume without loss of generality that f, g, and h are nonnegative. Following Lieb's argument in [19], we consider the chain of inequalities

$$\begin{aligned}\iint f(y)g(x-y)h(x)\,dydx &\leq \iint f^*(y)g^*(x-y)h^*(x)\,dydx \\ &\leq \|g\|_{w,q} \iint f^*(y)\,|x-y|^{-\lambda}\,h^*(x)\,dydx \\ &\leq C(p,\lambda,n)\,\|f\|_p\,\|g\|_{w,q}\,\|h\|_r \ ,\end{aligned}$$

where the first line is the Riesz rearrangement inequality (1.2), the second follows from the definition of the weak norm, and the third is the Hardy-Littlewood-Sobolev inequality. For equality in (1.9), there has to be equality in all three lines. By Theorem 2.3(ii) from [19], equality in the Hardy-Littlewood-Sobolev inequality implies that f^* and h^* are strictly spherically decreasing. Equality in the second line implies that $g^*(x) = \|g\|_{w,p}\,|x|^{-\lambda}$. By Theorem 2, equality in the first line implies the claim. ∎

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