Cases of Equality in the Riesz Rearrangement Inequality

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Contents

Summary

Rearrangement inequalities are classical tools for solving variational problems with symmetry. Here we will deal with the *spherically decreasing rearrangement*, which replaces a given function with a function that is rotationally symmetric and decreases with increasing radius, while preserving certain properties such as the norm of the function in some function spaces. In particular, it replaces the characteristic function of a measurable set with the characteristic function of the centered ball of equal measure. Another example is the Steiner symmetrization procedure, which will also be used in this thesis. A functional is said to satisfy a *rearrangement inequality*, if it always decreases or always increases under rearrangement. Such rearrangement inequalities can be used to show the existence of a rotationally symmetric solution of a given rotationally symmetric variational problem.

Many geometric inequalities can be seen as rearrangement inequalities. For example the isoperimetric inequality says that the surface of a body may only decrease, if the set is replaced by its spherical rearrangement, the centered ball of equal measure. The Brunn-Minkowski inequality says that the volume of the pointwise sum of two sets will not increase if the two sets are replaced by their spherical rearrangements. Similarly, the Faber-Krahn inequality says that the first Eigenvalue of the Laplacian with Dirichlet boundary conditions decreases under spherical rearrangement. The Riesz rearrangement inequality, which we study in this thesis, belongs to this family of inequalities with geometric meaning.

Classical applications of rearrangement techniques include variational problems arising from functional inequalities such as the Sobolev, the Hardy-Littlewood-Sobolev, and the Young inequalities, as well from as physical problems, such as the shape of a fluid body in equilibrium, the shape of the conducting body with smallest capacitance for a given volume, the shape of the drum with lowest fundamental frequency, and the symmetry properties of ground states of Hamiltonians with spherically symmetric potentials.

It is often important to identify the optimizers in such variational problems. For instance, the most direct way to prove a functional inequality with the best possible constant may be to evaluate the functional at an optimizer. A complete list of the solutions is clearly of interest in physical problems as well.

In these situations it is useful to know all the cases of equality in the rearrangement inequality that was applied in the solution of the problem. In other words, while a rearrangement inequality can be used to construct symmetric solutions of symmetric variational problems, a sharp version will be useful for questions of uniqueness.

Finally, simple curiosity leads us to ask when equality occurs in a given rearrangement inequality. One might suspect that optimizers of a rearrangement inequality should not change much under rearrangement. This is, however, not generally true. It is true for the isoperimetric inequality and the Faber-Krahn inequality, where the optimizers are essentially balls (that need not be centered). It is not true for the Brunn-Minkowski inequality, where every pair of homothetic convex sets produces equality.

The Riesz rearrangement inequality, which is the subject of this thesis, states that the functional on the left hand side of Young's inequality may only increase under spherical rearrangement of the three functions.

Our main result is a discussion of the cases of equality in the Riesz rearrangement inequality. In some important special cases, we show that equality can occur only for triples of functions that are equivalent to triples of symmetrically decreasing functions under the symmetries of the functional. We show that in other cases, there are many optimizers that are not of this form. We combine these results to a general characterization of the optimizers of the inequality.

Our other results are some related inequalities in sharp form, and an application to the weak Young inequality.

In an attempt to make this thesis as coherent and self-contained as possible, we briefly describe the history of the inequality, its various proofs, its applications, some generalizations, and its relation to the well-known Brunn-Minkowski inequality. In particular, we will present a proof of the inequality, and take care to develop the classical tools.

Chapter 1

Introduction

1.1 The main results

The Riesz rearrangement inequality states that the functional

$$\mathcal{I}(f,g,h) := \iint f(y)g(x-y)h(x)\,dy\,dx \tag{1.1}$$

(where f, g, and h are nonnegative real-valued functions on $\mathbf{R}^n$) can only increase under spherical rearrangement.

A function is called **spherically decreasing**, if it can be written as a nonincreasing function of the modulus. The **spherically decreasing rearrangement**, f^*, of a nonnegative measurable function f is the spherically decreasing function equimeasurable to f. It can be defined by

$$f^*(x) = \sup\Big\{s > 0 \;\Big|\; \mu(\mathcal{N}_s(f)) \geq \omega_n\,|x|^n\Big\}\,, \tag{1.2}$$

where

$$\mathcal{N}_s(f) := \Big\{x \in \mathbf{R}^n \;\Big|\; f(x) > s\Big\}$$

is the level set of f at height s, and ω_n denotes the measure of the unit ball in $\mathbf{R}^n$. That is, the level sets of f^* are the centered balls of the measure as the corresponding level sets of f. This definition makes sense if all level sets corresponding to positive values of f have finite measure, for example, if f is in L^p for some $p < \infty$. Then the Riesz rearrangement inequality says that

$$\mathcal{I}(f,g,h) \leq \mathcal{I}(f^*,g^*,h^*) \tag{1.3}$$

for any triple (f,g,h) of nonnegative measurable functions on $\mathbf{R}^n$ whose spherically decreasing rearrangements (f^*,g^*,h^*) can be defined by (1.2).

The cases of equality in inequality (1.3) are the main subject of this thesis. A triple of functions that satisfies (1.3) with equality will be called an optimizing triple, or **optimizer**, of the inequality. There are many optimizers of inequality (1.3). One reason this occurs is that the functional $\mathcal{I}$ is invariant under a large group of affine transformations: For any linear map, $\mathcal{L}$, of determinant ± 1, and vectors a, b, and $c = a+b$ in $\mathbf{R}^n$, and any three real-valued functions f, g, and h on $\mathbf{R}^n$, the functional satisfies

$$\mathcal{I}(f,g,h) = \iint f(\mathcal{L}^{-1}y - a)g(\mathcal{L}^{-1}(x-y) - b)h(\mathcal{L}^{-1}x - c)\,dydx\ . \tag{1.4}$$

The permutation symmetry

$$\mathcal{I}(f,g,h) = \mathcal{I}(g,f,h) = \mathcal{I}(h,g^-,f) \tag{1.5}$$

will also be useful. Here g^- denotes the function defined by $g^-(x) := g(-x)$. Clearly, any triple of functions that is equivalent to a triple of spherically decreasing functions under these symmetries is an optimizer.

There is a second reason to expect many optimizers. Consider the case when f and g have compact support. Then the convolution $f * g$ has compact support. If h is the characteristic function of a set that contains the support of $f * g$, then there is equality in (1.3) regardless of the shape of that set.

We will show that the non-uniqueness of the optimizers is due only to these two reasons. In particular, we will give conditions on the three functions that guarantee that any optimizing triple either consists of spherically decreasing functions, or is equivalent to such a triple under the symmetries given in (1.4).

The main result, which is stated in the following theorem, goes back to a conjecture by Lieb and Loss [21]. It describes all cases of equality in (1.3) where f, g, and h are characteristic functions of measurable sets. The spherical rearrangement, A^*, of a measurable set A of finite measure, is defined to be the open centered ball of the same measure as A. With this definition, $\mathcal{X}_{A^*} = (\mathcal{X}_A)^*$, and all level sets of a measurable function f and its spherical rearrangement f^* are related by $\mathcal{N}_s(f^*) = (\mathcal{N}_s(f))^*$ (see Section 3.1). Define the functional

$$\mathcal{J}(A,B,C) := \mathcal{I}(\mathcal{X}_A, \mathcal{X}_B, \mathcal{X}_C) = \iint \mathcal{X}_A(y)\mathcal{X}_B(x-y)\mathcal{X}_C(x)\,dy\,dx\ . \qquad (1.6)$$

By equation (1.4), the functional $\mathcal{J}$ is symmetric under the following affine transformations: For any linear transformation, $\mathcal{L}$, of determinant ± 1 on $\mathbf{R}^n$, and vectors a, b, and $c = a + b$, the functional satisfies

$$\mathcal{J}(a+\mathcal{L}A, b+\mathcal{L}B, c+\mathcal{L}C) = \mathcal{J}(A,B,C)\ . \qquad (1.7)$$

Moreover, by equation (1.5),

$$\mathcal{J}(A,B,C) = \mathcal{J}(B,A,C) = \mathcal{J}(C,-B,A)\ , \qquad (1.8)$$

where $-B := \left\{-x \mid x \in B\right\}$.

In the statement of the following theorem, $A + B$ will denote the Minkowski sum of two sets A and B in $\mathbf{R}^n$. It is defined by

$$A + B \;=\; \left\{a + b \mid a \in A, b \in B\right\}.$$

Theorem 1 (Sharp Riesz rearrangement inequality for characteristic functions) *Let A, B and C be measurable sets of finite measure in $\mathbf{R}^n$, and denote by A^*, B^*, and C^* the centered balls of equal measure as A, B, and C, respectively. The following inequality holds for the functional $\mathcal{J}$ defined by (1.6).*

$$\mathcal{J}(A,B,C) \le \mathcal{J}(A^*,B^*,C^*) . \tag{1.9}$$

Assume that all three sets have positive measure, and denote by α, β, and γ the radii of the balls A^*, B^*, and C^*. If $|\alpha-\beta| < \gamma < \alpha+\beta$, then there is equality in (1.9) if and only if, up to sets of measure zero,

$$A = a + \alpha E , \quad B = b + \beta E , \quad C = c + \gamma E , \tag{1.10}$$

where E is a centered ellipsoid, and a, b, and $c = a + b$ are vectors in $\mathbf{R}^n$.

Otherwise, permute the three sets so that $\gamma \ge \alpha + \beta$, using (1.8). Then there is equality in (1.9) if and only if A, B, and C can be changed by sets of measure zero so that

$$C \supset A + B , \tag{1.11}$$

In particular, for $\gamma = \alpha + \beta$, there is equality in (1.9) if and only if, up to sets of measure zero,

$$A = a + \alpha M , \quad B = b + \beta M , \quad C = c + \gamma M = A + B , \tag{1.12}$$

where $M \subset \mathbf{R}^n$ is convex and open, and a, b, and $c = a + b$ are vectors in $\mathbf{R}^n$.

Definition Three positive numbers α, β, γ satisfy the **strict triangle inequality** if they could form the lengths of the sides of a nondegenerate triangle. That is,

$$|\alpha - \beta| < \gamma < \alpha + \beta . \tag{1.13}$$

Note that formula (1.13) is symmetric under permutation of α, β, and γ. We will say that the strict triangle inequality holds between three sets A, B, and C in $\mathbf{R}^n$, if it holds between the radii

$$\alpha = \left(\frac{\mu(A)}{\omega_n}\right)^{1/n}, \quad \beta = \left(\frac{\mu(B)}{\omega_n}\right)^{1/n}, \quad \gamma = \left(\frac{\mu(C)}{\omega_n}\right)^{1/n} \tag{1.14}$$

of the balls A^*, B^*, and C^*.

The main statement of Theorem 1 is conclusion (1.10), which says that any optimizing triple such that the radii of the rearrangements satisfy the strict triangle inequality is equivalent to a triple of centered balls under the symmetries of the functional $\mathcal{J}$. That is, there exist a linear map of determinant one, $\mathcal{L}$, and vectors a, b, and $c = a + b$ such that

$$A = a + \mathcal{L}A^* , \quad B = b + \mathcal{L}B^* , \quad C = c + \mathcal{L}C^* . \tag{1.15}$$

The set of optimizers for which the strict triangle inequality holds is as small as it could be.

The strict triangle inequality is violated, if one of the three sets is too large or too small relative to the other two. We have already discussed why one should expect many optimizers in this situation. This is confirmed by conclusion (1.11).

Conclusion (1.12) of Theorem 1 says that optimizers whose radii are in the critical size relation $\gamma = \alpha + \beta$ is (up to sets of measure zero) of the form $(A, B, C = A + B)$, where A and B produce equality in the Brunn-Minkowski inequality.

The full inequality (1.3) follows with classical methods from Theorem 1:

Theorem 2 (Riesz rearrangement inequality) *Inequality (1.3) holds for any three nonnegative measurable functions f, g, and h on $\mathbf{R}^n$ with spherically decreasing rearrangements f^*, g^*, and h^*.*

There is equality in inequality (1.3), if and only if for (Lebesgue) almost all triples (r, s, t) of positive numbers, the level sets $\mathcal{N}_r(f)$, $\mathcal{N}_s(g)$, $\mathcal{N}_t(h)$ produce equality in the inequality (1.9).

