ON THE GEOMETRY OF OPTIMAL WINDOWS, WITH SPECIAL FOCUS ON THE SQUARE

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Abstract. For the Laplace operator with mixed (Dirichlet and Neumann) boundary conditions, the dependence of the principal eigenvalue on the placement of the Dirichlet part is investigated. An optimal window is a Dirichlet part of the boundary that minimizes the principal eigenvalue among all competitors of the same area.

In the special case of a square, we provide both numerical evidence and rigorous partial results for the conjecture that optimal windows in a square are segments centered at either a corner or the midpoint of a side. In particular, we prove that the principal eigenvalue decreases as a window is shifted from a side-centereed position towards the corner. An optimal window contained in two sides of the square is connected and contains a corner in its interior. Optimal windows whose length does not exceed the length of one side break the symmetry of the square.

We also construct a starshaped domain whose optimal window(s) must be disconnected. Finally we give, for general domains in $\mathbb{R}^d$, continuity results for the eigenvalue as a function of the window, and examples of discontinuity when crucial hypotheses are violated. We also give a variation formula that relates the eigenvalue to the singularities of the eigenfunction (stress intensity coefficient) near the boundary of the window.

Methods are based on the variational problem and include rearrangement, Dirichlet Neumann Bracketing, capacity estimates, and deformation under a flow.

Keywords: optimal eigenvalue, Laplace operator, mixed boundary conditions, shape optimization, capacity, singular coefficient, rearrangement

AMS subject classifications: 49R50, 35J05, 35R05, 31C40

1. Introduction.

1.1. Overview over the Results.

Consider the first eigenvalue of the Laplace operator in a fixed domain $\Omega \subset \mathbb{R}^d$ (say, bounded Lipschitz)

$$-\Delta u = \lambda u \ , \quad u \geq 0 \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary conditions on some subset $D \subset \Omega$ and Neumann on the complement of D, i.e.,

$$u|_D = 0 \ , \quad \partial_\nu u|_{\partial\Omega \setminus D} = 0. \tag{1.2}$$

Technical questions of how these boundary conditions should be interpreted will be discussed below. We will call $\lambda = \lambda(D)$ the principal eigenvalue of the Laplacian under the *window boundary conditions on D*. The problem of optimal windows asks for minimization of this eigenvalue for prescribed surface area of the window.

As explained in [8], one may think of Ω as representing a room, with perfectly heat-conducting windows at D and insulating walls along $\partial\Omega \setminus D$. The principal eigenvalue

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$\lambda(D)$ gives the rate of exponential decay of any initial temperature distribution due to heat diffusion through the window as time becomes large, while the corresponding eigenfunction gives the asymptotic temperature profile. An optimal window minimizes long-term heat loss among all windows of a given size.

It has been shown in [9] that such optimal windows exist and that in the case of a ball of any dimension the optimal window is a spherical cap of the appropriate area. Similar results have been obtained independently by Cox and Uhlig [6], who treat windows as a singular limiting case of Robin boundary conditions.

Here we are concerned with the question what can be said about the geometry of optimal windows when Ω is not a ball. We suspect that an optimal window in a convex domain Ω should be connected, have some basic regularity properties, and lie in a region of $\partial\Omega$ with large mean curvature. This is certainly not the case for more general domains, as we show by constructing an example of a star-shaped domain with a disconnected optimal window. Heuristic evidence concerning the location of optimal windos has been discussed in [8], and is also corroborated by results of Harrell et. al. [13] on a different, but related problem.

As a model case for a convex domain, we study a square: The determination of the shape of optimal windows is already nontrivial in this case. Here we conjecture that the optimal window is a segment, centered either at the midpoint of a side or at a corner, depending on the prescribed boundary measure (length); and that there are no other optimal windows, up to sets of measure zero. This conjecture is corroborated by a number of rigorous partial results as well as numerical evidence. See Figure 2.1. In particular, we prove that the eigenvalue decreases as a short segment is moved from a side-centered position to a position adjacent to a corner; and that this monotonicity extends at least for some distance as the window is moved around the corner. We show the first part of this result by means of a Dirichlet-Neumann bracketing argument. The second part is proved by means of an Euler-Lagrange type variational formula, which we derive for any domain of sufficient regularity in arbitrary dimension.

Furthermore we show that some segment containing a corner in its interior is optimal among all windows lying on only two sides of the square (adjacent or not). The proof relies on discrete rearrangement arguments that are specific to the square (with some obvious, but maybe not too interesting, generalization to a cube or hypercube).

Numerical evidence shows that for segments whose length exceeds the sidelength by a certain small amount, up to slightly more than two sidelengths, the corner-centered position ceases to be optimal, with the side-centered position being better. This can be understood heuristically in terms of the fact that optimal windows prefer to use corners, as was already discussed in terms of a model problem in [8]. However, distributing the window evenly around the corners (sacrificing connectedness) is not advantageous and results in windows inferior to either the side-centered segment or the corner-centered segment. For small windows, we can even prove this analytically. On the other hand, our analytic results prove that the (non-optimal) window with four congruent corner-centered components is still better than any other window that has the full symmetry of the square.

Our study of the square also serves as a building block for an example of a star-shaped domain where any optimal window is disconnected. A related example was discussed heuristically in Figure 3 of [8].

The variation formula mentioned above is derived here for windows in general domains of any dimension. Its upshot is that the rate of change of the eigenvalue as a function of the window is determined by certain singular coefficients of the eigenfunction that show up at each interface of window and wall on the boundary. In a neighbourhood of such interface points, the eigenfunction (albeit in $W^{1,2}$) cannot be expected to be $W^{2,2}$.

In special geometries the singularities have been studied by Grisvard (e.g. [12]). In the situation of a simple interface of a wall and a window segment on a side of the square, the typically expected singular behavior of the eigenfunction is like $c\,\mathrm{Im}\sqrt{z}$ in a neighbourhood of 0, where the number c is the singular coefficient. We give a simple lower estimate for this coefficient, based on a maximum principle, to ensure that it does not vanish. In contrast, in a corner of the square, the singular coefficient vanishes. These two facts are responsible that the eigenvalue can be lowered by moving a segment a bit around the corner. For segments up to one sidelength, a better control of the singular coefficients appearing in the variational formula (which depend on global properties of the eigenfunction) should extend this monotonicity all the way until a minimum is achieved when the segment is centered at a corner, but a proof of this extended monotonicity has eluded us so far.

Contributions from geometric singularities (corners, ridges, conical points) have been studied by many authors, and in vast generality, e.g., Maz'ya and Plamenevskii [21]. Surveys are [16] and [23]. We are using only the very simplest case here.

The continuous dependence of the eigenvalue under shifts of the window and other reasonable modifications of its geometry is an intuitively plausible, but nontrivial result of relevance. For deformations of windows that can be achieved by the flow of a vector field, our Euler-Lagrange argument proves even differentiability. However, in the absence of good a-priori information on the window geometry, such flow type modifications are rather weak; this is why we include some continuity results for other modifications (in general Lipschitz domains). In this context, it is crucial to consider, in addition to the formulation of the EVP adopted in [9], [8], a more sophisticated definition that takes into account fine properties of eigenfunctions. It is easy to see that the results in [9], [8] carry over. We will argue this point specifically for the existence of optimal windows in Section 1.3. Both definitions coincide for *optimal* windows, as well as for windows of sufficient regularity, in particular for open windows.

1.2. Basic Facts, Context, and Notation; Variational Formulation. Let us introduce some notation. The symbol Ω will generally denote a bounded Lipschitz domain in $\mathbb{R}^d$, and the window D will be a measurable subset of $\partial\Omega$: as surface measure on $\partial\Omega$, we use $d-1$-dimensional Hausdorff measure, denoted here by σ. A point in $\overline{D}\cap\overline{\partial\Omega\setminus D}$ will be called an *interface point*.

The Laplacian with the window boundary conditions (1.2) will be denoted by Δ_D. The word 'eigenvalue' without adjective or ordinal will always denote the *lowest* eigenvalue, which is simple. This eigenvalue will be denoted by $\lambda(D)$.

Define the *optimal eigenvalue* for windows of a given surface measure by

$$\lambda_*(\ell) \;=\; \inf\{\lambda(D) \;\mid\; D\subset\partial\Omega,\ \sigma(D) \;=\; \ell\}\,. \tag{1.3}$$

A set $D\subset\partial\Omega$ will be called an *optimal window*, if $\lambda(D)=\lambda_*(\sigma(D))$, that is, if

$$\lambda(D) \;=\; \inf\{\lambda(D') \;\mid\; D'\subset\partial\Omega,\ \sigma(D') \;=\; \sigma(D)\}\,. \tag{1.4}$$

In [8] and [9], the eigenvalue $\lambda(D)$ in (1.1) and (1.3) was defined by the Courant-Hilbert variational problem (CHVP)

$$\lambda(D)=\min\left\{\int_\Omega |\nabla u|^2\,dx \;\middle|\; u\in W^{1,2}(\Omega)\,,\ \int_\Omega u^2\,dx=1\,,\ u|_D=0\right\}\,. \tag{1.5}$$

Here, the restriction $u|_D$ of a function $u \in W^{1,2}(\Omega)$ is to be understood as the *trace*. By general Sobolev space theory ([1, Thm. 5.4] or [10, 4.3, Thm 1]), the trace of a $W^{1,2}$-function is guaranteed to be an $L^2(\partial\Omega)$ function. The condition $u|_D = 0$, in the L^2-sense, will not distinguish sets D and D' if they differ by a set of $d-1$-dimensional measure zero, and therefore, the Dirichlet conditions for D in (1.1) need to hold only on a set D' thus differing from D. We refer to this definition of λ as the *coarse* formulation of the eigenvalue problem (1.1) and the CHVP (1.5).

The Neumann boundary conditions on $\partial\Omega \setminus D$ in equation (1.1) arise as natural boundary conditions for the variational problem in (1.5). The minimizing function u is a normalized eigenfunction corresponding to $\lambda(D)$, and can be chosen to be nonnegative. It agrees a.e. with an analytic function in the interior of Ω, but is not guaranteed to be continuous up to the boundary $\partial\Omega$, unless some assumptions are made on the geometry of D.

Clearly, the principal eigenvalue $\lambda_1(D)$ increases under inclusion of windows

$$D_1 \subset D_2 \implies \lambda(D_1) \subset \lambda(D_2),$$

since the minimizing function in the CHVP for $\lambda(D_1)$ is an admissible test function for the CHVP determining $\lambda(D_2)$.

1.3. Fine Variational Formulation. As mentioned above, there is another meaningful definition of the boundary conditions (1.2) and the corresponding variational problem (1.5). Since $W^{1,2}$-functions can actually be determined *quasi-everywhere*, that is, up to a set of zero *capacity*, one can insist that the Dirichlet boundary conditions in (1.1) and in the Courant-Hilbert variational problem (1.5) hold on a set D' that may differ from D only by a set of zero capacity. Since every set of capacity zero has $d-1$-dimensional measure zero, but not vice versa, this is a stronger condition. It corresponds to choosing a smaller domain for the quadratic form associated with Δ_D. We will refer to this definition of $\lambda(D)$ as the *fine* formulation of (1.1) or (1.5). When necessary, we distinguish the two definitions by superscripts, writing $\lambda^c(D)$ and $\lambda^f(D)$ for the coarse and fine eigenvalues, respectively. In general,

$$\lambda^f(D) \;\geq\; \lambda^c(D)\,, \tag{1.6}$$

and it is easy to construct examples where the inequality is strict: any (fractal) window with Hausdorff dimension between $d-2$ and $d-1$ has measure 0 and nonvanishing capacity [10, 4.7.2],[19, 2.1.7], hence coarse eigenvalue 0, but positive fine eigenvalue. The notion of sets of capacity 0 is well-defined even in two dimensions, where capacity can only be defined subject to some arbitrary choice. We can even completely avoid such subtleties, by replacing the window D in $\Omega \subset \mathbb{R}^2$ with the equivalent window $D \times [0,1]$ in $\Omega \times]0,1[\subset \mathbb{R}^3$.

In order to relate coarse and fine eigenvalues, we represent an element u of a Sobolev space by a function defined everywhere, which will be called the preferred representative. For any given Lipschitz domain Ω and any neighbourhood V of $\overline{\Omega}$, there is a linear bounded operator $\mathcal{E} : W^{1,2}(\Omega) \to \mathring{W}^{1,2}(V) \hookrightarrow W^{1,2}(\mathbb{R}^d)$ that extends Sobolev functions in Ω to the entire space as outlined in [10, 4.4], i.e., $\mathcal{E}u|_\Omega = u$. Then we choose the *preferred representative* as

$$\tilde{u}(x) \;:=\; \limsup_{r\to 0} \fint_{B_r(x)} \mathcal{E}u(y)\, dy \qquad \text{for} \qquad x \in \overline{\Omega}\,. \tag{1.7}$$

The lim sup is in fact a limit, except on a set of zero capacity, and $\tilde{u}$ is quasicontinuous (as defined in [19, 2.17]).