Thus, Theorem 1 determines all cases of equality in inequality (1.3). Unfortunately, since the condition in Theorem 2 involves the sizes and shapes of all level sets, it is hard to check in general. However, more can be said if additional information is available about the three functions f, g, and h.

We will say that a nonnegative real-valued function f on $\mathbf{R}^n$ is **strictly spherically decreasing,** if it can be written as a strictly decreasing function of the modulus. In many applications, the middle function is a given strictly spherically decreasing function, such as $g(x-y) = |x-y|^{-\lambda}$, or the heat kernel. The following result is due to Lieb [19].

Theorem 3 (Cases of equality, fixed middle function) (Lieb [19]) *Let f and h be nonnegative measurable functions with spherically decreasing rearrangements f^* and h^*, and let g be a strictly spherically decreasing function. Assume also that $\mathcal{I}(f, g, h)$ as defined by (1.1) is finite, and that none of the functions f, g, and h is zero almost everywhere.*

Then there is equality in inequality (1.3) if and only if f and h differ from f^* and h^* only by a common translation, in other words, if there exists a vector a in $\mathbf{R}^n$ so that

$$f(x) = f^*(x-a), \quad h(x) = h^*(x-a) \quad a.e. \tag{1.16}$$

Theorem 3 says that under the given assumptions, all optimizers are equivalent to rotationally symmetric ones under the symmetries of the functional which preserve the middle function. Similar statements holds if at least two of the three functions have no flat spots.

Theorem 4 (Cases of equality, two strictly decreasing rearrangements) *Let f, g, and h be three nonnegative measurable functions with spherically decreasing rearrangements f^*, g^*, and h^*. Assume that $\mathcal{I}(f,g,h)$ is finite, and that f, g, and h are not almost everywhere zero.*

If at least two of the three rearrangements f^*, g^*, and h^*, are strictly spherically decreasing, then there is equality in the Riesz rearrangement inequality (1.3) if and only if the triple (f,g,h) is equivalent to (f^*,g^*,h^*) under the symmetries (1.5). That is, there exist a linear map, $\mathcal{L}$, of determinant one, and vectors a, b, and $c = a + b$ such that

$$f(x) = f^*\left(\mathcal{L}^{-1}x - a\right), \quad g(x) = g^*\left(\mathcal{L}^{-1}x - b\right), \quad h(x) = h^*\left(\mathcal{L}^{-1}x - c\right) \qquad a.e. \tag{1.17}$$

Again, the set of optimizers is as small as possible.

The following example shows that the hypothesis that the rearrangements of two of the functions are strictly decreasing is essential. Let

$$\begin{aligned} f(x) &= \mathcal{X}_{(-2,2)}, \\ g(x) &= \mathcal{X}_{(-1,1)}, \\ h(x) &= \begin{cases} 0 & \text{if } x < -3 \\ 2 + x/2 & \text{if } -3 \le x < -1 \\ 2 - x/2 & \text{if } -1 \le x < 3 \\ e^{3-x}/2 & \text{if } x \ge 3. \end{cases} \end{aligned}$$

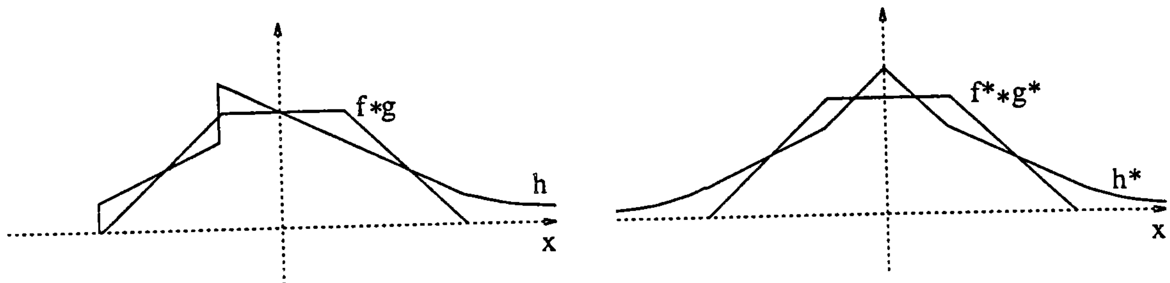

Figure 1.1: Example of an optimizing triple

Then f and g are already spherically decreasing, and h coincides with its rearrangement

$$h^*(x) = \begin{cases} 5/2 - |x| & \text{if } |x| < 1 \\ 2 - |x|/2 & \text{if } -1 \leq |x| < 3 \\ e^{6-2|x|}/2 & \text{if } |x| \geq 3 \end{cases}$$

on the union of the intervals $[-3,-1]$ and $[1,3]$. Since the convolution $f * g$ is locally constant on the complement of this union (see Figure (1.1), it follows that

$$\mathcal{I}(f,g,h) = \mathcal{I}(f^*,g^*,h^*) \, .$$

Note that h^* is strictly decreasing, but h and h^* are not related by a linear transformation.

Various consequences of Theorems 1–4 will be discussed in the last chapter of the thesis. In particular, as an application of Theorem 4, we will show that all the cases of equality in the the weak Young inequality

$$\left| \iint f(y)g(x-y)h(x)\, dy dx \right| \leq C(p,\lambda,n) \, \|f\|_p \, \|g\|_{w,q} \, \|h\|_r$$

can be obtained from the optimizers of a corresponding Hardy-Littlewood-Sobolev inequality.

1.2 The history of the inequality

The history of the inequality begins with Poincaré's work on the problem of identifying the possible shapes of a fluid body in equilibrium [11, 12]. The equilibrium shapes of the body are the critical points of the energy functional. If the total angular momentum of the body vanishes, then the energy functional is of the form (1.1) where $f = h$ is the characteristic function of the body, and the middle function $g(x - y) = |x - y|^{-1}$ is the potential of the gravitational attraction between two mass points located at x and y. Poincaré showed that, under some smoothness assumptions, the body assumes the shape of a ball. His main tools were potential theory, using the special properties of the kernel $|x - y|^{-1}$, and the isoperimetric inequality. Although he quoted Steiner's work on the isoperimetric inequality, his proof contains no concepts of rearrangement.

In the introduction of [11] he referred to an earlier proof by Liapunoff [17], which apparently did not use the isoperimetric inequality. He pointed out several shortcomings of Liapunoff's proof, which seems to have been more complicated, and possibly incomplete, as it did not show that minima of the energy functional existed.

All later proofs of inequality (1.3) (with or without requiring the middle function to be a fixed symmetrically decreasing integral kernel) are based on rearrangement ideas developed by Steiner for the isoperimetric inequality. Using symmetrization along lower-dimensional subspaces, known as **Steiner symmetrization**, the proof of the inequality can be split into two parts. The first part is a proof of the inequality in one dimension. The second part is the generalization to higher dimensions, which is typically done by induction. Applying the inequality to lower-dimensional cross sections and integrating shows that the left hand side satisfies an analogous inequality for Steiner symmetrization, while the right hand side stays the same.

To justify this approach, it is necessary to approximate the spherical rearrangement with repeated Steiner symmetrizations. It is a well known problem with Steiner's proofs of the isoperimetric inequality that he did not show that such an approximation procedure converges, thus leaving the possibility open that the functional in question may not attain its extremal value. The same problem occurs in some of the proofs of the Riesz rearrangement inequality discussed below.

The second universal tool is the '**layer-cake principle**'. Any nonnegative measurable function can be represented as an integral of the characteristic functions of its level sets. With this representation, inequality (1.3) follows easily from the corresponding inequality (1.9) for the level sets (see [25, 16, 19, 7]). We will discuss this in connection with the proof of Theorem 2 in the first section of Chapter 3.

Blaschke seems to have been the first to consider inequality (1.3) as a geometric inequality, and to use Steiner symmetrization to construct a proof [4]. He showed that the inequality holds if $f = h$ is the characteristic function of a convex set, and the middle function is spherically decreasing and concave. Since the function $1/x$ is not concave, his theorem does not apply to the problem considered by Liapunoff and Poincaré. With essentially the same methods, but a different argument for the one-dimensional case, Carleman showed that the inequality holds for any symmetrically decreasing middle function [8].

Both Blaschke and Carleman used techniques developed by Groß [14] for the isoperimetric inequality to extend the inequality to non-convex sets. Groß constructs Steiner symmetrization and proves the convergence result discussed above for closed sets whose boundary is described by curves. The discussion of the cases of equality in [4, 8] was not complete, as it was only shown that convex optimizers have to be balls, but not that all optimizers have to be convex sets. Finally, Lichtenstein [18] extended the inequality with the layer-cake principle to non-homogeneous fluids, that is, to the case where $f = h$ is a nonnegative function that need not be a characteristic function of a set.

Riesz first stated the full inequality, where all three functions are allowed to vary, in one dimension [25]. It is easy to read off the cases of equality from his proof. This proof, together with another proof of the inequality in one dimension can be found in Hardy, Littlewood, and Pólya [16]. In the second proof, inequality (1.9) is approximated with a discrete analog, which is, however, not easier to prove than inequality (1.9).

Riesz claimed that the inequality can be generalized easily to several dimensions [25]. He may have been thinking of approximating the spherical rearrangement with a sequence of Steiner symmetrizations as suggested above. Following work by Lusternik on the Brunn-Minkowski inequality [22], Sobolev took this approach for the Riesz inequality [26, 27]. Both proofs are incomplete, since it is not shown that the constructed sequence of Steiner symmetrizations converges to the spherical rearrangement in some suitable metric (Hausdorff metric for the Brunn-Minkowski inequality, and symmetric difference for the Riesz inequality). Sobolev was interested in inequality (1.3) because of an application to an inequality of the type of the Hardy-Littlewood-Sobolev inequality, where f and h are nonnegative measurable functions in some L^p-spaces, and g is the fixed integral kernel $|x-y|^{-\lambda}$.

Incidentally, Hadwiger and Ohmann proved the Brunn-Minkowski inequality and discussed the cases of equality for measurable sets with more direct geometric methods that do not involve Schwarz or Steiner symmetrization, and thus do not require such a convergence result.

Inequality (1.9) is closely related to the Brunn-Minkowski inequality. We have already mentioned that the optimizing triples of inequality (1.9) named in conclusion (1.12) of Theorem 1 are triples of the form $(A, B, A+B)$, where A and B produce equality in the Brunn-Minkowski inequality. The connection to the Brunn-Minkowski inequality is lost, if the middle function is restricted to be strictly spherically decreasing.

The first complete proof of inequality (1.3) is due to Brascamp, Lieb, and Luttinger [7]. They use the two standard tools, the layer-cake principle and Steiner symmetrization, to reduce the problem to the case where the three functions are characteristic functions of measurable sets in one dimension. They prove the convergence result that is so crucial for proof involving Steiner symmetrization. Moreover, they give a new proof of the inequality in one dimension: A deformation is constructed which gradually transforms the three sets into their rearrangements. To show that the functional cannot decrease under this deformation, the Brunn-Minkowski inequality is called upon in a surprising way, which is apparently not related to the aspects discussed in Section 2.1 of this thesis. This proof gives the more general inequality

$$\int \cdots \int \prod_{i=1}^{m} f_i \left(\sum_{j=1}^{k} a_{ij} x_j \right) dx_1 \ldots dx_k \leq \int \cdots \int \prod_{i=1}^{m} f_i^* \left(\sum_{j=1}^{k} a_{ij} x_j \right) dx_1 \ldots dx_k \,. \quad (1.18)$$

The variables x_i are vectors in $\mathbf{R}^n$, and (a_{ij}) is a $m \times k$ matrix of scalars. Inequality (1.3) corresponds to the special case

$$m = 3\ , \quad k = 2\ , \quad (a_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$$

The full inequality (1.3), where all three functions are allowed to vary, has been applied to variants of Young's inequality, which relate the value of the functional $\mathcal{I}$ to the norms of f, g, and h in some function spaces. Beckner [3] proved a sharp generalized version of Young's inequality, using the analog of inequality (1.3) for multiple convolutions (see Chapter 3, Section 4). Brascamp and Lieb [5] proved a more general version of Young's inequality and determined best constants, based on (1.18). Since sharp versions of inequalities (1.3) and (1.18) were not available, both sources discussed the cases of equality with different methods.