The extension operator $\mathcal{E}$ is not unique, but depends on a choice of locally flattening coordinate charts, and we make such a choice once and for all, for each given Ω. The preferred representative on $\overline{\Omega}$ depends on the extension operator, but any two choices will only differ on a set of capacity zero. See [19, Thm 2.55&Rmk] or [10, 4.8, Thm 1]. The restriction of $\tilde{u}$ to $\partial\Omega$ represents the trace of u.

THEOREM 1.1. *For the CHVP (1.5) in the fine formulation, there exists a minimizer which is uniquely determined quasi-everywhere up to choice of a sign.*

PROOF: This is a slight modification of the classical argument for the existence of a minimizer for (1.5) in the (coarse) Sobolev sense. Let u_j be a minimizing sequence of quasicontinuous functions (in $W^{1,2}(\mathbb{R}^d)$, by extension) satisfying the boundary conditions in the fine sense. Extracting a subsequence (again denoted by u_j), we may assume weak convergence in $W^{1,2}(\Omega)$, strong convergence in $L^2(\Omega)$ and (by compactness of the trace map) strong convergence in $L^2(\partial\Omega)$, to a limit function u_*. We have to show that u_* inherits the fine boundary conditions from $\{u_j\}$. To this end, we replace the sequence $\{u_j\}$ by a sequence $\{\bar{u}_j\}$ of convex combinations that converges *strongly* in $W^{1,2}$, according to Mazur's theorem (see, e.g., [18, 2.13]). The normalized sequence $\hat{u}_j := \bar{u}_j/\|\bar{u}_j\|_{L^2(\Omega)})$ still converges strongly in $W^{1,2}(\Omega)$ because $\|\bar{u}_j\|_{L^2(\Omega)} \to 1$. The $\hat{u}_j$ inherit the fine boundary conditions from u_j and form therefore a sequence of legitimate competitors in the CHVP.

Now by convexity, we obtain

$$\int |\nabla u_*|^2 = \lim \int |\nabla \hat{u}_j|^2 = \lim \int |\nabla \bar{u}_j|^2 \le \lim \int |\nabla u_j|^2 = \inf \int |\nabla u|^2 \,.$$

We must show that u_* inherits the fine boundary conditions from the $\hat{u}_j$. This follows from the arguments in Sec. 2.1.3 of [19], which we sketch briefly, for the sake of being more self-contained:

(1) If a sequence of C_0^∞ functions v_j (with uniformly bounded support) converges strongly in $W^{1,2}(\Omega)$ to some u_*, then it holds for a subsequence (again called v_j):

$$\forall\varepsilon > 0\ \exists V_\varepsilon \text{ open}: \operatorname{cap}(V_\varepsilon) < \varepsilon\,,\ \|v_j - u_*\|_{C^0(\overline{\Omega}\setminus V_\varepsilon)} \to 0$$

(2) Every $W^{1,2}(\Omega)$ function u can be approximated in $W^{1,2}(\Omega)$ norm by C_0^∞ functions v_k (with uniformly bounded support), such that

$$\forall\varepsilon > 0\ \exists W_\varepsilon \text{ open}: \operatorname{cap}(W_\varepsilon) < \varepsilon\,,\ \|v_k - u\|_{C^0(\overline{\Omega}\setminus W_\varepsilon)} \to 0$$

(The existence of a quasi-continuous representative is actually a consequence of this.)

Now there are open sets V_j with $\operatorname{cap}(V_j) < 2^{-j}$ such that $\hat{u}_j$ is continuous on $\overline{\Omega} \setminus V_j$ and vanishes on $D \setminus V_j$, and there are smooth approximants $\hat{v}_j$ such that $\|\hat{v}_j - \hat{u}_j\|_{W^{1,2}(\Omega)} < 2^{-j}$ and $\|\hat{v}_j - \hat{u}_j\|_{C^0(\overline{\Omega}\setminus V_j\setminus W_j)} < 2^{-j}$ for appropriate open sets W_j with $\operatorname{cap}(W_j) < 2^{-j}$. Therefore, for every j_0, the sequence $\hat{v}_j$ converges uniformly to u_* outside the set $\overline{V}_{j_0} := \bigcup_{j\ge j_0}(V_j \cup W_j)$, whose capacity is at most 2^{2-j_0}. Hence u_* vanishes on $D \setminus \overline{V}_{j_0}$, for every j_0.

(We could have simplified the argument by starting with a smooth minimizing sequence u_j from the very beginning, but prefer the generality for possible future convenience.)

Uniqueness and positivity follow from the strong maximum principle as in the classical argument. ■

The coarse and fine formulations of *optimal* windows and their eigenfunctions essentially agree:

PROPOSITION 1.2. *Let $\lambda_*^c(\ell)$ and $\lambda_*^f(\ell)$ be optimal eigenvalues for windows of* *size ℓ, as defined by (1.3) in the coarse and fine sense, respectively. Then*

$$\lambda_*^f(\ell) \;=\; \lambda_*^c(\ell) \qquad (0 \le \ell \le \lambda_{\mathrm{Dir}})\ . \tag{1.8}$$

Furthermore, D is an optimal window with respect to the coarse definition, if and only if it differs by a set of $d-1$-dimensional measure zero from an optimal window for the fine formulation.

PROOF: Clearly, from (1.6), we have

$$\lambda_*^f(\ell) \ge \lambda_*^c(\ell)\ .$$

To see the converse inequality, take an optimal window D for the coarse formulation, i.e, $\lambda^c(D) = \lambda_*^c(\ell)$, and let u^c be a minimizer of the corresponding CHVP (1.5). Let $\tilde{u}^c$ be the preferred representative of u^c, as defined above, and set

$$D' := \{x \in \partial\Omega \mid \tilde{u}^c(x) = 0\}\ .$$

We refer to this procedure as *refining* the window D. By definition, $\sigma(D') \ge \ell$ and $\lambda^c(D') = \lambda_*(\ell)$. Since $\tilde{u}^c|_{D'}$ vanishes identically, it is an admissible candidate for the CHVP (1.5) for $\lambda^f(D')$. It follows that

$$\lambda_*^f(\ell) \;\le\; \lambda^f(D') = \lambda^c(D) = \lambda_*^c(\ell)\ .$$

■

Note that we always have $\lambda^f(D') \le \lambda^c(D)$ since $\tilde{u}^c$ vanishes on D'. Whenever $\lambda^f(D) > \lambda^c(D)$ occurs, this is due to $D \setminus D'$ having positive capacity, which makes $\tilde{u}^c$ ineligible for the fine CHVP. Whenever de Giorgi's continuity argument applies at each point of D, i.e., when u^c has a representative that is continuous on $\Omega \cup D$, then $\tilde{u}^c$ is admissible for the fine CHVP and thus $\lambda^f(D) = \lambda^c(D)$. This holds in particular if D is open, notwithstanding possible discontinuities of u^c at interface points. A de Giorgi argument can also be used to show continuity of u^c, provided the window has positive density at every interface point $p \in \overline{D} \cap \overline{\partial\Omega \setminus D}$. We suspect that eigenfunctions for optimal windows should be continuous up to the boundary, but this is an unresolved question.

2. Numerical Results for the Square. We have used the `matlab` pdetool to calculate, by means of finite elements, the lowest eigenvalue for various window configurations. The calculation was done with a sequence of at least three subsequent mesh refinements so that numerical convergence within the precision of the graphics could be checked by inspection. In the accompanying Figure 2.1, we show the eigenvalue as a function of the length of the window, for five different simple geometric configurations.

As outlined in Section 1.1, we conjecture that the configurations giving the lowest eigenvalue in Fig 2.1 (namely either a side-centered or a corner-centered segment, depending on the length) is in fact the optimal configuration. As a rule of thumb, the better of the two choices of symmetric and connected windows is the one that contains more corners. Exceptions to this rule occur near integer multiples of a sidelength.

Figure 2.1 also displays a feature of the first variation formula: when the interface points are in the corner (which implies vanishing of the singular coefficients), the derivative of the eigenvalue vanishes. These are just the explicitly calculable cases marked in the figure. Our numerics does not resolve the modulus of continuity at

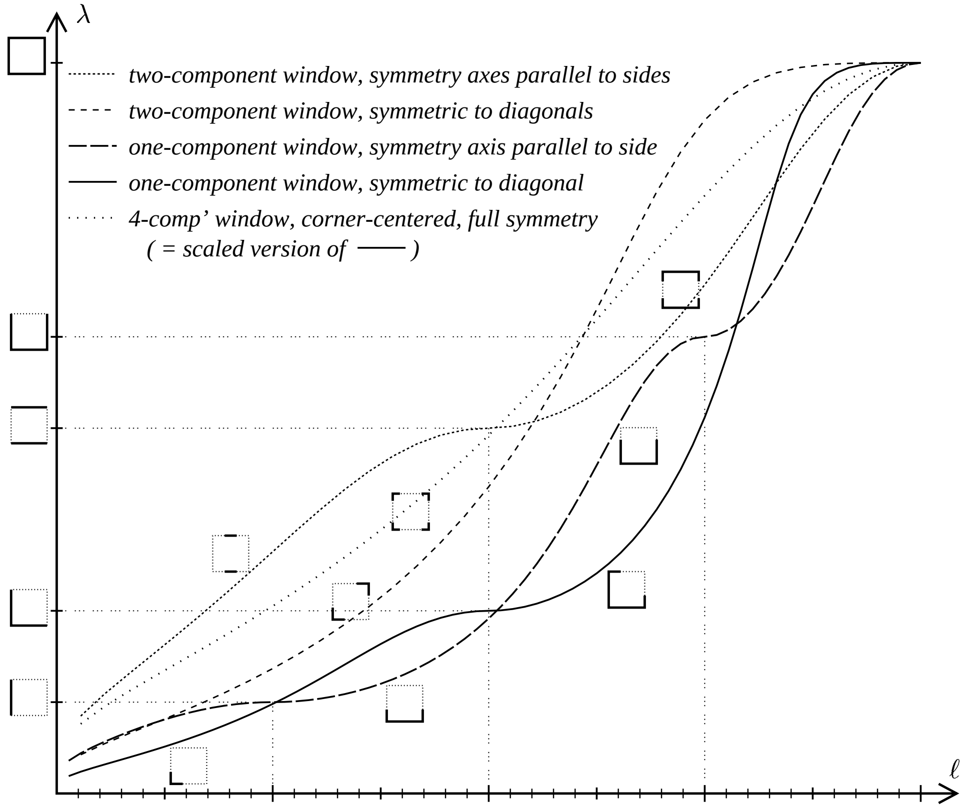


FIG. 2.1. *Five sections through the space of windows, each parametrized by length. The horizontal axis measures the length, in units of the perimeter of the square, the vertical axis gives the eigenvalue relative to the full Dirichlet eigenvalue. The labels mark those window configurations that can be calculated explicitly by separation of variables. In these pictograms, bold lines denote Dirichlet BCs.*

length 0. However, for each of the curves printed, it can be seen analytically that $c_1/\ln(1/\delta) \leq \lambda(\delta) \leq c_2/\ln(1/\delta)$ (with δ the total length of the window): the lower bound follows from Thm 5 in [8] (slightly modified for two dimensions, as pointed out there); the upper bound is an immediate consequence of our capacity estimate in Prop. 5.2 and eqn. (5.2) below. Sharp asymptotics for a different, but closely related problem, can be found in ch. 9 of [20].

We next study the dependence of the eigenvalue on the position of the window. In Figure 2.2, we shift windows of a given length from a side-centered to a corner-centered position. We cannot expect the eigenvalue to depend monotonically on the shift parameter for all lengths, because the side-centered and the corner-centered configuration yield the same eigenvalue for three particular lengths, close to 1.02, 2.04, 3.15 sidelengths, as seen in Figure 2.1. However, we observe in each case that even local minima only occur in symmetric positions, supporting our conjecture.