A similar inequality on spheres was proven by Luttinger [23] in one dimension, and by Baernstein and Taylor in connection with subharmonic maps [2]. It says that inequality (1.3) holds also, when f and h are nonnegative measurable functions on a sphere, and g is a fixed symmetrically decreasing integral kernel. The proof relies on a compression procedure that never decreases the value of the functional. Different from Steiner symmetrization, this compression procedure uses reflections at hyperplanes. A suitable sequence of such compressions will converge to the spherical rearrangement. Although it is not mentioned in [2], it is easy to identify the cases of equality from this proof. The same proof applies to inequality (1.3) in $\mathbf{R}^n$ in case the middle function is strictly spherically decreasing.

In [19], Lieb used inequality (1.3) to find solutions of a non-convex variational problem with rotational symmetry. He proved and used the sharp version given by

Theorem 3. He also showed that the classical rearrangement inequality for the L^2-norm of the gradient can be obtained with a limiting argument from (1.3) with the heat kernel as the middle function. His proof of Theorem 3 involves, as usual, the layer-cake principle and an induction over the dimension. The proof in one dimension is interesting, because it uses no approximation arguments, but direct geometric considerations for measurable sets. In particular, no regularity is assumed for the optimizers *a priori*. The generalization to higher dimensions is somewhat simpler, and the regularity problem is less serious than for the full inequality, because the one-dimensional case can be applied to *all* cross sections.

Apparently unaware of [19], Friedman and McLeod gave another proof of this result in [13].

Returning to the problem studied by Poincaré, Auchmuty [1] used the sharp form of the inequality with a fixed middle function to show that there are uniquely determined axisymmetric equilibrium shapes for rotating fluid bodies. Theorem 4 also plays a central role in Lieb's work on sharp constants in the Hardy-Littlewood-Sobolev inequalities [20] Similar arguments are needed to apply the *'Competing Symmetries'* method of Carlen and Loss [9, 10] to functional of the form (1.1).

1.3 An outline of the thesis

We will use the classical instruments discussed in the previous section to prove the main results. The entire second chapter is dedicated to the proof of the Riesz rearrangement inequality for characteristic functions, and the identification of its cases of equality (Theorem 1).

In the less interesting case, when the strict triangle inequality does not hold, Theorem 1 follows immediately from the Brunn-Minkowski inequality in the form proved by Hadwiger and Ohman [15].

In case the strict triangle inequality holds, we prove Theorem 1 by induction over the dimension. Key to the proof in one dimension is the observation that Riesz' original proof can be adapted to general measurable sets without approximation. As mentioned above, the cases of equality can be easily identified with this proof.

In the inductive step, we use the sequence of partial symmetrizations suggested by Lusternik [22] and Sobolev [26, 27]. To prove the inequality, we will show that this sequence converges to the spherical rearrangement. This is an example of the 'Competing symmetries' technique developed by Carlen and Loss [9] (the two operations are a rotation, and a pair of partial symmetrizations). The convergence proof is simpler, but otherwise similar to the proof by Brascamp, Lieb, and Luttinger [7], because we use a stronger partial symmetrization operation.

The discussion of the cases of equality is a new result. The proof takes up the major part of Chapter 2. The key points are a new technique of regularization by partial symmetrization, and the identification of optimizers as triple of ellipsoids, using local properties.

In the third chapter, we discuss some corollaries of Theorem 1. We prove Theorems 2–4 with the layer-cake principle. As an application of Theorem 4, we discuss the cases of equality in the weak Young inequality. We also prove an equivalent formulation, which we will call the dual, of inequality (1.3). We translate Theorems 1–4 into equivalent dual versions.

Finally, we generalize Theorems 1–4 to the inequality for multiple convolutions considered by Beckner [3]. In Theorem 1″, which is the analogue of Theorem 1, the strict triangle inequality is replaced by a size relation, which says that the radii of the rearranged sets could form the sides of a polygon with interior.

We are convinced that Theorems 1–4 can be generalized to include the inequality (1.18) proven by Brascamp, Lieb, and Luttinger in [7]. The strict triangle inequality in Theorem 1 has to be replaced by a different size condition which depends on the matrix (a_{ij}). However, Riesz' proof of inequality (1.9) does not work for inequality (1.18), and the proof by Brascamp, Lieb, and Luttinger uses approximation arguments in such a way that it seems impossible to read off the cases of equality.

Chapter 2

Proof of Theorem 1

The proof of Theorem 1 falls into several parts. In case one of the three sets is large enough compared to the other two, we deduce Theorem 1 from the result by Hadwiger and Ohmann [15] on the Brunn-Minkowski inequality and its cases of equality for measurable sets in $\mathbf{R}^n$ that was mentioned in the introduction.

In the most interesting case, when the strict triangle inequality holds between the sizes of the three sets, we will prove Theorem 1 by induction over the dimension. For the base case, dimension $n = 1$, we will formalize Riesz' proof and also use it to identify the cases of equality.

The idea for the inductive step is to use the fact that inequality (1.9) in $\mathbf{R}^{n+1}$ is just the inequality in a lower-dimensional space integrated over the cross sections. Hence, by the inductive assumption, symmetrization along lower-dimensional subspaces may only increase the value of the functional (Lemma 3). We already showed that the functional is invariant under linear transformations. We will combine a pair of symmetrizations along subspaces with a linear transformation to yield a basic symmetrization operation. We will show that under repeated applications of this operation, any measurable set approaches its spherical rearrangement, and the functional converges monotonically to its maximal value (Lemma 6). This will complete the proof of the inequality.

The cases of equality are a more difficult question. It is easy to show that almost all triples of n-dimensional cross sections of the three sets (at heights satisfying a certain relation) are optimizers of inequality (1.9) in $\mathbf{R}^n$ (Lemma 4). Since the functional $\mathcal{J}$ is invariant under rotations, Theorem 1 can be applied to intersections of the three sets with hyperplanes in arbitrary directions.

There are two major difficulties. First, since Theorem 1 only makes a strong statement if the strict triangle inequality holds, it is crucial to find sufficiently many triples of cross sections that satisfy this inequality. Since not much can be said about the measures of cross sections of general measurable sets, some regularity is needed.

Second, the information about the cross sections obtained from the inductive assumption has to be pieced together to draw conclusions about the entire sets. The three sets have to be identified as ellipsoids by local properties. Eventually, we will derive and solve a differential equation for the boundaries of the three sets of an optimizing triple. Additional regularity will be needed to do this.

It turns out that the symmetrization procedure constructed for the proof of the inequality can also be used for regularization. We will show that it has the following properties.

(R1) It transforms optimizers of (1.9) into optimizers of (1.9).

(R2) It is rather simple, so that we can show that optimizers that satisfy the conclusions of Theorem 1 after regularization must have satisfied them to begin with.

(R3) It gives enough regularity so that we can find many cross sections satisfying the strict triangle inequality .

In Lemma 4 we will prove that regularization has the property (R1). In Lemma 11, we will show that it has the property (R2). In the proof of Lemma 11, we will need the inductive assumption in form of Lemma 9. Property (R3) follows from Lemmas 7 and 8.

It follows from properties (R1) and (R2), that we need to prove conclusion (1.10) of Theorem 1 only for optimizing triples that have been regularized. For such triples, we find cross sections that satisfy the strict triangle inequality, using Lemmas 1, 2 and 8. We apply the inductive hypothesis in form of Lemma 9, and derive some properties of these cross sections. From these properties, we identify the sets as ellipsoids that differ only by scale factors, using Lemmas 12 and 13. This will complete the proof of Theorem 1.

2.1 Some simple observations

In this section, we deal with those parts of the proof of Theorem 1 that do not require the inductive hypothesis. In the first subsection, we discuss how Theorem 1 relates to the Brunn-Minkowski inequality. We describe the result by Hadwiger and Ohmann [15] on the Brunn-Minkowski inequality and its cases of equality for measurable sets, and use it to prove prove conclusions (1.10) and (1.11) of Theorem 1 in the cases where the strict triangle inequality does not hold.

In the second subsection, we discuss a simple property of optimizers, which is easy to prove, but plays a central role in the inductive step: It will be used to find at least some cross sections of a given optimizing triple satisfying the strict triangle inequality, for which the strict triangle inequality holds as well.

2.1.1 The Brunn-Minkowski inequality

Consider a triple of measurable sets of finite positive measure in $\mathbf{R}^n$. Since neither the functional $\mathcal{J}$, nor the spherical rearrangements A^*, B^*, and C^*, nor the radii α, β, and γ change, if A, B, and C are changed by sets of measure zero, it seems natural to make the following definition. Two measurable sets will be called **equivalent** if they differ by a set of measure zero, in other words, if their characteristic functions represent the same element of L^1.

To avoid dealing with equivalence classes of measurable sets, we will often choose the set of Lebesgue points of a measurable set as a particular representative of its equivalence class. We will call a point a **Lebesgue point** of a measurable set A in $\mathbf{R}^n$, if

$$\lim_{\rho\downarrow 0} \frac{\mu(A \cap B_\rho(a))}{\omega_n \rho^n} = 1 ,$$

where $B_\rho(a)$ is the ball of radius ρ centered at a, and ω_n is the measure of the unit ball in $\mathbf{R}^n$. It is well known that any measurable set differs by a set of measure zero from the set of its Lebesgue points. Any point in the interior of a set is a Lebesgue point, and any Lebesgue point of a set is contained in its closure.

Lemma 1 (The support of the convolution) *Let A and B be two measurable sets A and B in $\mathbf{R}^n$, and denote by $\mathcal{X}_A$ and $\mathcal{X}_B$ their characteristic functions. If A and B consist exactly of their Lebesgue points, then*

$$\left\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) > 0\right\} = A + B .$$

Proof "$\supset$": Suppose that at some point x, the convolution takes a positive value. Then, by definition of the convolution,

$$\mu((x-A)\cap B) = \int \mathcal{X}_A(x-y)\mathcal{X}_B(y)\,dy = \mathcal{X}_A * \mathcal{X}_B(x) > 0 \ ,$$

that is, the intersection of $x-A$ with B has positive measure. Any Lebesgue point of this intersection is of the form $b = x - a$ for some Lebesgue points $a \in A$, $b \in B$.

"$\subset$": Any point of the form $x = a + b \in A + B$ is a Lebesgue point of both $a + B$ and $b + A$, so it is a Lebesgue point of the intersection. Consequently, $b = x - a$ is a Lebesgue point of $(x-A)\cap B$. Hence this intersection has positive measure, and the convolution takes a positive value at x. ■

Lemma 1 establishes a connection between Theorem 1 and the Brunn-Minkowski inequality. The Brunn-Minkowski inequality for measurable sets in $\mathbf{R}^n$ as proved by Hadwiger and Ohmann in [15] states that the measure of the pointwise sum of any two nonempty measurable sets A and B in $\mathbf{R}^n$ satisfies the inequality

$$\mu(A+B)^{1/n} \geq \mu(A)^{1/n} + \mu(B)^{1/n} \ . \tag{2.1}$$

There is equality if and only if either A is a point, or B is a point, or A and B are of the form

$$A = \bar{A} \setminus N_A, \quad B = \bar{B} \setminus N_B \tag{2.2}$$

where N_A and N_B are sets of measure zero, and $\bar{A}$ and $\bar{B}$ are convex sets that can be mapped to each other by scaling and translation. Inequality (2.1) for convex sets is due to Brunn. The discussion of the cases of equality, assuming that the two sets are convex, is due to Minkowski [24].