The slope of the shift curves is proportional to the difference of the singular coefficients at the endpoints of the segment. In the symmetric configurations, this slope vanishes by symmetry. When both endpoints lie in a corner, where the singular coefficients vanish, the derivative appears to vanish to a higher order, indicating further

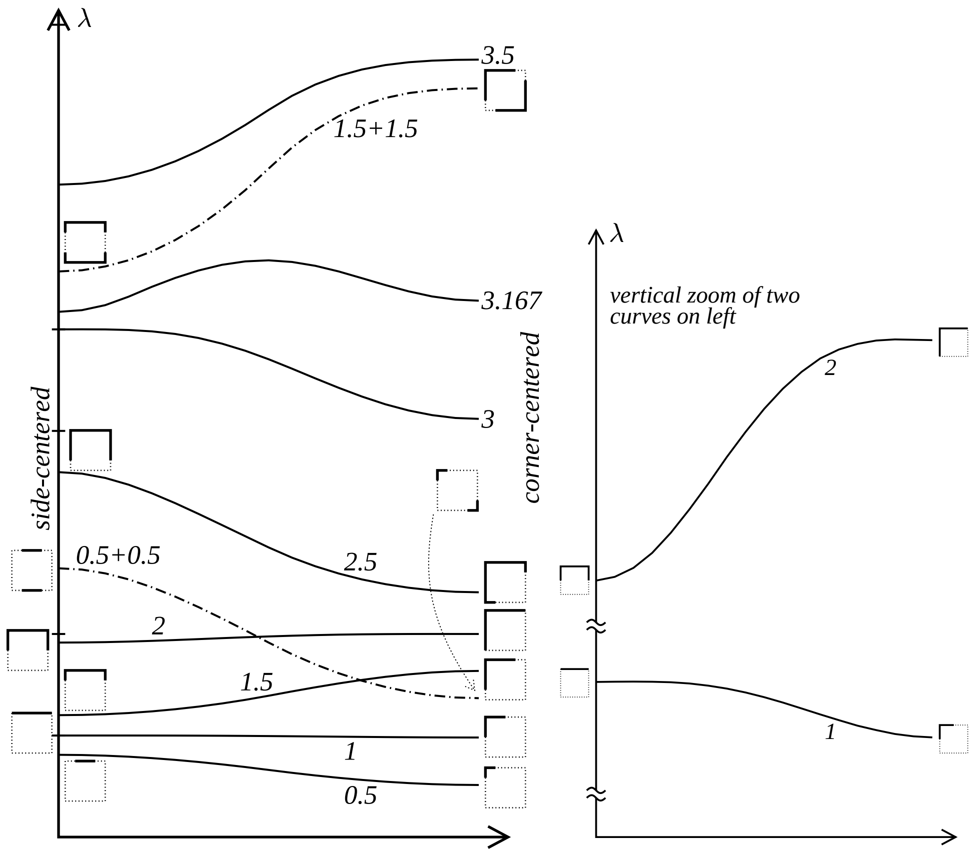


FIG. 2.2. *The eigenvalue for connected windows of nine different lengths and two-component windows of two different lengths, as a function of a shift parameter. The horizontal axis indicates the distance to a side-centered position.*

cancellations. This can be plausibly observed for lengths 1 and 3 in the side-centered configuration, and for length 2 in the corner-centered configuration.

Finally, we observe the effect of tearing apart a connected window into two pieces. See Figure 2.3. Note the competition between corner positions and connectedness as geometric features favoring low eigenvalues.

3. Rigorous Results for the Square. In this section, we collect some inequalities and monotonicity results that are specific to the square.

3.1. Monotonicity of Shifting (Rectangle). For the geometric situation, see the top of Figure 3.1

THEOREM 3.1. *Let Ω be a rectangle. The principal eigenvalue of a connected window D which is contained entirely in one side of $\partial\Omega$ is a continuous, strictly decreasing function of the distance of D from the side-centered position.*

PROOF: By scaling, rotating, and translating, we may assume that $\Omega =]0,1[\times]0,h[$, and that the window is contained in the bottom side of the rectangle. See Figure 3.1. For $0<\ell<1$ and $|t|\le(1-\ell)/2$, let $D(t) =]\frac{1}{2}+t-\frac{\ell}{2}, \frac{1}{2}+t+\frac{\ell}{2}[\times\{0\}$ be the window of length ℓ that has been shifted by t from the side-centered position, and

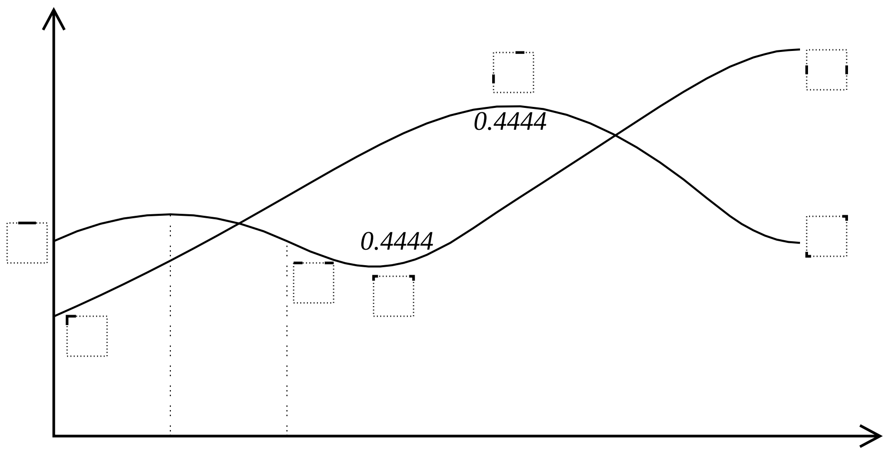


FIG. 2.3. *Examples of competition between connectedness and corner position for small windows. Dotted lines: The first hump for the side-centered window is actually axially symmetric, due to the argument used in Section 3.1.*

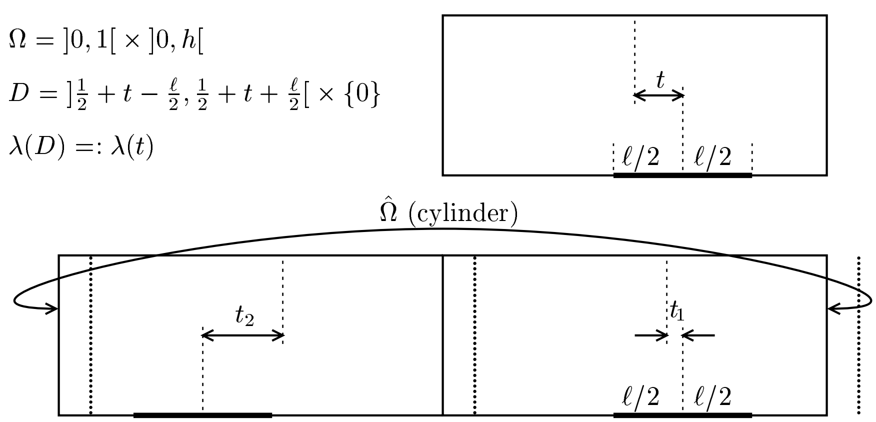


FIG. 3.1. *Shifting windows in a rectangle after doubling it to obtain a cylinder*

denote the corresponding eigenvalue by $\lambda(t)$. By symmetry, λ is an even function of t. Continuity of $\lambda(t)$ follows most easily from Prop. 5.2. To prove the last assertion, we will show that

$$\lambda\big(\tfrac{1}{2}(t_1+t_2)\big) \;>\; \min\{\lambda(t_1),\lambda(t_2)\} \tag{3.1}$$

holds for any pair $t_1 < t_2$. Setting $t_1 = -t$, $t_2 = t$ in Eq. (3.1) and using that $\lambda(-t) = \lambda(t)$ shows that λ takes its global maximum at $t = 0$. Setting $t_{1/2} = t \mp \varepsilon$ in Eq. (3.1) shows that $\lambda(t)$ cannot assume a local minimum on the open interval $]0,(1-\ell)/2[$. We conclude that $\lambda(t)$ is strictly decreasing on $[0,(1-\ell)/2]$, as claimed.

In order to prove claim (3.1), we combine a doubling trick with a special case of Dirichlet–Neumann Bracketing (see [22], XIII.15). Fix $t_1 \le t_2$ with $|t_1|,|t_2| \le (1-\ell)/2$, set $t = (t_1+t_2)/2$, and let u be the positive normalized eigenfunction for the rectangle $\Omega =]0,1[\times]0,h[$ with the window $D(t)$.

Consider the the CHVP on the cylinder $\hat{\Omega} = (\mathbb{R}/2\mathbb{Z})\times]0,h[$, with window $\hat{D}(t) =$

$D(t)\cup(-D(t))$, which is obtained by gluing a copy of Ω with window $D(t)$ to its mirror image along the vertical edges. Since the minimizing function $\hat{u}$ is automatically symmetric by simplicity of the principal eigenvalue in a connected domain, it follows that this CHVP on $\hat{\Omega}$ with window $\hat{D}(t)$ is equivalent to the CHVP on Ω with window $D(t)$. In particular,

$$\hat{u}(x,y) \;=\; \frac{1}{\sqrt{2}}u(|x|,y) \quad (-1\le x\le 1, 0\le y\le h),$$

and the corresponding principal eigenvalue coincides with $\lambda(t)$.

On the other hand, with the understanding that x coordinates are interpreted modulo 2, the cylinder $\hat{\Omega}$ contains the disjoint union of the rectangles $\Omega_1 =](t_2-t_1)/2, 1+(t_2-t_1)/2[\times]0,h[$ and $\Omega_2 =]-1+(t_2-t_1)/2,(t_2-t_1)/2[\times]0,h[$, which are copies of Ω with windows $D(t_1)$ and $D(t_2)$, respectively. By restricting $\hat{u}$ to $\Omega_1\cup\Omega_2$ we obtain a test function for $\Omega_1\cup\Omega_2$ with window $D(t_1)\cup D(t_2)$. Since the principal eigenvalue for a disjoint union of domains is the smaller of the two eigenvalues, it follows that

$$\lambda(t)\ge \min\{\lambda(t_1),\lambda(t_2)\}.$$

The functions $u_1=\hat{u}|_{\Omega_1}$ and $u_2=\hat{u}|_{\Omega_2}$ cannot be eigenfunctions for $\lambda(t_1)$ and $\lambda(t_2)$, because the gradient of $\hat{u}$ vanishes at those boundary points of Ω_1 or Ω_2 that were corners of Ω, in violation of the Hopf boundary point lemma for u_1 and u_2. This completes the proof of Eq. (3.1). ■

The doubling argument used in the proof shows that two connected window segments of length ℓ each, placed symmetrically with distance $2s$ from the center of a rectangle of side length 2 yield the same eigenvalue as two such windows placed symmetrically with distance $2(1-\ell-s)$ apart. This can be observed in the curve for the side-centered configurations in Figure 2.3.

3.2. Optimality Among Windows on One or Two Sides. The monotonicity argument in the previous subsection implies in particular that among all connected windows contained in *one* side of a rectangle, the one touching a corner produces the minimal eigenvalue. We next consider windows contained in two sides of a square.

THEOREM 3.2. *Among all windows that lie on only two sides of the square (adjacent or not), the optimal window is connected and contains a corner of the square in its interior.*

The proof is based on rearrangement techniques, which have been widely used for geometric inequalities (see [14, 18] for a general reference). Here we will use two rearrangements adapted to the square: the increasing rearrangement and polarization.

For a nonnegative measurable function u on a rectangle, we define the *increasing rearrangement in the x-direction*, $\mathcal{R}u$, by replacing the restriction of u to each line $y=$ *const* with the unique non-decreasing left-continuous function which is equimeasurable with $u(\cdot,y)$. By Fubini's theorem, $\mathcal{R}u$ is equimeasurable with u.

LEMMA 3.3. *Let u be a nonnegative $W^{1,2}$-function on a rectangle $\Omega =]0,1[\times]0,h[$, and let $\mathcal{R}u$ be its increasing rearrangement in the x-direction. If the trace of u vanishes σ-a.e. on a window*

$$D \;=\; \Big(\{0\}\times D_l\Big)\cup\Big(\{1\}\times D_r\Big)\cup\Big(D_b\times\{0\}\Big)\cup\Big(D_t\times\{h\}\Big)\ ,$$

then $\mathcal{R}u$ vanishes σ-a.e. on the window $\mathcal{R}D$ defined by

$$(3.2)\quad (\mathcal{R}D)_l = D_l \cup D_r,\ (\mathcal{R}D)_r = \emptyset,\ (\mathcal{R}D)_b =]0, \sigma(D_b)[,\ (\mathcal{R}D)_t =]0, \sigma(D_t)[.$$

In general,

$$\sigma(D) \geq \sigma(\mathcal{R}D)\ ,$$

with equality certainly when $D_r = \emptyset$. The corresponding principal eigenvalues satisfy

$$\lambda(D) \geq \lambda(\mathcal{R}D)\ ,$$

with equality only when $\mathcal{R}D$ agrees σ-a.e. with either D or its mirror image.

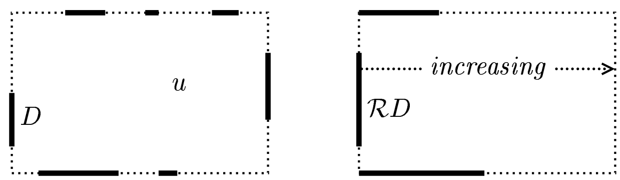


FIG. 3.2. *The effect of the symmetric increasing rearrangement on a window with components on all four sides of a rectangle.*

PROOF: If u is continuous up to the boundary of the rectangle and vanishes on D, then its rearrangement $\mathcal{R}u$ is also continuous and vanishes on the window $\mathcal{R}D$. To see that the trace of $\mathcal{R}u$ vanishes on $\mathcal{R}D$ for any nonnegative nonnegative function u in $W^{1,2}$ vanishing on D, we note that the increasing rearrangement is closely related with Steiner symmetrization. In fact, if we extend both u and $\mathcal{R}u$ by reflection across the line $x = 1$ to functions $\hat{u}$ and $\hat{\mathcal{R}u}$ on the doubled rectangle $\hat{\Omega} =]0, 2[\times]0, h[$, then $\hat{\mathcal{R}u}$ is just the Steiner symmetrization of $\hat{u}$. Since Steiner symmetrization is continuous on $W^{1,2}$ [3], $\mathcal{R}$ is continuous as well, and the first claim follows by a density argument.

For the second claim we use that $\mathcal{R}$ preserves the L^2-norm but reduces the norm of the gradient. In particular, $\mathcal{R}$ can only decrease the Rayleigh quotient. Choosing u to be the principal eigenfunction of the CHVP corresponding to the window D, we see that

$$(3.3)\qquad \lambda(D) = \frac{\int |\nabla u|^2\,dx}{\int |u|^2\,dx} \geq \frac{\int |\nabla \mathcal{R}u|^2\,dx}{\int |\mathcal{R}u|^2\,dx} \geq \lambda(\mathcal{R}D)\ .$$

By analyticity, the partial derivative $\partial_x u$ vanishes only on a set of zero measure. It follows from a theorem of Brothers and Ziemer [4] that the rearrangement inequality in Eq. (3.3) is strict unless u is already either increasing or decreasing in x on each line $y = const.$ ■

The second rearrangement exploits the symmetry of the square under reflections at the diagonals. Let $\Omega =]0,1[\times]0,1[$ be the unit square, and let $\tau(x,y) = (y,x)$ denote the reflection at the diagonal joining the lower left with the upper right hand corner. For any function u on Ω, the *polarization* $\mathcal{P}u$ of u with respect τ is given by

$$\mathcal{P}u(x,y) = \begin{cases} \max\{u(x,y), u(\tau(x,y))\}, & \text{if } y \geq x, \\ \min\{u(x,y), u(\tau(x,y))\}, & \text{if } y \leq x. \end{cases}$$

For a comprehensive account of polarization we refer to [2]. We have the following lemma:

LEMMA 3.4. *Let u be a nonnegative $W^{1,2}$-function on the unit square $\Omega =]0,1[\times]0,1[$, and let $\mathcal{P}u$ be its polarization, as defined above. If (the trace of) u vanishes σ-a.e. on a window*

$$D \;=\; \Big(\{0\}\times D_l\Big)\cup\Big(\{1\}\times D_r\Big)\cup\Big(D_b\times\{0\}\Big)\cup\Big(D_t\times\{1\}\Big)\ ,$$

then $\mathcal{P}u$ vanishes σ-a.e. on the window $\mathcal{P}D$ with

$$\mathcal{P}D_l = D_l\cap D_b,\ \mathcal{P}D_r = D_r\cup D_t,\ \mathcal{P}D_b = D_l\cup D_b,\ \mathcal{P}D_t = D_r\cap D_t.$$

In general,

$$\sigma(D)=\sigma(\mathcal{P}D)\ ,$$

and the principal eigenvalues satisfy

$$\lambda(D)\geq\lambda(\mathcal{P}D),$$

with equality only if $\mathcal{P}D$ agrees σ-a.e. with either D or $\tau(D)$.

PROOF: The form of $\mathcal{P}D$ is immediate from the definition of $\mathcal{P}$. To see the second claim, choose u to be the principal eigenfunction corresponding to the window D. Since $\mathcal{P}u$ is equimeasurable with u, and $|\nabla\mathcal{P}u|$ is equimeasurable with $|\nabla u|$ by definition of the polarization, we have

$$\lambda(D)=\frac{\int|\nabla u|^2\,dx}{\int|u|^2\,dx}=\frac{\int|\nabla\mathcal{P}u|^2\,dx}{\int|\mathcal{P}u|^2\,dx}\geq\lambda(\mathcal{P}D)\ .$$

Unless $\mathcal{P}u$ agrees with either u or $u\circ\tau$, it cannot be real analytic, and hence is not the eigenfunction corresponding to $\lambda(\mathcal{P}D)$. We conclude that then the last inequality is strict. ■

PROOF OF THM. 3.2:
Within the class of windows contained in two sides of the square, there clearly exists an optimal one. By Lemma 3.4, a window consisting of two non-empty parts contained in two opposite sides of the square cannot be optimal, since it can be improved by polarization.

If D is contained in two adjacent sides (say, left and bottom) of the square, Lemma 3.3 implies that replacing D with $\mathcal{R}D$ strictly reduces the principal eigenvalue, unless the bottom part of the window is connected and contains a corner. Note that in this case, $\mathcal{R}D$ has the same length as D. Repeating this argument for the vertical direction, we see that also the part of D on the left hand side must be connected and contain the lower left corner.

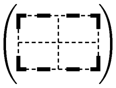
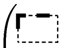
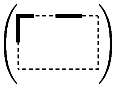
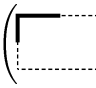
It remains to show that a corner must lie in the *interior* of the window. If the length of D happens to equal the length of one side of the square, we refer to the numerical result, which shows that the corner-centered position improves over the one-side position. Otherwise, we refer to Cor. 6.3 below to show that moving the segment a short distance round the corner improves the eigenvalue. ■

3.3. Non-Optimality of $\mathbb{Z}_2\times\mathbb{Z}_2$-Symmetric Windows. We have the following theorem:

THEOREM 3.5. *In a rectangle, any window of sufficiently small length that has the full symmetry group of a rectangle is not optimal. In particular in a square, a symmetric window whose length does not exceed the length of one side is not optimal.*

As mentioned before, numerical results for the square indicate that the length restriction is not needed.

PROOF: In self-explanatory pictogram notation, we reason that

(3.4) $\lambda\Big($  $\Big) = \lambda\Big($  $\Big) = 4\lambda\Big($  $\Big) \geq 4\lambda\Big($  $\Big)$,

exploiting symmetry, scaling, and the rearrangement of Lemma 3.3 in turn. The last inequality is strict unless the window consists of four L-shaped windows in the corners to begin with, showing that an optimal window having full symmetry must be of that form. In Eq. (3.4), we have gained a factor 4, but lost half of the window length. We now double the window using Lemma 6.1.

Assume that the rectangle has the form $\Omega =]0,a[\times]0,b[$ and the L-shaped window (called D_L) has lengths $q_x a$ and $q_y b$ on the horizontal and vertical parts respectively. An admissible test function for the CHVP for an L-shaped window with side lengths $2q_x a$ and $2q_y b$ is given by $u\circ\psi$, where $\psi(x,y) = (h(x),k(y))$ with h,k piecewise linear such that $h(0)=k(0)=0$, $h(a)=a$, $k(b)=b$, and $h(2q_x a)=q_x a$, $k(2q_y b)=q_y b$. It is easy to see that $\psi:\Omega\to\Omega$ is bi-Lipschitz. The largest value for the spectral radius of $(D\psi)(D\psi)^T/\det D\psi$ is $2(1-q)/(1-2q)$ with $q=\max\{q_x,q_y\}$, and the largest value for $\det D\psi$ is $(1-q_x)/(1-2q_x)\times(1-q_y)/(1-2q_y)$. By Lemma 6.1, the window $\psi^{-1}(D_L)$ is an improvement over the original window, whenever $2(1-q)^3/(1-2q)^3 \leq 4$, which happens for $q < 0.17$ and translates to a smallness condition on the window size, depending on the side lengths of the rectangle.

In the square $]0,1[\times]0,1[$, an optimal window which is symmetric under reflections at the vertical and horizontal axes must be symmetric under reflection in the diagonals as well, since otherwise a better window is obtained by polarization; this gives $q_x = q_y =: q$. We can now get a better quantitative estimate in Eq. (3.4) for the square. Define a bi-Lipschitz map by setting

$$\psi:(x,y)\mapsto\begin{cases}\Big(x,1-\frac{1-q}{1-2q}(1-y)\Big) & \text{if } y\geq 1-(1-2q)(1-x) \qquad \text{(I)}\\ \big(x,\frac{1}{2}(x+y)\big) & \text{if } x\leq y\leq 1-(1-2q)(1-x) \quad \text{(II)}\end{cases}$$

above the diagonal, and an analogous formula below the diagonal. The spectral radius of $(D\psi)(D\psi)^T/\det D\psi$ is $(1-q)/(1-2q)$ in domain (I) and $\frac{1}{2}(3+\sqrt{5})$ in (II). The Jacobian $\det D\psi$ is largest in (I), namely $(1-q)/(1-2q)$. Lemma 6.1 asserts that the window can be doubled with a factor ≤ 4 in the eigenvalue, provided $\sigma(D) = 4q \leq 4(5-\sqrt{5})/(13-\sqrt{5}) \approx 1.027$. ■

COROLLARY 3.6. *The result of Thm. 3.5 holds, for sufficiently small windows in a rectangle $]-a,a[\times]-b,b[$, under the weaker assumption that either (a) there is equal window area in each of the four quadrants, or (b) the window is symmetric under the $180°$ rotation $(x,y)\mapsto(-x,-y)$.*

PROOF: For (a), the first step in (3.4) can be replaced with an inequality, where that quarter is selected that contributes the smallest Rayleigh quotient. For (b), note that the symmetry is inherited by the eigenfunction, and we have $u(0,y) = u(0,-y)$. So we can define $\hat{u} \in W^{1,2}$ by: $\hat{u}(x,y) = u(x,y)$ for $x \leq 0$, and $\hat{u}(x,y) = u(x,-y)$ for $x \geq 0$. $\hat{u}$ represents another window $\hat{D}$ with the same area as D, has the same Rayleigh quotient, and is not the optimizer yet, unless $\hat{D} = D$; this reduces the corollary to the theorem again. ■

4. A Star-Shaped Domain With Disconnected Optimal Window. We here prove the properties of the following

EXAMPLE 4.1. *There exists a starshaped Lipschitz domain Ω in $\mathbb{R}^2$ and a length ℓ such that a connected window of length ℓ in Ω cannot be optimal.*

PROOF: In a one-parameter family of domains Ω_ε, we calculate an upper bound for the eigenvalue of a certain window D_2 with two components. Then we establish a larger lower bound for the eigenvalue of any connected window D. These estimates, based on Dirichlet-Neumann bracketing, work for sufficiently small ε, and can be made quantitative.

Ω_ε is the union of a 'torso' rectangle T_ε and a pair of 'handles' H_ε, $-H_\varepsilon$:

$$T_\varepsilon := \,]-1,1[\,\times\,]-1-\varepsilon,1+\varepsilon[\,, \quad H_\varepsilon := [1,9-\varepsilon[\,\times\,]-\varepsilon,\varepsilon[\,. \tag{4.1}$$

See the top left part of Figure 4.1. We choose $D_2 := (\partial\Omega_\varepsilon) \setminus \overline{T}_\varepsilon$, with $\sigma(D_2) = 32$. The remaining boundary $W := (\partial\Omega_\varepsilon) \setminus D_2$ has measure $\sigma(W) = 8$.

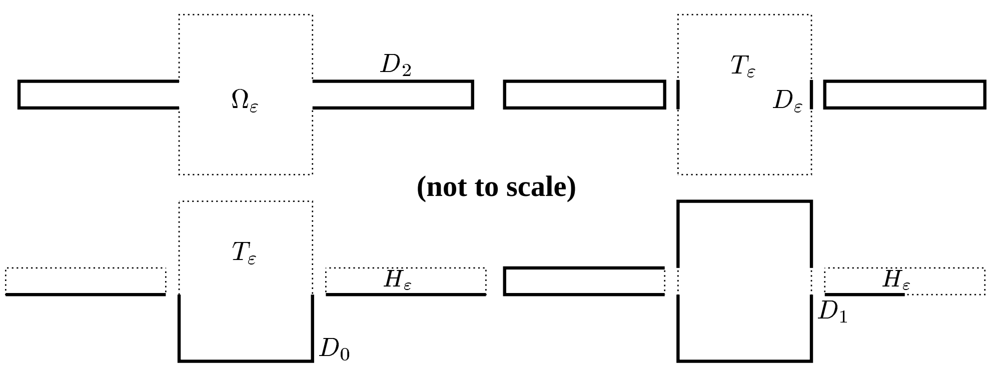


FIG. 4.1. *Top left: A starshaped Lipschitz domain whose optimal window(s) of a certain length ℓ cannot be connected. Top right: upper bound for eigenvalue of disconnected window. Bottom: Lower bounds for connected windows.*

For comparison, disconnect the handles from the torso by means of extra Dirichlet boundary $D_\varepsilon = \{\pm 1\} \times [-\varepsilon,\varepsilon]$, as in the top right of Fig. 4.1. With fewer competitors in the CHVP (1.5), we get an upper bound. In self-explaining notation, we conclude

$$\lambda(\Omega_\varepsilon, D_2) < \min\Big\{\lambda_{\mathrm{Dir}}(H_\varepsilon), \lambda(T_\varepsilon, D_\varepsilon)\Big\} = \lambda(T_\varepsilon, D_\varepsilon)\ .$$

By testing the EVP for T_ε with $\sin\frac{\pi}{2}(|y|-\varepsilon)_+$, one can see that the evaluation of the minimum is valid for all $\varepsilon \leq 1$.