In view of Lemma 1, we can define the **essential sum** of two measurable sets by

$$A +_{(ess.)} B := \left\{ x \mid \mathcal{X}_A * \mathcal{X}_B(x) > 0 \right\} .$$

Clearly, the essential sum is open. If A and B are changed by sets of measure zero, then the essential sum does not change. In other words, the essential sum is the natural definition for the Minkowski sum of equivalence classes of measurable sets.

Applying the Brunn-Minkowski inequality (2.1) to the sets of Lebesgue points of A and B shows that inequality (2.1) also holds for the essential sum $A +_{(ess.)} B$. There is equality, if and only if A and B differ by sets of measure zero from convex sets that are related by scaling and translation.

On the other hand, it is not hard to show that the essential sum is equivalent to a subset of the Minkowski sum, since $a+b$ is a Lebesgue point of $A+B$, whenever a and b are Lebesgue points of A and B. This subset is often a subset of smaller measure. Hence the Brunn-Minkowski inequality for the essential sum implies inequality (2.1) for general measurable sets [6].

Before we use the Brunn-Minkowski inequality to prove Theorem 1 in case the strict triangle inequality is violated, we will make a short deviation and show that, conversely, inequality (1.9) implies (2.1), and Theorem 1 implies that the optimizers of (2.1) satisfy (2.2). Note that (2.1) is more precise than (1.12).

Riesz implies Brunn-Minkowski. Suppose that Theorem 1 has been proven for measurable sets in $\mathbf{R}^n$. Let A and B be nonempty measurable sets in $\mathbf{R}^n$ consisting of Lebesgue points only. It is always true that

$$\mathcal{J}(A,B,C) \le \int \mathcal{X}_A * \mathcal{X}_B \, dx = \mu(A)\mu(B) . \qquad (2.3)$$

Set $C = A + B$. Since C contains the support of $\mathcal{X}_A * \mathcal{X}_B$ by Lemma 1, we have equality in (2.3). Inequality (1.9) implies that also (A^*, B^*, C^*) produce equality in (2.3). It follows from Lemma 1 that $C^* \supset A^* + B^*$, that is, $\gamma \geq \alpha + \beta$ by definition (1.14). This proves (2.1) for such sets.

Assume now additionally that A and B produce equality in the Brunn-Minkowski inequality (2.1), and set $C = A + B$. Then

$$\mu(C)^{1/n} = \mu(A)^{1/n} + \mu(B)^{1/n} ,$$

that is, $\gamma = \alpha + \beta$ by definition (1.14). Moreover,

$$\mathcal{J}(A, B, C) = \mu(A)\mu(B) = \mathcal{J}(A^*, B^*, C^*) .$$

That is, (A, B, C) produces equality in (1.9). By conclusion (1.12) of Theorem 1, A and B satisfy (2.2).

Proof of Theorem 1, $\gamma \geq \alpha + \beta$ Let A, B, and C be measurable sets of finite positive measure. In case the strict triangle inequality does not hold, inequality (1.9) is trivial: Since $\gamma \geq \alpha + \beta$ is equivalent to $C^* \supset A^* + B^*$, it follows that

$$\begin{aligned} \mathcal{J}(A, B, C) &\leq \int \mathcal{X}_A * \mathcal{X}_B \, dx \\ &= \mu(A)\mu(B) \\ &= \int \mathcal{X}_{A^*} * \mathcal{X}_{B^*} \, dx \\ &= \mathcal{J}(A^*, B^*, C^*) \end{aligned}$$

Since the convolution $\mathcal{X}_A * \mathcal{X}_B$ is a continuous function, it is easy to see that there is equality in the first line if and only if C contains the support of $\mathcal{X}_A * \mathcal{X}_B$ up to a set of measure zero. This proves assertion (1.11) .

If $\gamma = \alpha + \beta$, assume that A, B, and C consist exactly of their Lebesgue points. We have just shown that equality implies that C contains the Minkowski sum $A + B$. It follows from the Brunn-Minkowski inequality (2.1) that

$$\mu(C) \geq \mu(A + B) , \tag{2.4}$$

that is, $\gamma \geq \alpha + \beta$ by definition (1.14) of α, β, and γ. Hence A and B produce equality in the Brunn-Minkowski inequality (2.1). Consequently, there is equality in (2.4), and C differs by a set of measure zero from its subset $A + B$. It follows from (2.2) that A, B, and consequently $A + B$, are convex sets differing only by scaling and translation. This proves assertion (1.12). ■

2.1.2 Optimizers and level sets

Fix two sets A and B in $\mathbf{R}^n$. The convolution $\mathcal{X}_A * \mathcal{X}_B$ of their characteristic functions is continuous, even if A and B are only measurable. Fix the measure of the third set, C. The well-known '**bathtub principle**' says that the integral of a function over a set of a given size is minimized, if the sets is chosen to be a sub-level set of the function. Similarly, it is easy to see that the functional $\mathcal{J}$ defined by (1.6) is maximized, if C is a level set of $\mathcal{X}_A * \mathcal{X}_B$ It is the purpose of this subsection to describe this level set more precisely.

Definition The number s will be called a **degenerate** value of $\mathcal{X}_A * \mathcal{X}_B$ if the level 'surface'

$$\left\{x \mid \mathcal{X}_A * \mathcal{X}_B = s\right\}$$

has positive n-dimensional Lebesgue measure, and **non-degenerate**, if it has measure zero.

Consider the symmetrized set A^* and B^*. By definition, these sets A^* and B^* are the open balls of radius α and β. The convolution

$$\mathcal{X}_{A^*} * \mathcal{X}_{B^*}(x) = \mu\Big(B_\alpha(0) \cap B_\beta(x)\Big)$$

is a nonincreasing function of the radius $|x|$ (see Figure 2.1). It is positive on the open ball of radius $\alpha + \beta$, and it achieves its maximum on all points of the closed ball of radius $|\alpha - \beta|$. Its maximal value is either $\mu(A)$ or $\mu(B)$, whichever is smaller. It is strictly spherically decreasing for radii between $\alpha + \beta$ and $|\alpha - \beta|$. Hence, if α, β, γ satisfy the strict triangle inequality, then C^* is a level set corresponding to a nondegenerate value of $\mathcal{X}_{A^*} * \mathcal{X}_{B^*}$. If $\gamma \geq \alpha + \beta$, then C^* contains the support of the convolution. If $\gamma \leq |\alpha - \beta|$, then C^* is a subset of the set where the convolution takes its maximal value.

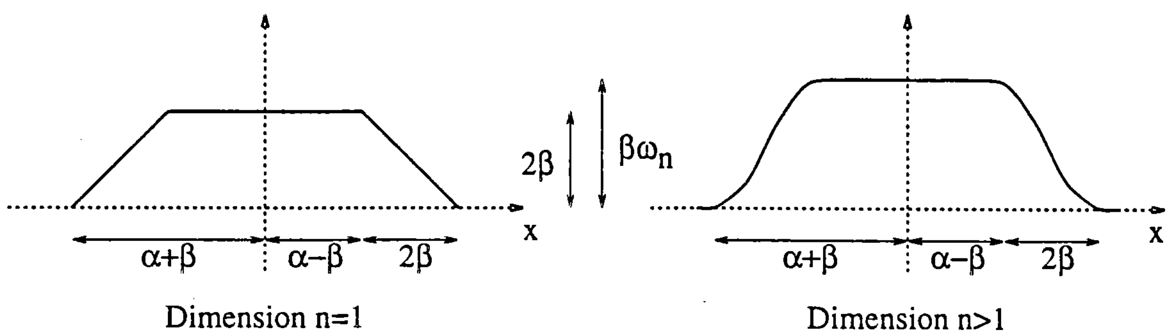

Figure 2.1: Convolution of the characteristic functions of two balls

The following lemma shows that similar statements hold for general optimizing triples. It can also be seen as a regularity result – it implies that the interior and the closure of each of the three sets are equivalent. Note that it is necessary to assume that inequality (1.9) holds.

Lemma 2 (Optimizers and level sets) *Suppose that inequality (1.9) has been proven for measurable sets of finite measure in* $\mathbb{R}^n$.

Consider an optimizing triple (A, B, C) of inequality (1.9) in $\mathbf{R}^n$. If α, β, and γ satisfy the strict triangle inequality, then C differs by a set of measure zero from the level set

$$\left\{x \mid \mathcal{X}_A * \mathcal{X}_B > s\right\}, \tag{2.5}$$

where

$$s := \inf_{x \in C^*} \mathcal{X}_{A^*} * \mathcal{X}_{B^*}(x)$$

is a nondegenerate value of $\mathcal{X}_A * \mathcal{X}_B$. If $\gamma \geq \alpha + \beta$, then C contains (except for a set of measure zero) the level set

$$\left\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) > 0\right\},$$

if $\gamma = \alpha + \beta$, then C differs by a set of measure zero from this level set. If $0 < \gamma \leq |\alpha - \beta|$, then C is contained (up to a set of measure zero) in

$$\left\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) = \min(\mu(A), \mu(B))\right\},$$

if $0 < \gamma = |\alpha - \beta|$, then C differs by a set of measure zero from this level set.

Proof Assume without loss of generality that $\mu(A) \geq \mu(B)$, that is, $\alpha \geq \beta$. The strict triangle inequality between α, β, and γ guarantees that

$$0 < s < \max\left(\mathcal{X}_{A^*} * \mathcal{X}_{B^*}\right).$$

Clearly, s is a nondegenerate value of $\mathcal{X}_{A^*} * \mathcal{X}_{B^*}$. Define the sets

$$\begin{aligned} C_+ &:= C \cap \left\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) > s\right\}, \\ C_- &:= C \cup \left\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) \geq s\right\}. \end{aligned}$$

Then

$$C_+ \subset C \subset C_- \,, \tag{2.6}$$

and

$$C_+ \subset \Big\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) > s\Big\} \subset \Big\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) \geq s\Big\} \subset C_- \,.$$

If C differs by a set of positive measure from the level set (2.5), or if s is a degenerate value of $\mathcal{X}_A * \mathcal{X}_B$, then at least one of the sets $C \setminus C_+$, $C_- \setminus C$ has positive measure. If $\mu(C \setminus C_+) > 0$ then

$$\begin{aligned}
\mathcal{J}(A,B,C) &\leq \mathcal{J}(A,B,C_+) + s\mu(C \setminus C_+) \\
&\qquad\qquad \text{because } \mathcal{X}_A * \mathcal{X}_B \leq s \text{ on } C \setminus C_+ \\
&\leq \mathcal{J}(A^*,B^*,C_+^*) \qquad \text{by inequality (1.9) and (2.6)} \\
&< \mathcal{J}(A^*,B^*,C^*) \qquad \text{because } \mathcal{X}_{A^*} * \mathcal{X}_{B^*} > s \text{ on } C^* \,.
\end{aligned}$$

Similarly, if $\mu(C_- \setminus C) > 0$ then

$$\begin{aligned}
\mathcal{J}(A,B,C) &\leq \mathcal{J}(A,B,C_-) - s\mu(C_- \setminus C) \\
&\qquad\qquad \text{because } \mathcal{X}_A * \mathcal{X}_B \geq s \text{ on } C_- \setminus C \\
&\leq \mathcal{J}(A^*,B^*,C_-^*) - s\mu(C_-^* \setminus C^*) \qquad \text{by (1.9) and (2.6)} \\
&< \mathcal{J}(A^*,B^*,C^*) \\
&\qquad\qquad \text{because } \mathcal{X}_{A^*} * \mathcal{X}_{B^*} < s \text{ outside the closure of } C^* \,.
\end{aligned}$$

In either case, it follows that (A,B,C) produces strict inequality in (1.9).