For any connected window D of length 32, it can easily be seen that, except for reflection symmetry, either $D \supset D_0$ or $D \supset D_1$, where

$$D_0 = \{(x,y) \in \partial\Omega_\varepsilon \mid y \leq -\varepsilon\}$$
$$D_1 = \{(x,y) \in \partial\Omega_\varepsilon \mid x \leq 1\} \cup [1,5] \times \{-\varepsilon\}$$

To get lower bounds for $\lambda(\Omega_\varepsilon, D_0)$ and $\lambda(\Omega_\varepsilon, D_1)$, disconnect the handles from the torso by means of extra Neumann boundary $\{\pm 1\}\times[-\varepsilon,\varepsilon]$: In slight abuse of notation, we write $\lambda(T_\varepsilon, D_i)$ for $\lambda(T_\varepsilon, D_i \cap \partial T_\varepsilon)$, and similarly for H_ε. We have either

$$\lambda(\Omega_\varepsilon, D) > \min\Big\{\lambda(H_\varepsilon, D_0), \lambda(T_\varepsilon, D_0)\Big\} = \lambda(T_\varepsilon, D_0) > (\tfrac{\pi}{4+4\varepsilon})^2 \tag{4.2}$$

or

$$\lambda(\Omega_\varepsilon, D) > \min\Big\{\lambda(H_\varepsilon, D_1), \lambda(T_\varepsilon, D_1)\Big\} = \lambda(H_\varepsilon, D_1) \tag{4.3}$$

The evaluation of the minimum in (4.2), for any ε, relies on a test function that vanishes for $y \le -\varepsilon$. The evaluation of the minimum in (4.3) is valid for all $\varepsilon < \frac{3}{2}$, since then, using comparison functions $\cos(\pi y/(2+2\varepsilon))$ and $\sin \pi(x-5)_+/2(4-\varepsilon)$,

$$\lambda(T_\varepsilon, D_1) > (\pi/(2+2\varepsilon))^2 \ge (\pi/(2(4-\varepsilon)))^2 > \lambda(H_\varepsilon, D_1)\ .$$

For $\varepsilon < \frac{2}{3}$, we can also conclude that

$$\lambda(H_\varepsilon, D_1) < (\pi/(8-2\varepsilon))^2 < (\pi/(4+4\varepsilon))^2 < \lambda(T_\varepsilon, D_0)\ .$$

It therefore only remains to prove the middle inequality in

$$\lambda(\Omega_\varepsilon, D) > \lambda(H_\varepsilon, D_1) > \lambda(T_\varepsilon, D_\varepsilon) > \lambda(\Omega_\varepsilon, D_2)\ .$$

But as $\varepsilon \to 0$, one has $\lambda(T_\varepsilon, D_\varepsilon) \to 0$, whereas $\lambda(H_\varepsilon, D_1) \to (\pi/8)^2$. This intuitively clear fact can be proved in a straightforward way by writing the quadratic form $\int_{H_\varepsilon}(u_x^2+u_y^2)\,dx\,dy$ as a quadratic form $\int(\varepsilon\chi(\xi)^{-1}u_\xi^2 + \varepsilon^{-1}\chi(\xi)u_\eta^2)\,d\xi\,d\eta$ on L^2 with measure $\varepsilon\chi(\xi)d\xi\,d\eta$ in a fixed reference domain $]0,8[\,\times\,]-1,1[$. Here $\chi(\xi)=1$ for $\xi<4$, and $\chi(\xi) = 1-\varepsilon/4$ for $\xi > 4$. If we carry out the limit $\varepsilon \to 0$ in the CHVP with the appropriate eigenfunctions, we have the uniform upper bound $(\pi/(8-2\varepsilon))^2$ for the eigenvalue, as mentioned before. This controls the $W^{1,2}$ norm in the fixed domain, and actually enforces $u_\eta \to 0$. The limiting function will indeed not depend on the η coordinate and solve the one-dimensional eigenvalue problem $-u_{\xi\xi} = \lambda u$ on $[4,8] \ni \xi$, with $u(4)=0$, $u_\xi(8)=0$. ∎

5. Some Continuity Results. In this section, we study how the eigenvalue changes if a window of a particular size is added at a particular location. The basic philosophy is that windows can be added more cheaply at locations where the eigenfunction was already small before the addition. In the second subsection, we discuss related continuity properties of the corresponding eigenfunctions.

5.1. Continuity of Eigenvalues. Our first result is an estimate for the increase of the principal eigenvalue, if a set of small capacity is added to a given window.

LEMMA 5.1. *Let $D_2 \supset D_1$ and let u_1 be the normalized eigenfunction for D_1. Let G be a domain containing $D_2 \setminus D_1$; in case the dimension $d=2$, assume additionally that G is bounded. Then*

$$\lambda(D_2) - \lambda(D_1) \le \frac{\lambda(D_1)\,\mathrm{vol}\,(G\cap\Omega) + \mathrm{cap}(D_2\setminus D_1, G)}{1-(\sup_{G\cap\Omega} u_1)^2\,\mathrm{vol}\,(G\cap\Omega)}\,\big(\sup_{G\cap\Omega} u_1\big)^2\,, \tag{5.1}$$

where cap *is the capacity defined in [19, 2.2.1], namely:*

$$\mathrm{cap}(D_2\setminus D_1, G) := \inf\Big\{\int_G |\nabla v|^2 \ \Big|\ v = 1 \textit{ in a nbhd of } D_2\setminus D_1\,;\ v \in C_0^\infty(G)\Big\}\ .$$

Proof: Let $M := \sup_{G\cap\Omega} u_1$. As explained in [9] near (3.2), it follows from de Giorgi's argument (see formula (5.12) in chapter 2 of Ladyzhenskaya–Ural'tseva [17]) that $\sup_\Omega u_1$ is finite, and can even be chosen to depend only on Ω, not on D_1. To obtain a test function for the CHVP which determines $\lambda(D_2)$, we modify u_1 in G: In $\overline{G}\cap\Omega$, let $u_2 := \min\{u_1, M(1-v)\}$, where v is one of the functions that approximate the capacity of $D_2\setminus D_1$; outside (if any), let $u_2 = u_1$. Since $u_2 = u_1$ on $\Omega\cap\partial G$, this does not introduce discontinuities, and u_2 is an admissible test function for $\lambda(D_2)$.

Clearly

$$\int_\Omega u_2^2(x) \geq \int_{\Omega\setminus G} u_1^2 \geq 1 - M^2\,\mathrm{vol}\,(G\cap\Omega)$$

and

$$\int_\Omega |\nabla u_2|^2 \leq \int_\Omega |\nabla u_1|^2 + M^2 \int_G |\nabla v|^2 \to \int_\Omega |\nabla u_1|^2 + M^2\,\mathrm{cap}(D_2\setminus D_1, G)$$

as v runs through a minimizing sequence for the capacity functional. We conclude (5.1) immediately. ■

In applications of the lemma, G should be a small neighbourhood of $D_2\setminus D_1$, so that in the numerator on the right hand side of (5.1), the capacity term dominates the volume term. It can be used to establish continuity of the eigenvalue under deformations of sufficiently regular windows. The following simplified estimate suffices to show the continuous dependence of the eigenvalue on the length and position of a segment in a square:

Proposition 5.2. *For a given bounded Lipschitz domain $\Omega\subset\mathbb{R}^d$, there exists a nonnegative continuous function η with $\eta(0) = 0$ so that for any pair of windows $D_1\subset D_2\subset\partial\Omega$,*

$$\lambda(D_2) \;\leq\; \lambda(D_1) + \eta\,(\mathrm{diam}(D_2\setminus D_1))\ ,$$

where the η is a continuous function with $\eta(0) = 0$ which depends only on Ω but not on D_1 and D_2. The result applies to the coarse as well as to the fine definition of the eigenvalue.

Proof:
We assume $D_2\setminus D_1\subset B_\delta(x_0)$ where $\delta := \mathrm{diam}(D_2\setminus D_1)$ and use the Green's function as a legitimate limiting case for v in the capacity functional; namely, for dimension $d = 2$, let $G = B_R(x_0)$ and $1 - v := \ln_+(|x-x_0|/\delta)/\ln(R/\delta)$, with, say, $R = \sqrt{\delta}$ when $\delta < 1$. For $d\geq 3$, we can take $G = B_R(x_0)$ with $R := \delta^{(d-2)/d}$ and let $1 - v := (\delta^{-(d-2)} - |x-x_0|^{-(d-2)})_+/(\delta^{-(d-2)} - R^{-(d-2)})$. For simplicity, we can take $M := \sup_\Omega u_1$ as an upper bound for $\sup_{G\cap\Omega} u_1$, and obtain the claim with

$$\eta(\delta) := \begin{cases} \pi M^2\,\dfrac{\delta\lambda_{\mathrm{Dir}} + 2/\ln(\delta^{-1/2})}{1-\pi M^2\delta} & \text{for } d = 2 \\[2ex] \dfrac{(d-1)^2\omega_d M^2\delta^{d-2}}{1-\omega_d M^2\delta^{d-2}} & \text{for } d > 2 \end{cases} \tag{5.2}$$

where ω_d is the volume of the unit ball. ■

It should be noted that the modulus of continuity of the eigenvalue cannot be expressed in terms of $\sigma(D_2 \setminus D_1)$ alone. This is due to the fact [8, Thm. 8] that for any ε, there exists a window of measure $< \varepsilon$ with eigenvalue $> \lambda_{\mathrm{Dir}} - \varepsilon$. This observation also implies, in view of the a-priori estimate for $\|u\|_\infty$ and Hölder's inequality, that an estimate in terms of $\|u_1\|_p$ is not possible for any $p < \infty$.

THEOREM 5.3. *The optimal eigenvalue λ_* depends continuously on the prescribed boundary measure of the window.*

PROOF: We will prove that in dimensions $d > 2$, the function $\ell \mapsto \lambda_*(\ell)$ is Hölder continuous with exponent $(d-2)/(d-1)$, for $\ell < \sigma(\partial\Omega)$. In $d = 2$ dimensions, we will obtain a logarithmic estimate for the modulus of continuity.

Fix $\ell_1 < \sigma(\partial\Omega)$, and let D_1 be an optimal window with $\sigma(D_1) = \ell_1$. It follows from Prop. 5.2 that

$$\lambda\big(D_1 \cup (B_\delta(x_0) \cap \partial\Omega)\big) - \lambda(D_1) < \eta(\delta)$$

for any choice of $x_0 \in \partial\Omega$ and $\delta > 0$. We want to choose x_0 so that $\sigma\big((B_\delta(x_0) \cap \partial\Omega) \setminus D_1\big)$ is bounded away from zero. To do this, we use Fubini's theorem to estimate

$$\begin{aligned}
\fint_{\partial\Omega} \sigma((B_\delta(x) \cap \partial\Omega) \setminus D_1) d\sigma(x) &= \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega} \int_{\partial\Omega\setminus D_1} \mathbf{1}_{|x-y|<\delta}\, d\sigma(y) d\sigma(x) \\
&= \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega\setminus D_1} \sigma(B_\delta(y))\, d\sigma(y) \\
&\geq \frac{\sigma(\partial\Omega) - \sigma(D_1)}{\sigma(\partial\Omega)} \inf_{y\in\partial\Omega} \sigma(B_\delta(y)) \ .
\end{aligned}$$

Since Ω is a bounded Lipschitz domain, there exists a constant c, depending only on Ω, such that $\sigma(B_\delta(x_0)) \geq c\delta^{d-1}$. We conclude that for any value of δ there exists a point $x_0 \in \partial\Omega$ such that

$$\sigma\big(B_\delta(x_0) \setminus D_1\big) \geq \left(1 - \frac{\ell_1}{\sigma(\partial\Omega)}\right) c\delta^{d-1} \ .$$

For $\ell_2 > \ell_1$, set

$$\delta = \left(\frac{\ell_2 - \ell_1}{c(1 - \ell_1/\sigma(\partial\Omega))}\right)^{1/(d-1)} ,$$

and let $D_2 = D_1 \cup (B_\delta(x_0) \cap \partial\Omega)$. Since $\sigma(D_2) \geq \ell_2$, it follows that

$$\lambda_*(\ell_2) - \lambda_*(\ell_1) \leq \lambda(D_2) - \lambda(D_1) \leq \eta(\delta) \ .$$

The claim now follows from the expression for η given in Prop. 5.2. ■

The punchline of Thm. 5.3 is that we get a uniform modulus of continuity without extra regularity assumptions on the boundary. For smoother $\partial\Omega$, stronger results could be obtained using the tools of Sec. 6. We conjecture (but have not pursued) that the window D_2 in Ex. 4.1 is actually optimal, and that the modulus of continuity at that length in Ex. 4.1 is precisely $O(\delta^{2/3})$. This intuition is based on the $r^{1/3}$ singularity of the eigenfunction at the re-entrant corner, the role of singularities revealed in Sec. 6, and the estimate from Lemma 5.1. A Lipschitz estimate for $\ell \mapsto \lambda_*(\ell)$ should not be expected without further assumptions on $\partial\Omega$, but smoothness (a.e.) of $\partial\Omega$ will improve upon Thm. 5.3.

The following simple lemma estimates the change of the eigenvalue under increase of a window solely in terms of the eigenfunction on the smaller window.