For $\gamma \geq \alpha + \beta$, define

$$\begin{aligned}
C_+ : &= C \cap \Big\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) > 0\Big\} \,, \\
C_- : &= C \cup \Big\{x \mid \mathcal{X}_A * \mathcal{X}_B(x) > 0\Big\} \,.
\end{aligned}$$

If $C_- \setminus C$ is a set of positive measure, then

$$\begin{aligned} \mathcal{J}(A,B,C) &< \mathcal{J}(A,B,C_-) \qquad \text{because } \mathcal{X}_A * \mathcal{X}_B > 0 \text{ on } C_- \setminus C \\ &\leq \mu(A)\mu(B) \\ &= \mathcal{J}(A^*,B^*,C^*) \,. \end{aligned}$$

If $\gamma = \alpha + \beta$, and $C \setminus C_+$ is a set of positive measure, then

$$\begin{aligned} \mathcal{J}(A,B,C) &= \mathcal{J}(A,B,C_+) \qquad \text{because } \mathcal{X}_A * \mathcal{X}_B = 0 \text{ on } C \setminus C_+ \\ &\leq \mathcal{J}(A^*,B^*,C_+^*) \qquad \text{by inequality (1.9)} \\ &< \mathcal{J}(A^*,B^*,C^*) \qquad \text{because } \mathcal{X}_{A^*} * \mathcal{X}_{B^*} > 0 \text{ on } C^* \,. \end{aligned}$$

Finally, for $0 < \gamma \leq \alpha - \beta$, define

$$\begin{aligned} C_+ : &= C \cap \left\{ x \mid \mathcal{X}_A * \mathcal{X}_B(x) = \mu(B) \right\} , \\ C_- : &= C \cup \left\{ x \mid \mathcal{X}_A * \mathcal{X}_B(x) = \mu(B) \right\} . \end{aligned}$$

If $C \setminus C_+$ has positive measure, then

$$\begin{aligned} \mathcal{J}(A,B,C) &< \mathcal{J}(A,B,C_+) + \mu(C \setminus C_-)\mu(B) \\ &\qquad \text{because } \mathcal{X}_A * \mathcal{X}_B < \mu(B) \text{ on } C \setminus C_+ \\ &\leq \mathcal{J}(A^*,B^*,C_+^*) + \mu(C \setminus C_+)\mu(B) \\ &= \mathcal{J}(A^*,B^*,C^*) \,. \end{aligned}$$

If $\gamma = \alpha - \beta$, and $C_- \setminus C$ has positive measure, then

$$\begin{aligned} \mathcal{J}(A,B,C) &= \mathcal{J}(A,B,C_-) - \mu(C_- \setminus C)\mu(B) \\ &\qquad \text{because } \mathcal{X}_A * \mathcal{X}_B = \mu(B) \text{ on } C_- \setminus C \\ &\leq \mathcal{J}(A^*,B^*,C_-^*) - \mu(C_- \setminus C)\mu(B) \qquad \text{by inequality (1.9)} \end{aligned}$$

$$< \ \mathcal{J}(A^*, B^*, C^*)$$

because $\mathcal{X}_{A^*} * \mathcal{X}_{B^*} < \mu(B)$ outside the closure of C^* .

In all these cases, A, B, and C cannot produce equality in inequality (1.9). ∎

Remark Combining Lemma 2 with Lemma 1 shows that if (A, B, C) is an optimizer of inequality (1.9) and the strict triangle inequality holds between α, β, and γ, then A, B, and C differ by sets of measure zero from open sets satisfying

$$\text{closure}(C) \subset A + B \ . \tag{2.7}$$

2.2 Proof of Theorem 1 in one dimension

In this section, we adapt Riesz' original proof of inequality (1.9) in one dimension [25] (see also [16]) to general measurable sets, and use it to determine the cases of equality when the strict triangle inequality holds between the measures of the three sets.

The idea of the proof is to replace two of the three sets by smaller sets, so that the three sets are in the critical size relation $\gamma = \alpha + \beta$. Although it not hard to prove Theorem 1 in this case directly, we will refer to the previous section instead.

Proof of Theorem 1, $n = 1$, $|\alpha - \beta| < \gamma < \alpha + \beta$ For $\delta \geq 0$, define

$$\begin{aligned} A_\delta &= \left\{x \in A \mid \int_{-\infty}^{x} \mathcal{X}_A(s)\, ds > \delta/2, \int_{x}^{\infty} \mathcal{X}_A(s)\, ds > \delta/2\right\} \\ B_\delta &= \left\{x \in B \mid \int_{-\infty}^{x} \mathcal{X}_B(s)\, ds > \delta/2, \int_{x}^{\infty} \mathcal{X}_B(s)\, ds > \delta/2\right\} . \end{aligned}$$

The sets A_δ and B_δ are obtained by cutting off subsets of measure $\delta/2$ from both ends of A and B. In general,

$$\mu(A_\delta) = (\mu(A) - \delta)_+ \, .$$

If we choose

$$\delta := \frac{1}{2}(\mu(A) + \mu(B) - \mu(C)) = \alpha + \beta - \gamma \, ,$$

then

$$\begin{aligned} \mu(A_\delta) &= \mu(A) - \frac{1}{2}\Big(\mu(A) + \mu(B) - \mu(C)\Big) &= \alpha + \gamma - \beta \, , \\ \mu(B_\delta) &= \mu(B) - \frac{1}{2}\Big(\mu(A) + \mu(B) - \mu(C)\Big) &= \beta + \gamma - \alpha \, . \end{aligned}$$

Both expressions are positive because α, β, and γ satisfy the strict triangle inequality. Moreover, the measures of A_δ, B_δ, and C are in the critical size relation

$$\mu(A_\delta) + \mu(B_\delta) = \mu(C)$$

of Theorem 1.

To estimate how the value of the functional $\mathcal{J}$ changes when A and B are replaced by A_δ and B_δ, observe that for any two measurable sets,

$$\begin{aligned} (A \cap B)_\delta &= \left\{ x \in A \cap B \mid \int_{-\infty}^{x} \mathcal{X}_A(s)\mathcal{X}_B(s)\, ds > \delta/2, \int_{x}^{\infty} \mathcal{X}_A(s)\mathcal{X}_B(s)\, ds > \delta/2 \right\} \\ &\subset A_\delta \cap B_\delta \, , \end{aligned}$$

and hence

$$\mu(A_\delta \cap B_\delta) \geq \mu((A \cap B)_\delta) \geq \mu(A \cap B) - \delta \, . \tag{2.8}$$

Applying inequality (2.8) to the intersection of A with $x - B$ and integrating gives

$$\begin{aligned} \mathcal{J}(A,B,C) - \mathcal{J}(A_\delta, B_\delta, C) &= \int_C \mu((A \cap (x - B)) - \mu(A_\delta \cap (x - B_\delta))\, dx \\ &\leq \delta \mu(C) \, . \end{aligned} \tag{2.9}$$

It is easy to calculate directly, that the triple (A^*, B^*, C^*) produces equality in (2.9), and hence, that

$$\mathcal{J}(A, B, C) - \mathcal{J}(A_\delta, B_\delta, C) \leq \mathcal{J}(A^*, B^*, C^*) - \mathcal{J}(A_\delta^*, B_\delta^*, C^*) . \tag{2.10}$$

Adding inequality (1.9) for (A_δ, B_δ, C)

$$\mathcal{J}(A_\delta, B_\delta, C) \leq \mathcal{J}(A_\delta^*, B_\delta^*, C^*) \tag{2.11}$$

to inequality (2.10) gives (1.9) for (A, B, C).

For equality in (1.9), it is necessary that A_δ, B_δ, and C produce equality in (2.11). The second case of Theorem 1, which was proved in the previous section, implies that C differs from an interval by a set of measure zero. By the permutation symmetry (1.8), A and B have to be intervals up to sets of measure zero as well. It is easy to see that the centers must satisfy $a + b = c$. ∎

2.3 Tools for the proof in higher dimensions

The main tools for the inductive step in the proof of Theorem 1 are symmetrization procedures along lower dimensional subspaces. We define a special partial symmetrization procedure in the first subsection. We prove that spherical rearrangement can be approximated by a suitable sequence of such symmetrizations.

In the second subsection, we show that partial symmetrization can also be used for regularization. We show that the procedure has the three properties (R1)–(R3) announced at the beginning of the chapter.

In the third subsection, we prove the lemmas we will use to actually identify the optimizers as ellipsoids.

2.3.1 Symmetrization along subspaces

Let A be a measurable set in $\mathbf{R}^{n+1}$. Write points in $\mathbf{R}^{n+1}$ as $\mathbf{x} = (x^0, x)$ with $x^0 \in \mathbf{R}$ and $x \in \mathbf{R}^n$. Denote the n-dimensional cross section of A at $x^0 = z$ by

$$A(z) := \{x \in \mathbf{R}^n \mid (z, x) \in A\} .$$

The **Schwarz symmetrization**, $\mathcal{S}_1 A$, of A is defined by the property that the n-dimensional cross sections of $\mathcal{S}_1 A$ perpendicular to the x^0-axis are centered open balls whose measures equal the measures of the corresponding cross sections of A (see Figure 2.2, left). In short, for all $z \in \mathbf{R}$,

$$(\mathcal{S}_1 A)(z) = (A(z))^* , \tag{2.12}$$

where * denotes symmetrization of the cross section in $\mathbf{R}^n$. If a cross section is not measurable or does not have finite measure, the corresponding cross section of $\mathcal{S}_1 A$ is defined to be $\mathbf{R}^n$.

Similarly, the **Steiner symmetrization**, $\mathcal{S}_2 A$, of A is defined by the property, that the intersections of $\mathcal{S}_2 A$ with lines parallel to the x^0-axis are intervals of the same lengths as the measures of the corresponding intersections with A (see Figure 2.2, right). In short, for all $x \in \mathbf{R}^n$,

$$(\mathcal{S}_2 A)(x) = (A(x))^* , \tag{2.13}$$

where * denotes symmetrization in $\mathbf{R}$ of the linear cross section at x. If a cross section is not measurable, or does not have finite measure, the corresponding cross section of $\mathcal{S}_2 A$ is defined to be $\mathbf{R}$.

The basic symmetrization operation $\mathcal{S}$ that will be used in the inductive step consists of a Schwarz symmetrization followed by a Steiner symmetrization, composed

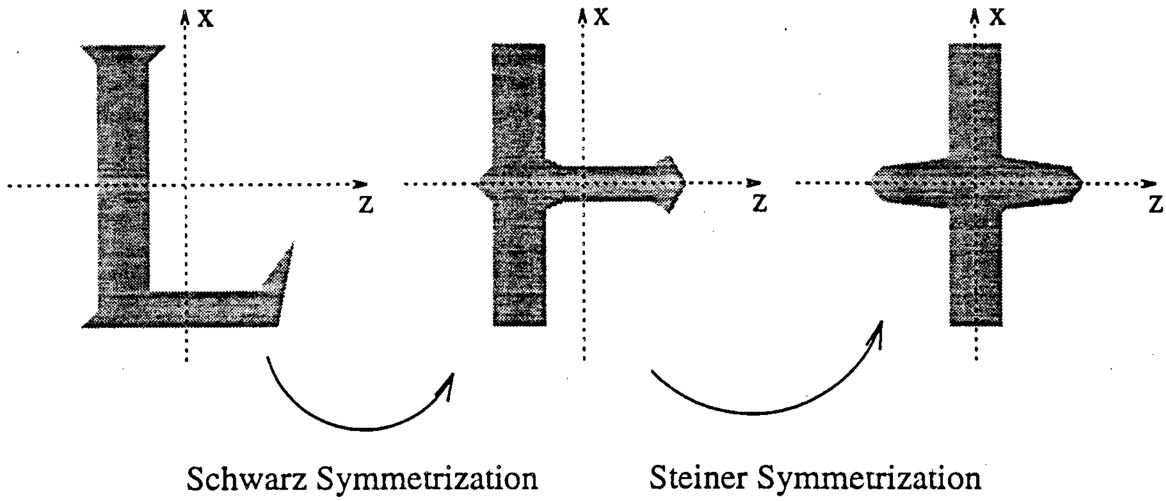

Figure 2.2: Partial symmetrizations

with a suitable rotation: Let $\mathcal{R}$ be a rotation by an angle that is not a rational multiple of $\pi/2$ in the x^0-x^1-plane, which leaves the other coordinates fixed, and define $\mathcal{S}$ by

$$\mathcal{S}A := \mathcal{S}_2\mathcal{S}_1\mathcal{R}A \ . \tag{2.14}$$

Schwarz and Steiner symmetrization, and consequently $\mathcal{S}$, are uniquely determined by properties (2.12) and (2.13). All three operations preserve the measure of A by Fubini's theorem, so that

$$(\mathcal{S}A)^* = (\mathcal{S}_1A)^* = (\mathcal{S}_2A)^* = A^* \ .$$

Moreover, if A is changed by a set of measure zero, then A^* does not change, and $\mathcal{S}_1A$ and $\mathcal{S}_2A$ change by sets of measure zero. In other words, all these symmetrizations can also be defined as operations on equivalence classes of measurable sets.