LEMMA 5.4. *Given $\Omega \subset \mathbb{R}^d$ and two windows $D_1 \subset D_2 \subset \partial\Omega$. Let u_1 be the normalized eigenfunction corresponding to $\lambda(D_1)$. Then*

$$\lambda(D_2) - \lambda(D_1) \le \lambda(D_1) \frac{\sqrt{\mathrm{vol}\,(\Omega)}\,\sup_{D_2\setminus D_1} u_1}{1 - \sqrt{\mathrm{vol}\,(\Omega)}\,\sup_{D_2\setminus D_1} u_1} \tag{5.3}$$

PROOF: Let $\varepsilon := \sup_{D_2\setminus D_1} u_1$. Then $v_\varepsilon = (u_1 - \varepsilon)_+$ is an admissible test function for both the CHVP's defining $\lambda(D_2)$ and $\lambda(D_1)$. We compute

$$\|\nabla v_\varepsilon\|_2^2 = \int \nabla u_1 \cdot \nabla (u_1 - \varepsilon)_+ = \lambda(D_1) \int u_1 (u_1 - \varepsilon)_+ \ , \tag{5.4}$$

where we have used the weak form $\int \nabla u \nabla \varphi = \lambda \int u \varphi$ of the eigenvalue equation $\Delta u = -\lambda u$, with $\varphi := v_\varepsilon$. It follows that

$$\begin{aligned} \lambda(D_2) - \lambda(D_1) &\le \frac{\int |\nabla v_\varepsilon|^2}{\int v_\varepsilon^2} - \lambda(D_1) \\ &\le \lambda(D_1) \frac{\int \varepsilon (u_1 - \varepsilon)_+}{\int (u_1 - \varepsilon)_+^2} \\ &\le \lambda(D_1) \frac{\varepsilon}{\|(u_1 - \varepsilon)_+\|_2} \left(\mathrm{vol}\,(\Omega)\right)^{1/2} \ . \end{aligned}$$

The triangle inequality $\|(u_1 - \varepsilon)_+\|_2 \ \ge\ 1 - \varepsilon \left(\mathrm{vol}\,(\Omega)\right)^{1/2}$ now yields the claim. ■

For a given window $D \subset \partial\Omega$, denote by

$$D_\delta := \Big(\bigcup_{x \in D} B_\delta(x) \Big) \cap \partial\Omega \quad (\delta > 0) \ , \quad D_0 := D \tag{5.5}$$

the relative δ-neighborhood of D in $\partial\Omega$. Continuity of the eigenfunction up to the boundary is sufficient for continuity of the eigenvalue function $\delta \mapsto \lambda(D_\delta)$:

THEOREM 5.5. *Let u be an eigenfunction for window boundary conditions on D, and assume that the preferred representative $\tilde{u}$ vanishes everywhere on D.*

(a) If $\tilde{u}$ is upper semi-continuous on $\overline{\Omega}$, then $\lambda(\cdot)$ is outer regular at D in the sense that for every $\varepsilon > 0$, there exists a relatively open subset $U \subset \partial\Omega$ containing D, with the property that

$$\lambda(U) \le \lambda(D_\varepsilon) \ .$$

(b) If u is continuous up to the boundary of Ω, then the map $\delta \mapsto \lambda(D_\delta)$ is right continuous at $\delta = 0$. We will show below (Thm. 5.7) that the hypothesis of part (a) is satisfied for $C^{1,\alpha}$ domains in $\mathbb{R}^2$, and at flat pieces of the boundary in any dimension. We conjecture that upper semicontinuity may hold at least for smooth domains in any dimension.

Concerning part (b), continuity up to the boundary can be shown for the eigenfunction by a careful analysis of de Giorgi's argument, under the assumption that the window D has positive Lebesgue density at every interface point $x_0 \in \overline{D} \cap \overline{\partial\Omega \setminus D}$.

We conjecture, but cannot prove, that eigenfunctions for optimal windows are continuous up to the boundary. Below, we show by an example that continuity of the eigenfunction is not necessary for continuity of $\delta \mapsto \lambda(D_\delta)$.

Proof of Thm. 5.5:
If the preferred representative $\tilde{u}$ is upper semicontinuous on the closure of Ω, then the set

$$U = \{x \in \partial\Omega \ \mid \ u(x) < \eta\}$$

is a (relatively) open set containing D. By Lemma 5.4, we have that

$$\lambda(D) \leq \lambda(U) \leq \frac{\lambda(D)}{1 - \eta\mu(\Omega)^{1/2}} < \lambda(D) + \varepsilon \ ,$$

if $\eta = \eta(\varepsilon)$ is chosen sufficiently small (e.g., $\eta := \varepsilon(\lambda_{\mathrm{Dir}}(\Omega)\,\mathrm{vol}\,(\Omega))^{-1})$. This proves outer regularity. If $\tilde{u}$ is continuous, then D is compact, and hence there exists a $\delta > 0$ so that $D_\delta \subset U$, which proves the second claim. ∎

Note that assuming that $\tilde{u}$ vanishes everywhere on D amounts to replacing D with its refinement, and selecting the fine eigenvalue. Since coarse and fine eigenvalues agree for the open windows D_δ, continuity of λ^c certainly fails at any window D for which $\lambda^c(D) < \lambda^f(D)$. Cantor sets of zero measure but positive capacity provide examples of such windows.

However, $\delta \mapsto \lambda^f(D_\delta)$ cannot be continuous in general either. For an open-dense window D of small measure, we clearly have $\lambda(D_\delta) = \lambda_{\mathrm{Dir}}$ for all $\delta > 0$. However, we claim that $\lambda_D < \lambda_{\mathrm{Dir}}$. To see this, note that u_D cannot agree with u_{Dir}, since eigenfunctions do not take on 'extra' Dirichlet boundary conditions, as was shown near Fig. 1 in [9]. Since u_{Dir} is an admissible candidate for the CHVP for λ_D, it follows from the uniqueness of the minimizer that $\lambda_{\mathrm{Dir}} > \lambda_D$. We have hereby found an example of a window whose eigenfunction is discontinuous at 'most' (in terms of measure) of the boundary.

Example 5.6. *There exists an open window D with discontinuous eigenfunction, such that still $\delta \mapsto \lambda(D_\delta)$ is right continuous.*

Proof: In a planar domain, parametrize a portion of the boundary by arclength and refer to segments on the boundary as intervals in this parameter. We will construct two decreasing sequences $x_n \searrow 0$ and $\delta_n \searrow 0$ and let $I_n :=]x_n - \delta_n, x_n + \delta_n[$. The sequences x_n and δ_n will be specified later. The window will be $D := \bigcup_{n=1}^{\infty} I_n$, and we will also define $D_N := \bigcup_{n=1}^{N} I_n$, with the eigenvalues and normalized eigenfunctions λ, λ_N, u, u_N respectively. If N is the first index such that $x_N < \delta$, then

$$D_\delta \setminus D \subset]{-\delta}, \delta[\cup \bigcup_{n=1}^{N} [x_n + \delta_n, x_n + \delta_n + \delta[\cup \bigcup_{n=1}^{N-1}]x_n - \delta_n - \delta, x_n - \delta_n] \,.$$

It follows from Prop. 5.2 that $\lambda(D_\delta) - \lambda(D) < (2N+1)\eta(\delta) < (2N+1)\eta(x_{N-1})$. Choosing the sequence (x_n) such that $(2N+1)\eta(x_{N-1}) \to 0$ as $N \to \infty$ ensures the right continuity of $\delta \mapsto \lambda(D_\delta)$.

With (x_n) thus fixed, we introduce the compact set $K := \{0\} \cup \{y_n \mid n \in \mathbb{N}\}$, where $y_n = (x_n + x_{n+1})/2$ and construct the sequence (δ_n) inductively. Let $\delta_1 = (x_1 - y_1)/2$.

Since D_1 has positive Lebesgue density at all interface points, it follows from de Giorgi's argument that the corresponding eigenfunction u_1 is Hölder continuous up to the boundary. Let $a := \inf_K u_1 > 0$ and define $a_n := (1/2 + 1/2^n)a$. We will choose δ_N in such a way that $\inf_K u_N \geq a_N$. Assume $\delta_1, \ldots, \delta_{N-1}$ have been constructed. The interval I_N and thus D_N and u_N, will depend on the choice of δ_N. But as $\delta_N \to 0$, the local de Giorgi estimates near K remain uniform, because the L^∞ estimate for u_N does not depend on the window and the interface stays away from K. Then $u_N^{(\delta_N)}$ converges weakly in $W^{1,2}(\Omega)$, strongly in $L^2(\Omega)$, and strongly in $L^2(\partial\Omega)$ by the usual compactness arguments. It also converges strongly in $W^{1,2}(\Omega)$ to u_{N-1} since $\lambda_N^{(\delta_N)} \to \lambda_{N-1}$; the convergence is uniform in a neighbourhood of K by the equicontinuity obtained from de Giorgi. Since $u_{N-1} \geq a_{N-1}$ on the compact set K, we can achieve $u_N \geq a_{N-1} - \varepsilon$ for any $\varepsilon > 0$ by making δ_N small; in particular we can achieve $u_N \geq a_N$.

It is now easy to show that u is discontinuous at 0. Indeed, as $N \to \infty$, $u_N \to u$ in the Sobolev spaces mentioned above. Again, the convergence is uniform in a neighbourhood of each single y_n. Therefore $u(y_n) \geq a/2$ for each n, whereas $u(x_n) = 0$. Hence u is discontinuous at 0. ■

We finally refer to Lemma 6.1, which gives continuity estimates under distortion of a window by means of a bi-Lipschitz homeomorphism. Due to the similarity of proofs, we conveyed it to Section 6.

5.2. On Upper Semicontinuity of Eigenfunctions. Here, we will prove semicontinuity of eigenfunctions as a consequence of a subharmonicity argument.

THEOREM 5.7. *If $\Omega \subset \mathbb{R}^2$ has a $C^{1,\alpha}$ boundary, then for any measurable window $D \subset \partial\Omega$, the eigenfunction u has an upper semicontinuous preferred representative $\tilde{u}$. If $\Omega \subset \mathbb{R}^d$ with $d > 2$, then $\tilde{u}$ is upper semicontinuous at any boundary point where the boundary is locally part of a hyperplane.*

PROOF: Let u be the solution of the CHVP (1.5) for D, the eigenvalue being $\lambda(D)$. Fix $x_0 \in \partial\Omega$. We will show that if the $\partial\Omega$ coincides with a hyperplane in some neighborhood of x_0, then the limit

$$\tilde{u}(x) := \lim_{r\to 0} \fint_{B_r(x)\cap\Omega} u(y)\, dy \tag{5.6}$$

exists for all points in this neighbourhood and defines an upper semicontinuous function. This limit agrees with the preferred representative defined in (1.7). In the special case of two dimensions, the conclusion holds assuming only that $\partial\Omega$ is of regularity $C^{1,\alpha}$ neat x_0. We note that u is always smooth in the interior of Ω, and there is nothing to show.

The basic idea is as follows: When the boundary is locally part of a hyperplane, extend u by *even* reflection, regardless of the type of boundary conditions. The nonnegative function u, thus extended, has only such discontinuities as are possible for a subharmonic distribution, and this fact is shown by means of the test function $(u - t\varphi)_+$ in the CHVP, where φ is smooth nonnegative. Subharmonicity implies upper semicontinuity according to Thm. 9.3 in [18]. For curved boundary in 2D, the Riemann mapping theorem locally provides an analog of the reflection.

Consider first the case where there exists a neighborhood V of x_0 such that $\partial\Omega \cap V$ is contained in a hyperplane. We may assume that the hyperplane is given by $x_d = 0$, that Ω lies above the hyperplane, and that V is symmetric under the reflection $(x', x_d) \mapsto (x', -x_d)$. Let φ be a smooth nonnegative function with support in V. Since $(u-t\varphi)_+$ is a legitimate candidate for the CHVP when $t \geq 0$, we have

$$\frac{A(t)}{B(t)} \geq \frac{A(0)}{B(0)}\,, \tag{5.7}$$

where

$$A(t) := \int_\Omega |\nabla(u-t\varphi)_+|^2\,, \quad B(t) := \int_\Omega |(u-t\varphi)_+|^2\,.$$

We calculate from the weak Euler equations

$$\begin{aligned} A(t) &= \int_\Omega \nabla(u-t\varphi)_+\nabla u - t\int_\Omega \nabla(u-t\varphi)_+\nabla\varphi \\ &= \lambda\int_\Omega (u-t\varphi)_+ u - t\int_{u>t\varphi}\nabla u\nabla\varphi + t^2\int_{u>t\varphi}|\nabla\varphi|^2\,, \end{aligned} \tag{5.8}$$

and expand

$$B(t) = \int_\Omega u(u-t\varphi)_+ - t\int_\Omega \varphi(u-t\varphi_+)\,. \tag{5.9}$$

Inserting (5.8) and (5.9) into (5.7) and using that $A(0)=\lambda$ and $B(0)=1$, we obtain for $t>0$:

$$\begin{aligned} 0 &\leq t^{-1}(A(t)B(0)-A(0)B(t)) \\ &= \int_{u>t\varphi}\Big[-\nabla u\nabla\varphi + \lambda u\varphi\Big] + t\int_{u>t\varphi}\Big[|\nabla\varphi|^2 - \lambda\varphi^2\Big] \end{aligned}$$

Since all integrals over sets $u>t\varphi$ converge to integrals over Ω by Lebesgue's dominated convergence theorem, we obtain for $t\to 0+$ that

$$0 \leq \int_{V\cap\Omega}\Big[-\nabla u\nabla\varphi + \lambda u\varphi\Big] \tag{5.10}$$

We now extend u by even reflection $u(x',-x_d) := u(x',x_d)$ and use (5.10) for the likewise reflected test function φ. Adding the reflected and the original (5.10), we obtain

$$0 \leq \int_V\Big[-\nabla u\nabla\varphi + \lambda u\varphi\Big] = \int_V\Big[u\Delta\varphi + \lambda u\varphi\Big] \tag{5.11}$$

where we have used that φ is C^2 and supported in V.