Another characterizing property of spherical, Schwarz, and Steiner symmetrization is the following. Let h be a bounded nonnegative strictly spherically decreasing function on $\mathbf{R}^{n+1}$. Then, for any measurable set A the following inequalities hold.

$$\int_A h(x)\,dx \leq \int_{A^*} h(x)\,dx \tag{2.15}$$

$$\int_A h(x)\,dx \leq \int_{\mathcal{S}_1 A} h(x)\,dx \tag{2.16}$$

$$\int_A h(x)\,dx \leq \int_{\mathcal{S}_2 A} h(x)\,dx \tag{2.17}$$

If the integrals on the right hand side are finite, then there is equality in (2.15), (2.16), or (2.17), if and only if A differs by a set of measure zero from A^*, $\mathcal{S}_1 A$, or $\mathcal{S}_2 A$, respectively. The proof is another simple application of the bathtub principle.

Notation Write $\mathcal{S}_1(A,B,C)$, $\mathcal{S}_2(A,B,C)$, and $\mathcal{S}(A,B,C)$ for the triples consisting of the corresponding symmetrizations of A, B, and C.

The following two lemmas are the starting point of the inductive step for the proof of Theorem 1. The first lemma is needed in the proof of the inequality. The second shows that cross sections of optimizers must be optimizers. It also shows that partial symmetrization has property (R1).

Lemma 3 ($\mathcal{J}$ cannot decrease under partial symmetrization) *Assume that inequality (1.9) has been proven for dimensions up to n. Then for any triple of measurable sets (A,B,C) of finite positive measure in $\mathbf{R}^{n+1}$ the following inequalities hold.*

$$\mathcal{J}(A,B,C) \leq \mathcal{J}(\mathcal{S}_1(A,B,C))\,, \tag{2.18}$$

$$\mathcal{J}(A,B,C) \leq \mathcal{J}(\mathcal{S}_2(A,B,C))\,, \tag{2.19}$$

$$\mathcal{J}(A,B,C) \leq \mathcal{J}(\mathcal{S}(A,B,C))\,. \tag{2.20}$$

Proof The functional $\mathcal{J}$ can be written as

$$\mathcal{J}(A,B,C) = \int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{J}(A(w), B(z-w), C(z))\, dwdz \,.$$

Inequality (1.9) applied to the n-dimensional cross sections gives

$$\mathcal{J}(A,B,C) \le \int_{\mathbf{R}}\int_{\mathbf{R}} \mathcal{J}((A(w))^*, (B(z-w))^*, (C(z))^*)\, dwdz = \mathcal{J}(\mathcal{S}_1(A,B,C)) \,.$$

Similar arguments for Steiner symmetrization $\mathcal{S}_2$ show inequality (2.19). Combining the two with (1.7) as in (2.14) gives inequality (2.20). ∎

Lemma 4 (Partial symmetrization preserves optimizers) *Assume that inequality (1.9) has been proven for dimensions up to $n+1$. Let A, B, C be measurable sets of finite measure in $\mathbf{R}^{n+1}$. Then, if the triple (A, B, C) is an optimizer of inequality (1.9), then also $\mathcal{S}_1(A,B,C)$, $\mathcal{S}_2(A,B,C)$, and $\mathcal{S}(A,B,C)$ are optimizers. Moreover, almost all n-dimensional cross sections of any optimizing triple (A,B,C) at z_A, z_B, and $z_C = z_A + z_B$ form optimizing triples for the inequality in $\mathbf{R}^n$.*

Proof Equality follows immediately from inequalities (2.18), (2.19), and (2.20) with inequality (1.9) in $\mathbf{R}^{n+1}$. The statement about the cross sections follows with inequality (1.9) in $\mathbf{R}^n$ from Fubini's theorem. ∎

The following lemma plays a role in proving that repeated application of $\mathcal{S}$ approximates the spherical rearrangement.

Lemma 5 *Let $A \subset \mathbf{R}^{n+1}$ be a measurable set of finite positive measure, and let $\mathcal{S}$ be the symmetrization operation defined in (2.14). Then*

$$\mathcal{S}A = \left\{ \mathbf{x} \in \mathbf{R}^{n+1} \;\middle|\; |\hat{x}| < \alpha(x^0) \right\} . \tag{2.21}$$

where α is even, nonnegative, and nonincreasing for positive arguments.

Remark Since a monotone function can have at most countably many discontinuities, we can make α lower semicontinuous by changing $\mathcal{S}A$ by a set of measure zero. It is generally true that

$$\lim_{z\to\infty} \alpha(z) = 0 ,$$

but, unless A is bounded, α may be unbounded at 0 and need not have compact support.

Proof Define the function α by

$$\alpha(z) = \sup_{x\in\mathcal{S}A(z)} |x| = \inf_{x\notin\mathcal{S}A(z)} |x| , \tag{2.22}$$

where the supremum and infimum of an empty set are taken to be 0, and ∞, respectively. The two expressions in (2.22) coincide, because by definition of $\mathcal{S}$, for any point $\mathbf{x} \in \mathcal{S}A$, the solid cylinder spanned by $\mathbf{x}$ is contained in $\mathcal{S}A$, that is,

$$\left\{ \mathbf{y} \in \mathbf{R}^{n+1} \mid |y^0| \leq |x^0|, |\hat{y}| \leq |\hat{x}| \right\} \subset \mathcal{S}A \tag{2.23}$$

It follows that

$$2\omega_n |z| (\alpha(z))^n \leq \mu(A) ,$$

which shows that $\alpha(z)$ is finite when $z \neq 0$, and $\lim_{z\to\infty} \alpha(z) = 0$. The function α is nonincreasing for positive z, because for any given $z > 0$, all points $\mathbf{x}$ with $x^0 < z$ and $|\hat{x}| < \alpha(z)$ are contained in $\mathcal{S}A$ by the cylinder property (2.23). ∎

Next we prove the convergence lemma, which we discussed in the introduction.

Lemma 6 (Convergence) *For any measurable set A, in $\mathbf{R}^{n+1}$, the sequence*

$$\left\{ \mathcal{S}^k A \right\}_{k\geq 0}$$

converges to A^* with respect to symmetric difference, that is,

$$\lim_{k\to\infty}\left(\mu(\mathcal{S}^k A \setminus A^*) + \mu(A^* \setminus \mathcal{S}^k A)\right) = 0 .$$

Proof Consider the functional

$$\mathcal{H}(M) := \int_M h(x)\,dx$$

where h is a fixed nonnegative, bounded, strictly spherically decreasing function. The functional takes a finite value for any measurable set of finite measure, because h is uniformly bounded.

We will apply the 'Competing Symmetries' technique of [9] to the functional $\mathcal{H}$. In our proof, $\mathcal{S}_2\mathcal{S}_1$ plays the role of the transformation $\mathcal{U}$, and $\mathcal{R}$ plays the role of $\mathcal{R}$ in [9].

Note that the functional depends continuously on M with respect to symmetric difference. Consider the sequence of sets defined by

$$\begin{aligned} A_0 &= A \\ A_{k+1} &= \mathcal{S}A_k \quad (k \geq 0) . \end{aligned}$$

The sequence $\{\mathcal{H}(A_k)\}_{k\geq 0}$ is nondecreasing by inequalities (2.16) and (2.17). Inequality (2.15) shows that it is bounded above by $\mathcal{H}(A^*)$. It follows that it is convergent. Denote the limit by

$$\mathcal{H}_\infty := \lim_{k\to\infty} \mathcal{H}(A_k) .$$

By equation (2.21) of Lemma 5, the sets A_k are for $k \geq 1$ given as rotational solids of nonincreasing functions α_k. By Helle's theorem, there exists a subsequence α_{k_j} which converges on all rational points to a limiting nonincreasing function α_∞. Let

$$A_\infty := \left\{ x \in \mathbf{R}^{n+1} \mid |\hat{x}| < \alpha_\infty x^0 \right\} .$$

Convergence of the sequence $\left\{\alpha_{k_j}\right\}_{j\geq 0}$ implies that

$$\lim_{j\to\infty} \left(\mu(A_{k_j} \setminus A_\infty) + \mu(A_\infty \setminus A_{k_j}) \right) = 0 .$$

By construction, A_∞ is symmetric under rotation about the x^0-axis. It follows from the definition (2.14) of $\mathcal{S}$, inequalities (2.17) and (2.16), and equation (1.7) that

$$\mathcal{H}(\mathcal{S}A_\infty) = \mathcal{H}(\mathcal{S}_2\mathcal{S}_1\mathcal{R}A_\infty) \geq \mathcal{H}(\mathcal{S}_1\mathcal{R}A_\infty) \geq \mathcal{H}(\mathcal{R}A_\infty) = \mathcal{H}(A_\infty) .$$

Since the sequence $\{\mathcal{H}(A_k)\}_{k\geq 0}$ is convergent and $\mathcal{H}$ depends continuously on its arguments,

$$\mathcal{H}(\mathcal{S}A_\infty) = \lim_{j\to\infty} \mathcal{H}\left(\mathcal{S}A_{k_j}\right) = \lim_{j\to\infty} \mathcal{H}\left(A_{k_j+1}\right) = \mathcal{H}(A_\infty) ,$$

and consequently

$$\mathcal{H}(\mathcal{S}_1\mathcal{R}A_\infty) = \mathcal{H}(\mathcal{R}A_\infty) .$$

By (2.16), we have that

$$\mathcal{S}_1\mathcal{R}A_\infty = \mathcal{R}A_\infty . \tag{2.24}$$

Clearly, this implies that $\mathcal{R}A$ is symmetric under rotation about the x^0-axis. By the choice of $\mathcal{R}$, A is simultaneously symmetric under rotation about two axes enclosing an irrational angle. It follows that A_∞ is symmetric under arbitrary rotations. Using (2.24) again, we see that the n-dimensional cross sections of $\mathcal{R}A_\infty$ must be balls. Consequently A_∞ is a ball, and we have shown that

$$A_\infty = A^* .$$

Since every subsequence of $\{A_k\}$ has a subsequence converging to A^*, the whole sequence converges to A^*. ∎

Remark If A is bounded, then the sequence converges to A^* with respect to Hausdorff distance as well. To see this, observe that the sequence $\{\alpha_i\}_{i\geq 0}$ converges uniformly to the limit $\sqrt{(\alpha^2 - x^2)_+}$ outside a neighborhood of the origin. By the properties of the Hausdorff metric, any subsequence converges. The limit can only differ from A^* by a 'spike' at the origin, or by a disc at the hyperplane $x^0 = 0$. Since a single partial symmetrization transforms such a set into a ball, the limit set itself must be a ball.

2.3.2 Regularization

In this subsection, we will prove that $\mathcal{S}^2$ has the properties announced at the beginning of the chapter and can be used as a regularization procedure.

We have already shown (Lemma 4) that this regularization procedure has property (R1), that is, it transforms optimizers of inequality (1.9) into optimizers.

Concerning the regularity properties claimed in (R3), we have already shown that $\mathcal{S}^2$ transforms any measurable set into the rotational solid of an even and nonincreasing function (Lemma 5). Lemma 8 shows that this function is bounded. Lemma 7 says that the measures of the intersections of such a set with most hyperplanes depends continuously on the hyperplane.

Finally, we will show in Lemma 11 that the procedure also has property (R2), so it is enough to prove conclusion (1.10) of Theorem 1 for optimizing triples that have been regularized with $\mathcal{S}^2$.