We have shown that $\Delta u + \lambda u$ is nonnegative in the sense of distributions. If $v := u + \frac{M\lambda}{2d}|x|^2$, where $M := \|u\|_\infty < \infty$, then $\Delta v \geq 0$ in the sense of distributions. By [18, Thm. 9.3]), v is subharmonic, that is,

$$v(x) \;\leq\; ⨍_{B_r} v \tag{5.12}$$

for almost every $x\in V$, provided $B_r(x)\subset V$. Furthermore, the preferred representative $\tilde v$ of v is upper semicontinuous, and satisfies the subharmonicity condition (5.12) for all x and r so that $B_r(x)\subset V$. Since $\tilde u$ differs from $\tilde v$ by a continuous function, it is upper semicontinuous as well. This settles the case where $\partial\Omega \cap V$ is contained in a hyperplane, in some neighbourhood of x_0.

In the case where $\Omega \subset \mathbb{R}^2$ we use complex notation. Let V be a neighborhood of $z_0 \in \partial\Omega$ such that $\partial\Omega \cap V$ is of class $C^{1.\alpha}$, and let V_+ be the intersection of V with Ω. Replacing V by a subset, we may assume that there exists a conformal map ψ from a semidisc B_+ to V_+ such that the diameter of the semidisc maps onto $V \cap \partial\Omega$. The function $\bar{u} = u \circ \psi$ on the semidisc satisfies $\Delta\bar{u} = |\psi'|^2 (\Delta u) \circ \psi$. Our argument will rely on the boundedness of $|\psi'|$ (shown below). By reflection, we can extend $\bar{u}$ into the full disc B. The extended function $\bar{u}$ is still in $W^{1,2}(B_+)$ since $\psi' \in L^\infty$; and as before, the extended function remains in $W^{1,2}(B)$. From (5.10), we conclude, using the conformal invariance of the Dirichlet integral, that

$$0 \leq \int_{B_+} \left[-\nabla(u \circ \psi)\nabla(\varphi \circ \psi) + \lambda|\psi'|^2(u \circ \psi)(\varphi \circ \psi) \right]$$

for all $0 \leq \varphi \in W^{1,2}(V_+)$ that vanish on $\Omega \cap \partial V_+$; in particular for all $\varphi := \bar{\varphi} \circ \psi^{-1}$ with $\bar{\varphi} \in C_0^2(B)$. As with (5.11), we can now conclude that $v := \bar{u} + M\lambda 2d \sup|\psi'|^2$ is subharmonic, and finish up the argument as before.

We still need to explain why $|\psi'|$ remains bounded near $\partial\Omega$: this is where the $C^{1,\alpha}$ regularity of the boundary enters. Refer to Figure 5.1. Choose U to be the intersection of a neighbourhood of $z_0 \in \partial\Omega$ with Ω, such that U is simply connected. Choose a point $p \in U$. The Green's function of U can be obtained in the form $\ln|z-p| + \xi(z)$ with ξ harmonic subject to boundary values $-\ln|z-p|$. Near z_0, this harmonic function ξ is $C^{1,\alpha}$ up to the boundary, because the boundary has this regularity there. This result follows from the Schauder estimates given in [11]; namely their Thm. 5.1 in connection with Lemma 2.1. If η is a conjugate harmonic to ξ (namely $\eta_y = \xi_x$, $\eta_x = -\xi_y$), then $w : z \mapsto (z-p)\exp[\xi(z) + i\eta(z)]$ is a conformal map of U onto a disc. (For more details, see [5, Sec. I.7].) The mapping w inherits the $C^{1,\alpha}$ regularity from ξ. With a conformal mapping μ from the disc onto a half plane, we select an appropriate semidisc B_+ from this half plane and let $\psi := (\mu \circ w)^{-1}|_{B_+}$ with $V_+ := \psi(B_+) \subset U$. ■

It is worth noting that a C^1 boundary is not sufficient for the bounded derivatives of a Riemann map, as can be seen from the map $w(z) = z \ln z$ and its inverse, which map neighbourhoods of 0 in the half planes $\operatorname{Re} z > 0$ or $\operatorname{Re} w > 0$ respectively onto domains bounded by a C^1 curves.

6. First Variation, and the Role of Singular Coefficients in Optimality. In this section, we study how the principal eigenvalue of the Laplacian with window boundary conditions changes under deformations of the window. The first lemma contains some estimates for distortions by bi-Lipschitz maps.

Lemma 6.1. *Let $\psi : \overline{\Omega}_1 \to \overline{\Omega}_2$ be a bi-Lipschitz map. Then for any window D in $\overline{\Omega}_2$, it holds*

$$\lambda(\psi^{-1}(D)) \leq \lambda(D) \sup_{\Omega_1} \rho\Big((D\psi)(D\psi)^T (\det D\psi)^{-1} \Big) \sup_{\Omega_1} (\det D\psi)$$

where ρ denotes the spectral radius. In terms of the distortion ratios

$$a(x) := \limsup_{y \to x} \frac{|\psi(y) - \psi(x)|}{|y - x|}\ , \quad b(x) := 1 / \liminf_{y \to x} \frac{|\psi(y) - \psi(x)|}{|y - x|}$$

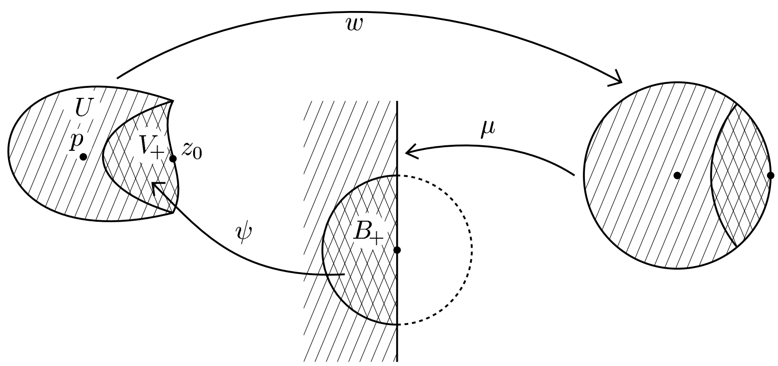


FIG. 5.1. *The Riemann mappings used in the proof of Thm 5.7*

we have the simpler (but weaker) estimates

$$\frac{\lambda(\psi^{-1}(D))}{\lambda(D)} \le \sup(ab^{d-1}) \sup(a^{d-1}/b) \le (\sup a)^{d+1} (\sup b)^{d-1} .$$

PROOF: For any two differentiable functions h_1, h_2 on Ω and any diffeomorphism ψ, we have the transformation formulas

$$\int_\Omega h_1(y)\, h_2(y)\, dy = \int_{\psi^{-1}(\Omega)} (h_1 \circ \psi)(x)\, (h_2 \circ \psi)(x)\ \det D\psi(x)\, dx \tag{6.1}$$

and

$$\int_\Omega \nabla h_1(y) \cdot \nabla h_2(y)\, dy = \int_{\psi^{-1}(\Omega)} \nabla_x (h_1 \circ \psi)(x)^T M(x) \nabla_x (h_2 \circ \psi)(x)\, dx\ , \tag{6.2}$$

where the matrix M is given by

$$M(x) = D\psi(x)^{-1}\, D\psi(x)^{-T}\ \det D\psi(x)\ . \tag{6.3}$$

Let u be the nonnegative normalized eigenfunction for window $D \subset \partial\Omega$, and take $u \circ \psi$ as a test function in the CHVP for $\psi^{-1}(D)$. The first claim follows from (6.1)–(6.3) by setting $h_1 = h_2 = u$, and using that the smallest eigenvalue of $M(x)$ is the reciprocal of the spectral radius of $M(x)^{-1}$. The distortion ratio estimates follow for $\psi \in C^1$ from $\rho(D\psi(x)^T D\psi(x)) \le a(x)^2$ and $a(x)^2/b(x)^{2(d-1)} \le \det(D\psi(x)^T D\psi(x)) \le a(x)^{2(d-1)}/b(x)^2$, as calculated in an eigenbasis of this symmetric matrix. Both estimates extend to bi-Lipschitz maps by approximation. ■

Our main result in this section describes the change of the principal eigenvalue under a diffeomorphism generated by a flow.

THEOREM 6.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$, D a window, u its normalized eigenfunction, and X a vector field of regularity $C^1(\Omega) \cap C^0(\overline{\Omega})$ that is 'parallel' to the boundary in the sense that Ω is the union of an increasing sequence of smoothly bounded subdomains Ω_δ, with $\delta \searrow 0$, such that X is tangential on $\partial\Omega_\delta$ for δ sufficiently small.*

Let ψ_t be the flow of X. Consider the dependence of the first eigenvalue λ as D changes under the flow. Then it holds:

$$\frac{d}{dt}\lambda(\psi_t(D))\bigg|_{t=0} = -2\lim_{\delta\to 0}\int_{\partial\Omega_\delta} \partial_\nu u\, L_X u\ , \tag{6.4}$$

where $L_X u$ denotes the directional derivative of u in direction X.

REMARK: The assumptions guarantee that X is tangential to the boundary of Ω at smooth boundary points, and that X vanishes in those boundary points where the boundary is not C^1. Moreover, the flow on the boundary is defined uniquely as the continuous extension of the flow in the interior.

PROOF: Let $\psi_t : x \mapsto \psi_t(x) = y\ ,\ \Omega \to \Omega$ be the bi-Lipschitz homeomorphism arising from the vector field X, i.e., $\frac{d}{dt}\psi_t(x)|_{t=0} = X(\psi_t(x))$, $\psi_0(x) = x$. Since $X \in C^1$, ψ is a C^1-diffeomorphism in the interior of Ω and satisfies a Lipschitz estimate up to the boundary.

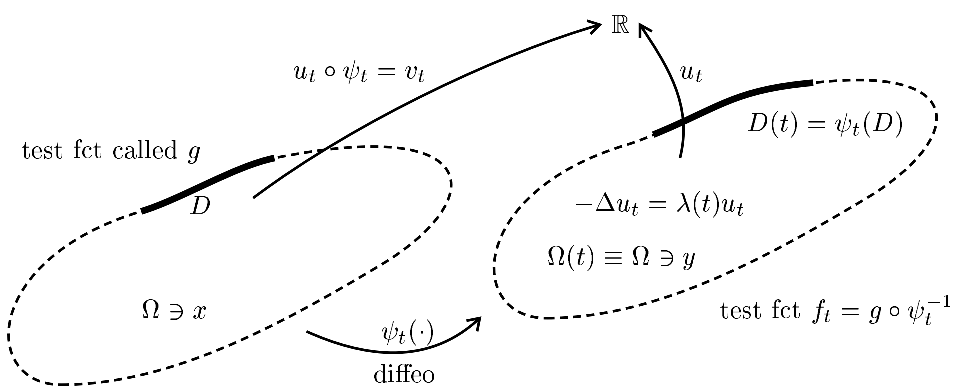


FIG. 6.1. *The mappings in the proof of Thm. 6.2*

Let $u_t(\cdot)$ and $\lambda(t)$ be the eigenfunctions and eigenvalue for $D(t) := \psi_t(D)$, and let g be a test function on Ω whose trace vanishes on D. The variation of geometry will be expressed as a variation of the operator by referring all windows back to the coordinates x.

We will denote the pullback of the eigenfunction u_t to Ω with window boundary conditions on D as $u_t \circ \psi_t =: v_t$. Similarly $f_t := g \circ \psi_t^{-1}$ the pushforward of the test function g. The weak eigenvalue equation for $u_t(\cdot)$ is

$$\int_{\Omega(t)} \nabla_y u_t(y) \cdot \nabla_y f_t(y)\, dy = \lambda(t) \int_{\Omega(t)} u_t(y) f_t(y)\, dy$$

where, in our case, $\Omega(t) \equiv \Omega$, g vanishes on D, and f_t vanishes on $D(t)$.