Lemma 7 (Continuity of intersections) *Consider an open set of the form*

$$A = \left\{ \mathrm{x} = (x^0, \hat{x}) \in \mathrm{R}^{n+1} \;\mid\; |\hat{x}| < \alpha(x^0) \right\}$$

where α is an even, nonnegative function that is nonincreasing for positive arguments. Assume that the measure of A is finite and positive. Consider the intersection of A with hyperplanes of the form

$$x^0 = mx^1 + t \, , \tag{2.25}$$

where m and t are scalars. If $m \neq 0$ then the n-dimensional measure of the intersections is uniformly bounded in t, and jointly continuous in (m,t) at $(m,0)$.

If additionally α is bounded and continuous at 0, then the measure of the intersections is also jointly continuous in (m,t) at $(0,0)$.

Proof Write points in R^{n+1} as $\mathrm{x} = (x^0, x^1, \hat{x})$ if $n > 1$, and points in R^2 as $\mathrm{x} = (x^0, x^1)$. The intersection of a hyperplane of the form (2.25) with A is given by the equations

$$x^0 = mx^1 + t \, , \quad |\hat{x}|^2 < \alpha^2\left(x^0\right) - (x^1)^2 \, ,$$

for $n = 1$ set $|\hat{x}| = 0$. Integrating over $\hat{x}$, the measure of this intersection is $(1 + m^2)^{n/2} I(m,t)$, where

$$I(m,t) = \omega_{n-1} \int_{-\infty}^{\infty} \left(\alpha^2(ms+t) - s^2\right)_+^{(n-1)/2} \, ds \, ,$$

with the convention that $\omega_0 = 1$, and $0^0 = 0$. The integrand cannot be positive unless $\alpha(ms+t) > 1$ or $|s| < 1$ or both. For $m \neq 0$, decompose

$$I \leq I_1 + I_2 \, ,$$

where I_1 is given by

$$\begin{aligned} I_1 &= \frac{\omega_{n-1}}{m} \int_{-s_0}^{s_0} \left(\alpha^2(s) - ((s-t)/m)^2\right)_+^{(n-1)/2} ds \\ &\leq \frac{\omega_{n-1}}{m} \int_{-s_0}^{s_0} \alpha^n(s)\, dt \\ &= \frac{\omega_{n-1}}{m\,\omega_n} \mu(A) \end{aligned}$$

with $s_0 = \inf\{s > 0 \mid \alpha(s) \leq 1\}$. The other part of the integral is

$$I_2 = \omega_{n-1} \int_{-1}^{1} \alpha^{n-1}(ms+t)\, ds \ .$$

It is easy to see from these representations that I_1, I_2, and I are bounded uniformly in t, and that $\lim_{|t|\to\infty} I(m,t) = 0$.

Continuity for $n > 1$ follows from the fact that the integrand in

$$I(m,t) = \omega_{n-1} \int_{-\infty}^{\infty} m^{-1} \left(\alpha^2(s) - ((s-t)/m)^2\right)_+^{(n-1)/2} ds$$

depends continuously on (m,t), and vanishes for fixed (m,t) if s is outside a compact set. To show continuity for $n = 1$, note first that for $m > 0$, and $|t|$ small enough, the line (2.25) intersects the boundary of A only in the first and third quadrant, because A is open, and the intersection of the line with the second and the fourth quadrant is contained in a neighborhood of the origin. Since α is of bounded variation, the boundary of A can be parameterized continuously, for example over the arc length. Let P be any point on the boundary of A in the first quadrant. Since α is nonincreasing in the first quadrant, its graph is contained in the cone described by the equation

$$(x^0 - P^0)(x^1 - P^1) \leq 0 \ .$$

Any line with slope $m > 0$ through P intersects the boundary curve transversally, because it is contained in the complement of this cone. This shows continuity for $m > 0$. By symmetry, the claim also holds for $m < 0$.

Finally, if α is bounded and continuous at 0, then the integrand of I converges pointwise to

$$\left(\alpha^2(0) - s^2\right)_+^{(n-1)/2} ,$$

except possibly for $n = 1$ at $s = \pm\alpha(0)$. Dominated convergence shows that I is continuous at $(0,0)$. ∎

Lemma 8 *Let $A \subset \mathbb{R}^{n+1}$ be a measurable set of finite positive measure, and let $\mathcal{S}$ be the symmetrization operation defined in (2.14). Then,*

$$\mathcal{S}^2 A = \left\{ \mathrm{x} \in \mathbb{R}^{n+1} \mid |\hat{x}| < \alpha(x^0) \right\} .$$

where α is even, bounded, nonnegative, and nonincreasing for positive arguments.

Proof By Lemma 5, $\mathcal{S}^2 A$ can be written as

$$\mathcal{S}^2 A = \{ x \in \mathbb{R}^{n+1} \mid |\hat{x}| < \alpha(x^0) \} ,$$

where α is nonnegative, even, and nonincreasing for positive arguments. To show that α is bounded, recall that by definition (2.14),

$$\mathcal{S} = \mathcal{S}_2 \mathcal{S}_1 \mathcal{R} ,$$

where $\mathcal{R}$ rotates the x^0-axis by a non-integer multiple of $\pi/2$. By Lemma 5, $\mathcal{S}A$ can be written as

$$\mathcal{S}A = \{ x \in \mathbb{R}^{n+1} \mid |\hat{x}| < \alpha'(x^0) \} ,$$

where α' is nonnegative, even, and nonincreasing for positive arguments, but possibly unbounded at 0. The measures of the cross sections of $\mathcal{RS}A$ perpendicular to the x^0-axis are bounded by Lemma 7, because they correspond to intersections of $\mathcal{S}A$ with hyperplanes with $m \neq 0$. It follows that also the corresponding cross sections of $\mathcal{S}_1\mathcal{RS}A$ are bounded. The function α describing the cross sections of $\mathcal{S}^2A = \mathcal{S}_2\mathcal{S}_1\mathcal{RS}A$ is obtained by symmetric decreasing rearrangement from the function describing these cross sections. So α is bounded. ∎

For the proof of Lemma 11 in $\mathbf{R}^{n+1}$, we will need to apply Theorem 1 to the n-dimensional cross sections of an optimizing triple. We will reformulate the conclusions of Theorem 1 for cross sections in Lemma 9. This lemma will also be used later in the inductive step of the proof of Theorem 1 (see Section 2.4). Lemma 10 is only needed in the proof of Lemma 11.

Definition Let A, B, and C be three measurable sets of finite measure in $\mathbf{R}^{n+1}$. We will say that the strict triangle inequality holds for a triple of n-dimensional cross sections $(A(z_1), B(z_2), C(z_3))$, if the numbers

$$\left(\frac{\mu(A(z_1))}{\omega_n}\right)^{1/n}, \quad \left(\frac{\mu(B(z_2))}{\omega_n}\right)^{1/n}, \quad \left(\frac{\mu(C(z_3))}{\omega_n}\right)^{1/n}$$

satisfy inequality (1.13).

The inductive assumption will always be used in form of the following Lemma 9. It states that the intersections of the sets of an optimizing triple with hyperplanes perpendicular to the x^0-axis are ellipsoids whose midpoints lie on parallel lines, provided the strict triangle inequality holds for the cross sections.

Lemma 9 (Shape and midpoints of cross sections) *Assume that Theorem 1 has been proven for optimizing triples of the inequality in some fixed dimension n.*

Consider an optimizing triple (A, B, C) of inequality (1.9) in $\mathbf{R}^{n+1}$. Assume that there exist nonempty intervals I_A, I_B, and I_C such that for almost all $z_0 \in I_C$ there exists $w_0 \in I_A$ such that $z_0 - w_0 \in I_B$, and for almost all (z, w) in a neighborhood of (z_0, w_0), the n-dimensional cross sections $A(w)$, $B(z-w)$, and $C(z)$ satisfy the strict triangle inequality. Finally, assume that these assumptions also hold for any triple that can be obtained from (A, B, C) by the permutations (1.8).

Then there exists a centered ellipsoid $\hat{E}$ in $\mathbf{R}^n$, and vectors $\hat{a}$, $\hat{b}$, and $\hat{c} = \hat{a} + \hat{b}$ in $\mathbf{R}^n$, so that for almost all z in I_A, I_B, and I_C, respectively,

$$
\begin{aligned}
A(z) &= \hat{a} + z\hat{v} + \alpha(z)\hat{E} \ , \\
B(z) &= \hat{b} + z\hat{v} + \beta(z)\hat{E} \ , \\
C(z) &= \hat{c} + z\hat{v} + \gamma(z)\hat{E} \ ,
\end{aligned}
\tag{2.26}
$$

except for sets of n-dimensional measure zero.

Remark It is easy to see that

$$\hat{a} = \hat{b} = \hat{c} = 0$$

in equation (2.26) if A, B, and C are symmetric about the origin. In other words, the midpoints of the intersections of the three sets with hyperplanes perpendicular to the x^0-axis lie all on one line through the origin.

Proof Fix (z_0, w_0) as in the assumptions. By Lemma 4, almost all triples of cross sections $A(w)$, $B(z-w)$, $C(z)$ are optimizing triples for inequality (1.9) in $\mathbf{R}^n$. Since the strict triangle inequality holds for almost all cross sections with (z, w) near

(z_0, w_0), conclusion (1.10) of Theorem 1 shows that these cross sections are given by

$$\begin{aligned} A(w) &= a(w) + \alpha(w)E(z,w) \\ B(z-w) &= b(z-w) + \beta(z-w)E(z,w) \\ C(z) &= c(z) + \gamma(z)E(z,w) \end{aligned} \tag{2.27}$$

(up to sets of n-dimensional measure zero), where for each (z,w), $E(z,w)$ is a centered ellipsoid of the same measure as the unit ball in $\mathbf{R}^n$, and the vectors $a(w)$, $b(z-w)$, and $c(z)$ satisfy

$$c(z) = a(w) + b(z-w) . \tag{2.28}$$

Clearly, the ellipsoid $E(z,w)$ in equation (2.27) cannot depend on z or w. Therefore, there exists an ellipsoid $\hat{E}$ such that $E(z,w) = \hat{E}$ almost everywhere in a neighborhood of (z_0, w_0). Equation (2.28) implies that for small $|h|$

$$d(h) := c(z+h) - c(z) = a(w+h) - a(w) \tag{2.29}$$

is a function of h only. Moreover,

$$\begin{aligned} d(h_1+h_2) &= c(z+h_1+h_2) - c(z) \\ &= c(z+h_1+h_2) - c(z+h_2) + c(z+h_2) - c(z) \\ &= d(h_1) + d(h_2) . \end{aligned} \tag{2.30}$$

The function d is measurable, because $\hat{a}(z)$ is the center of gravity of the cross section $A(z)$,

$$a(z) = \mu(A(z))^{-1} \int_{\mathbf{R}^n} x \mathcal{X}_A(x,z)\, dx .$$

Therefore relation (2.30) implies that d coincides with a linear function except on a set of measure zero. By definition (2.29), for almost all values of z near w_0, $z_0 - w_0$,

z_0, respectively,

$$a(z) = \hat{a} + z\hat{v}\ , \quad b(z) = \hat{b} + z\hat{v}\ , \quad c(z) = \hat{c} + z\hat{v}\ ,$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{v}$ are vectors in $\mathbf{R}^n$, and $\hat{a} + \hat{b} = \hat{c}$. Since I_C is connected, the formula for $c(z)$ holds for almost all $z \in I_C$. Permute the three sets, using (1.8), to see that the formulas for $a(z)$ and $b(z)$ hold for almost all z in I_A and I_B, respectively. This proves assertion (2.26). ∎

Lemma 10 will be used to find cross sections satisfying the strict triangle inequality, when (A, B, C) are triples that can be transformed into ellipsoids by symmetrization. Thus Lemma 10 plays the same role for the proof of Lemma 11 as Lemmas 1 and 2, and in particular equation (2.7) did for the regularized triples.

Lemma 10 *Suppose that α, β, γ satisfy the strict triangle inequality. Then for any z with $|z| < \gamma$ there exists a nonempty open interval such that for w in this interval, the numbers*

$$\sqrt{\alpha^2 - w^2}\ , \quad \sqrt{\beta^2 - (z - w)^2}\ , \quad \sqrt{\gamma^2 - z^2}$$

satisfy the strict triangle inequality.