We now use (6.1)–(6.3) with $\psi = \psi_t$, $h_1 = u_t$, $h_2 = f_t$ and expand to first order in t. From $\frac{d}{dt}\psi_t(x) = X(\psi_t(x))$, $\psi_0(x) = x$, we obtain

$$\begin{aligned}
\psi_t(x) &= x + tX(x) + o(t) \\
D\psi_t(x)^j{}_i &= \delta_i^j + t\frac{\partial X^j}{\partial x^i} + o(t) \\
\left(D\psi_t(x)^{-1}\right)^j{}_i &= \delta_i^j - t\frac{\partial X^j}{\partial x^i} + o(t) \\
\det D\psi_t(x) &= 1 + t \operatorname{div} X + o(t)\ .
\end{aligned}$$

The estimates for the remainder terms are uniform in $x \in \Omega$. Inserting the first and last estimate into (6.1) with $\psi = \psi_t$, $h_1 = u_t$, $h_2 = f_t$ yields

$$u_t f_t\, dy = (v_t g(1 + t \operatorname{div} X) + o(t))\ dx\ ,$$

where the $o(t)$ term represents an L^1 function. Similarly, we obtain from (6.2)

$$\begin{aligned}
\nabla_y u_t(y) \cdot \nabla_y f_t(y)\, dy = \Big\{ &\nabla_x v_t(x) \cdot \nabla_x g(x) + \\
&+ t\Big((\operatorname{div} X)\nabla_x v_t \cdot \nabla_x g - \Big(\frac{\partial g}{\partial x^i}\frac{\partial v_t}{\partial x^j} + \frac{\partial v_t}{\partial x^i}\frac{\partial g}{\partial x^j}\Big)\frac{\partial X^j}{\partial x^i}\Big) + o(t)\Big\}\ dx\ ,
\end{aligned}$$

where the $o(t)$ term again represents an L^1 function. We have used the Einstein summation convention to express the sum over i and j.

If we truncate the bilinear forms by dropping the $o(t)$ terms, it is immediate that the eigenvalue will only change by $o(t)$. Since the truncated operators depend analytically on the perturbation parameter t, we may use results from Chapter VII of Kato [15] to estimate the eigenvalue up to errors of order $o(t)$. Kato's Thm. VII.4.2 and his discussion in VII §§6.2,4,5 ascertain, via spectral projections, and for any finite set of isolated eigenvalues, that the perturbation theory works as in finite dimensional spaces. In particular, a simple eigenvalue and its corresponding eigenfunction of the truncated operators depend analytically on t. We may therefore write down expansions $v_t = v_0 + tv_1 + O(t^2)$ of the eigenfunction for the truncated problem, and $\lambda(t) = \lambda_0 + t\lambda_1 + o(t)$ of the eigenvalue (for the truncated as well as for the full problem), and compare like powers of t.

Order t^0 yields

$$\int_\Omega \nabla v_0 \cdot \nabla g\, dx = \lambda_0 \int_\Omega v_0 g\, dx$$

which is just the weak Euler equation for v_0. Order t^1 yields

$$\begin{aligned}
\lambda_1 \int v_0 g\, dx + \lambda_0 \int \{v_1 g + (\operatorname{div} X) v_0 g\}\ dx = \\
= \int\Big\{ \nabla v_1 \cdot \nabla g + (\operatorname{div} X)\nabla v_0 \cdot \nabla g - \frac{\partial X^j}{\partial x^i}\Big(\frac{\partial g}{\partial x^i}\frac{\partial v_0}{\partial x^j} + \frac{\partial v_0}{\partial x^i}\frac{\partial g}{\partial x^j}\Big)\Big\}\ dx
\end{aligned}$$

These equations are valid for integration over any subdomain of Ω. We will integrate over Ω_δ, where Ω_δ runs through an increasing sequence of smoothly bounded domains compactly contained in Ω such that X is tangent to the boundary of Ω_δ. We write

$$⨌ := \int_{\Omega_\delta} \quad \text{and} \quad \oint := \int_{\partial\Omega_\delta}$$

for volume and surface integrals, respectively. Using $g = v_0$ as a test function, we obtain in first order

$$\lambda_1 \oint v_0^2 = \oint (\nabla v_1 \cdot \nabla v_0 - \lambda_0 v_0 v_1) + \oint \left\{ (\operatorname{div} X)\big(|\nabla v_0|^2 - \lambda_0 v_0^2\big) - 2\frac{\partial X^j}{\partial x^i}\frac{\partial v_0}{\partial x^i}\frac{\partial v_0}{\partial x^j} \right\} .$$

Since v_1 lies in $W^{1,2}(\Omega)$ and satisfies window boundary conditions for D, it is a valid test function in the Euler-Lagrange equation for v_0, and we conclude that the first integral vanishes as $\delta \to 0$. For the second integral, we use the identity

$$\begin{aligned} \frac{\partial X^j}{\partial x^i}\frac{\partial v_0}{\partial x^i}\frac{\partial v_0}{\partial x^j} &= \frac{\partial}{\partial x^i}\left(\frac{\partial v_0}{\partial x^i} L_X v_0\right) - X^j \frac{\partial}{\partial x^i}\left(\frac{\partial v_0}{\partial x^i}\frac{\partial v_0}{\partial x^j}\right) \\ &= \operatorname{div}(L_X v_0\, \nabla v_0) + \frac{1}{2} L_X(\lambda_0 v_0^2 - |\nabla v_0|^2) \end{aligned}$$

and Gauss' divergence theorem to compute

$$\begin{aligned} \oint \left\{ (\operatorname{div} X)\big(|\nabla v_0|^2 - \lambda_0 v_0^2\big) - 2\frac{\partial X^j}{\partial x^i}\frac{\partial v_0}{\partial x^i}\frac{\partial v_0}{\partial x^j} \right\} &= \\ &= \oint \operatorname{div}\Big((|\nabla v_0|^2 - \lambda_0 v_0^2) X\Big) - 2\operatorname{div}(\nabla v_0\, L_X v_0) \\ &= \oint (|\nabla v_0|^2 - \lambda_0 v_0^2)\, X \cdot \nu - 2\partial_\nu v_0\, L_X v_0 \, . \end{aligned}$$

The first term under the integral vanishes since X is tangential to the boundary of Ω_δ by assumption, and the claim follows as $\delta \to 0$. ■

We note that, at least formally, the integrand on the right hand side of (6.4) vanishes on both the Dirichlet and the Neumann parts of the boundary of Ω. The evaluation of the limit of the integral as $\delta \to 0$ is far from trivial in higher dimensions, but reasonably straightforward in two dimensions with nice window geometry. It amounts to the evaluation of certain singular coefficients at interface points between the Neumann and Dirichlet arts of $\partial\Omega$. It has been shown that in polygonal domains, in the neighbourhood of a corner, solutions of elliptic boundary problems lie locally in the direct sum of $W^{2,2}$ with a singular space, and in two dimensions, this singular space is one-dimensional. See, eg., Grisvard [12], in particular his Thm. 2.4.3. Indeed, functions in the singular space behave like the explicit harmonic functions $\operatorname{Re}(cz^\alpha)$ with α appropriate for the boundary conditions. In this context, it is understood that an interface point between Dirichlet and Neumann data is a corner even if (in particular if!) the geometric boundary is smooth there. As noted, corners that can be made disappear by means of the reflection principle (like the geometric corners of a rectangle) do not have a singular space. The singular coefficients (aka stress intensity coefficients) must be calculated (numerically) in practical situations. They depend on global information. For a wider background concerning singular contributions, see [7, 12, 16, 21, 23] and much other work by these authors and references given there.

In particular, the variational equation gives rise to the following

COROLLARY 6.3. *Consider a segment on the boundary of a rectangle, such that one endpoint of the segment is a corner of the rectangle, whereas the other endpoint is a point that is not a corner. Such a segment is not an optimal window, but can be improved infinitesimally by shifting in the direction that brings the corner point inside the window*

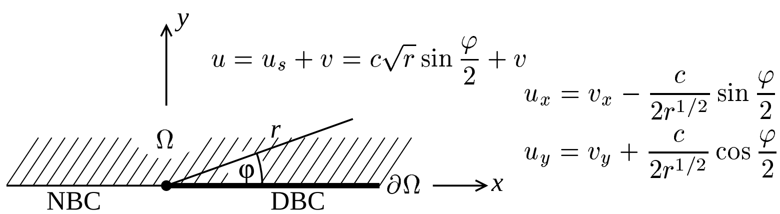
Proof: In self-explanatory notation, we refer to the windows as intervals, let $[a,b]$ be an interval with corner point b and non-corner point a; we will show (with some positive constants m, M):

$$\lambda([a+\varepsilon, b+\varepsilon]) \leq \lambda([a+\varepsilon, b]) + M\varepsilon^2 \qquad \text{and} \qquad \lambda([a+\varepsilon, b]) \leq \lambda([a,b]) - m\varepsilon\,.$$

From this the claim is immediate.

The first estimate (local near b) follows from Lemma 5.1, with G a ball of radius 2ε centered at the corner b. The eigenfunction is smooth near b, because reflection in the Neumann boundary removes the singularity: $|u| = O(\varepsilon)$ in G, and the estimate is uniform with respect to small changes at the other end a. The capacity term is bounded as $\varepsilon \to 0$, based on a radial test function $\ln_+(|x-b_1|/\varepsilon)/\ln 2$ as in the proof of Prop. 5.2.

The second estimate (local near a) follows from an evaluation of the singular boundary integral $\int_{\partial\Omega} L_X u \partial_\nu u$. In the particular case of an interface point on a straight line, the local behavior of a solution u is $u = c\sqrt{r}\sin(\varphi/2) + v$ with $v \in W^{2,2}$.



To evaluate the singular boundary integral in terms of the singular coefficient, define coordinates as in the above figure, with the boundary point a located at $(0,0)$. Let us assume that the C^1 vector field X is given by $f(x,y)\partial_x$ with the coefficient at the interface $f(0,0) = 1$. It can easily be seen that the regular function v does not contribute to the integral, nor do the mixed terms. We have

$$-2\int_{-t}^{t} L_X u\,\partial_\nu u\,dx = 2\int_{-t}^{t} \frac{\partial u_s}{\partial x}\frac{\partial u_s}{\partial y}\,dx = -\frac{c^2}{4}\int_{-t}^{t}\frac{y}{x^2+y^2}\,dx = -\frac{c^2}{2}\arctan\frac{t}{y}\,,$$

and this converges to $-\frac{c^2\pi}{4}$ as $y \to 0+$.

Finally, we estimate the singular coefficient. Choose r so small that $B_r(0)$ intersects $\partial\Omega$ in a straight line as in the above figure, with one radius (N_r) being Neumann boundary and one radius (D_r) Dirichlet boundary; let $S_r := (\partial B_r(0)) \cap \Omega$, and count φ from the Dirichlet to the Neumann boundary. Let

$$\begin{array}{ll} -\Delta h = 0 \;\text{ in } B_r(0)\cap\Omega, & \partial_\nu h = 0 \text{ on } N_r,\; h = 0 \text{ on } D_r,\; h = u \text{ on } S_r \\ -\Delta v = \lambda u \text{ in } B_r(0)\cap\Omega, & \partial_\nu v = 0 \text{ on } N_r,\; v = 0 \text{ on } D_r,\; v = 0 \text{ on } S_r \end{array}$$

Then $u = v + h$ with $v \geq 0$. Evaluation on the boundary implies that the singular coefficient of u is at least as large as the singular coefficient of h. Explicit calculation of the singular coefficient of h by means of Fourier analysis gives exactly

$$c \geq \frac{2}{\pi r^{1/2}}\int_0^{\pi} u(re^{i\varphi})\,\sin\frac{\varphi}{2}\,d\varphi > 0\,.$$

■

The above estimate of the singular coefficient is closely related to formula (2.3) in Dauge et al. [7], which actually gives the exact coefficient (in terms of u). However their formula is not designed to show non-vanishing (which relies on using the maximum principle), but is instead built on Fredholm properties. (The distinction that their formula is for a Dirichlet–Dirichlet corner, not a Dirichlet–Neumann corner, is a minor issue.)

Our argument shows that shortening a window infinitesimally at the interface decreases the eigenvalue by an amount proportional to the square of the singular coefficient at the end of the window. Moving a window amounts to shortening it at one end and lengthening it at the other end. To decrease the eigenvalue, the window should be moved in the direction of the smaller singular coefficient (i.e., towards the corner of the square, if it is already close to a corner). If the window consists of several intervals, nonlocal changes that lengthen one component at the expense of the other can also be studied in terms of the singular coefficients. Conversely, singular coefficients can be determined graphically from the slopes in Figure 2.1, for the geometric configurations depicted there.

As an immediate consequence of the role of singular coefficients, a window consisting of any number of equidistant and congruent arcs on the boundary of a circle is a critical point for the first eigenvalue. Since these arcs can now be moved independently, these are critical points of arbitrarily large index. The optimal window in a circle is known to be a single arc [9].

Limitations of our result should also be observed. The variations induced by the flow of vector fields correspond to the 'weak', C^1-small variations (as opposed to 'strong', C^0-small variations) that are exploited in the Euler-Lagrange equations of the classical Calculus of Variations. It is doubtful how significant a role such variations can play, if it comes to show, say, that a certain open-dense set of small measure is *not* an optimal window.

We have not established an analog of the fundamental lemma of the calculus of variations that would permit elimination of the vector field X. In the absence of a-priori regularity for optimal windows, such an attempt seems extremely difficult. There is however some hope to get nontrivial boundary regularity for the optimal eigenfunction by selecting vector fields constructed from the eigenfunction in some appropriate way. We plan a further investigation of this issue.

In spite of these limitations, Thm 6.2 does give some insight into the question of optimal windows, and in particular into the variation of windows with a given a-priori regularity.

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