Proof Fix z. We may assume without loss of generality that $\alpha \geq \beta$, and that $z \geq 0$. Consider the functions

$$\begin{aligned} h_+(w) &:= \sqrt{\alpha^2 - w^2} + \sqrt{\beta^2 - (z - w)^2} \\ h_-(w) &:= \left|\sqrt{\alpha^2 - w^2} - \sqrt{\beta^2 - (z - w)^2}\right| \end{aligned}$$

on the closed interval

$$I = \left\{ w \mid \alpha^2 - w^2 \geq 0,\ \beta^2 - (z - w)^2 \geq 0 \right\}$$

$$= \ [-\alpha, \alpha] \ \cap \ [z-\beta, z+\beta] \, .$$

By definition, the inequality

$$h_-(w) < h_+(w)$$

holds for w in I, with strict inequality in the interior of I. Since h_+ and h_- are continuous, it is enough to find one point w with

$$h_-(w) < \sqrt{\gamma^2 - z^2} < h_+(w) \, . \tag{2.31}$$

Pick

$$w_+ = \frac{\alpha z}{\alpha + \beta} \, , \quad z - w_+ = \frac{\beta z}{\alpha + \beta} \, ,$$

then w_+ lies in I, and

$$h_+(w_+) > \sqrt{\gamma^2 - z^2} \, . \tag{2.32}$$

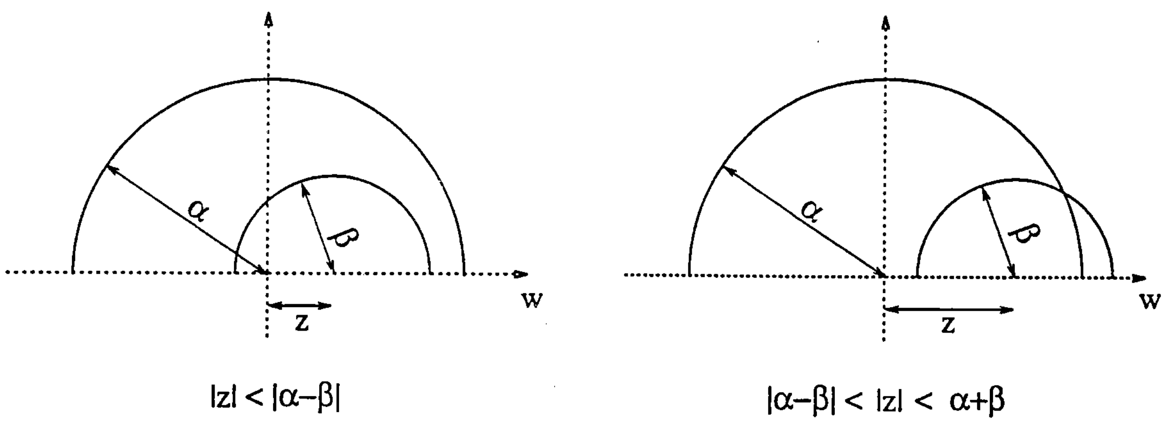

Figure 2.3: Determining w_+ and w_-

If $0 \leq z < \alpha - \beta$, choose

$$w_- := \frac{\alpha z}{\alpha - \beta} \, , \quad z - w_- = -\frac{\beta z}{\alpha - \beta} \, ,$$

if $\alpha - \beta \leq z < \alpha + \beta$, choose w_- to be the point of intersection of the semi-circles described by

$$y = \sqrt{\alpha^2 - w^2}\ , \quad y = \sqrt{\beta^2 - (z-w)^2}$$

(see Figure 2.3).

They do intersect, because the triangle inequality holds between the two radii and the distance of the midpoints. In both cases, w_- lies in I, and

$$h_-(w_-) < \sqrt{\gamma^2 - z^2}\ . \tag{2.33}$$

Define the function

$$h(w) := \frac{w \quad - w_-}{w_+ - w_-} h_+(w) \ + \ \frac{w_+ - w}{w_+ - w_-} h_-(w)$$

on I. Clearly, h is continuous, and

$$h(w_-) = h_-(w_-) < \sqrt{\gamma^2 - z^2} < h_+(w_+) = h(w_+)$$

by (2.32) and (2.33). By the intermediate value theorem there exists a point w between w_- and w_+ with

$$h(w) = \sqrt{\gamma^2 - z^2}\ .$$

Since h is a convex combination of h_- and h_+, inequality (2.31) follows. ■

The following lemma shows that $\mathcal{S}^2$ has the property (R2).

Lemma 11 (Optimizers may be regularized) *Assume that Theorem 1 has been proven for some fixed dimension $n \geq 1$. Let (A, B, C) be an optimizing triple $\mathbf{R}^{n+1}$ so that $\mathcal{S}^2(A, B, C)$ satisfies conclusion (1.10) of Theorem 1.9. Assume that α, β, and γ satisfy the strict triangle inequality.*

Then also (A, B, C) satisfies conclusion (1.10) of Theorem 1.

Proof We will first show that the claim

$$A = a + \alpha E \ , \quad B = b + \beta E \ , \quad C = c + \gamma E \tag{2.34}$$

holds with

$$a + b = c \tag{2.35}$$

under the stronger assumption that either

$$\mathcal{S}_1 A = a' + \alpha E' \ , \quad \mathcal{S}_1 B = b' + \beta E' \ , \quad \mathcal{S}_1 C = c' + \gamma E' \ , \tag{2.36}$$

or

$$\mathcal{S}_2 A = a' + \alpha E' \ , \quad \mathcal{S}_2 B = b' + \beta E' \ , \quad \mathcal{S}_2 C = c' + \gamma E' \ , \tag{2.37}$$

where a', b' and $c' = a' + b'$ are fixed vectors in R^{n+1}.

By the translation symmetry of the functional, we may assume that the centers of gravity of two of the three sets, say A and B, are at the origin, that is,

$$\mu(A)^{-1} \int_A x \, dx \ = \ \mu(B)^{-1} \int_B x \, dx = 0 \ . \tag{2.38}$$

Consider the case that (A, B, C) satisfies (2.36). By definition, the ellipsoid E' is symmetric under rotation about the x^0-axis, so

$$E' = \left\{ \mathrm{x} = (x^0, \hat{x}) \in \mathrm{R}^{n+1} \ \mid \ |\hat{x}|^2 < c_1^2 - c_2^2 x_0^2 \right\} ,$$

where c_1 and c_2 are constants. The cross sections of A have the same measure as the corresponding cross sections of $\mathcal{S}_1 A$. By assumption (2.36),

$$\left(\frac{\mu(A(z))}{\omega_n} \right)^{1/n} = \left(\frac{\mu(\mathcal{S} A(z))}{\omega_n} \right)^{1/n} = \sqrt{(c_1^2 \alpha^2 - c_2^2 z^2)_+} \ . \tag{2.39}$$

The same formula holds for B and C with α replaced by β and γ, respectively. It follows that

$$c_1^{-1}\left(\frac{\mu(A(zc_1/c_2))}{\omega_n}\right)^{1/n} = \sqrt{(\alpha^2 - z^2)_+}$$

and correspondingly for B and C. By Lemma 10, the triple (A, B, C) satisfies the assumptions of Lemma 9 with

$$I_A = (-\alpha c_1/c_2\,,\,\alpha c_1/c_2)\ ,\quad I_B = (-\beta c_1/c_2\,,\,\beta c_1/c_2)\ ,\quad I_C = (-\gamma c_1/c_2\,,\,\gamma c_1/c_2)\ .$$

Assumption (2.38) gives

$$\hat{a} = \mu(A)^{-1}\int_A \hat{x}\ d\mathrm{x}\ =\ 0\ ,\quad \hat{b} = \mu(B)^{-1}\int_B \hat{x}\ d\mathrm{x}\ =\ 0\ ,$$

so, by Lemma 9, also

$$\hat{c} = \hat{a} + \hat{b} = 0\ .$$

To show that A, B, and C are ellipsoids, write the ellipsoid $\hat{E}$ of (2.26) as

$$\hat{E} = \left\{\hat{x} \in \mathbf{R}^n\ \mid\ \hat{Q}(\hat{x}) < 1\right\}$$

where $\hat{Q}$ is a positive definite quadratic form on $\mathbf{R}^n$. Then, by equations (2.26) and (2.39), conclusion (1.10) holds with the ellipsoid

$$E = \left\{\mathrm{x} \in \mathbf{R}^{n+1}\ \mid\ \hat{Q}(\hat{x} - x^0\hat{v}) < c_1^2 - c_2^2 x_0^2\right\}\ ,$$

and $a = b = c = 0$.

In case (A, B, C) satisfies (2.37), note that Steiner symmetrization is a weaker operation than Schwarz symmetrization, and reduce to the first case: Fubini's theorem implies that Steiner symmetrization does not change the measures of n-dimensional cross sections perpendicular to the x^1-axis. Thus, if we take $\mathcal{R}_0$ to be the rotation by

$\pi/2$ that maps the x^0-axis to the x^1-axis and leaves the other coordinate axes fixed, then

$$\mathcal{S}_1\mathcal{R}_0 = \mathcal{S}_1\mathcal{R}_0\mathcal{S}_2 \ .$$

Assumption (2.37) implies

$$\mathcal{S}_1\mathcal{R}_0\, A = \mathcal{S}_1\mathcal{R}_0\mathcal{S}_2 A = \alpha E \ , \quad \mathcal{S}_1\mathcal{R}_0\, B = \beta E \ , \quad \mathcal{S}_1\mathcal{R}_0\, C = \gamma E \ .$$

In other words, the rotated triple $\mathcal{R}_0\,(A, B, C)$ satisfies assumption (2.36). Applying the first case shows that (2.34) holds.

Recall that by definition (2.14),

$$\mathcal{S} = \mathcal{S}_2\mathcal{S}_1\mathcal{R}$$

where $\mathcal{R}$ is a rotation. The claim (2.34) follows from the rotational invariance of $\mathcal{J}$, and the two results just proved. ∎

2.3.3 Identifying ellipsoids

The following two lemmas will be used to show that optimizers must be triples of ellipsoids that are related by scaling and translations. The first lemma is a local result. In the proof, we will derive and solve a differential equation for the boundaries of the three sets near the equatorial hyperplane $x^0 = 0$. The second lemma gives a continuation argument, which will be used to show that A, B, and C are globally ellipsoids.

Lemma 12 (Ellipses, local) *Let A be an open set in $\mathbf{R}^2$ of the form*

$$A = \Big\{(x, y) \in \mathbf{R}^2 \ \mid \ |y| < \alpha(x)\Big\} \tag{2.40}$$

where α is an even, bounded, nonnegative function which is nonincreasing $x \geq 0$, and continuous at 0. Consider the intersections of A with the family of lines

$$x = my + t \tag{2.41}$$

for $|m| < \varepsilon$, $|t| < \varepsilon$, where $\varepsilon > 0$ is a fixed number.

Assume that there exists a family of lines

$$y = b(m)x \tag{2.42}$$

with the following property: For all m with $|m| < \varepsilon$, the intersection of A with almost all lines (2.41) with $|t| < \varepsilon$ differs by a set of one-dimensional measure zero from a line segment whose midpoint lies on the line (2.42).

Then there exists a constant c, and $\delta > 0$ such that for $|x| < \delta$

$$\alpha^2(x) = \alpha^2(0) - c^2x^2 \ , \tag{2.43}$$

that is, the intersection of A with the vertical strip $|x| < \delta$ coincides with the intersection of an ellipsoid, or of a horizontal strip with this strip. The constant c, which determines the shape of the ellipsoid, is given by

$$c^2 = -\frac{b(m)}{m} \tag{2.44}$$

for any nonzero value of m.

Proof (see Figure 2.4) We will first show that

$$A_\delta = \left\{(x, y) \in A \ \mid \ |x| < \delta\right\} .$$

is convex if δ is small enough. We need to find through each boundary point of A_δ a line, so that A_δ is contained in one of the closed half-spaces defined by that line. Such

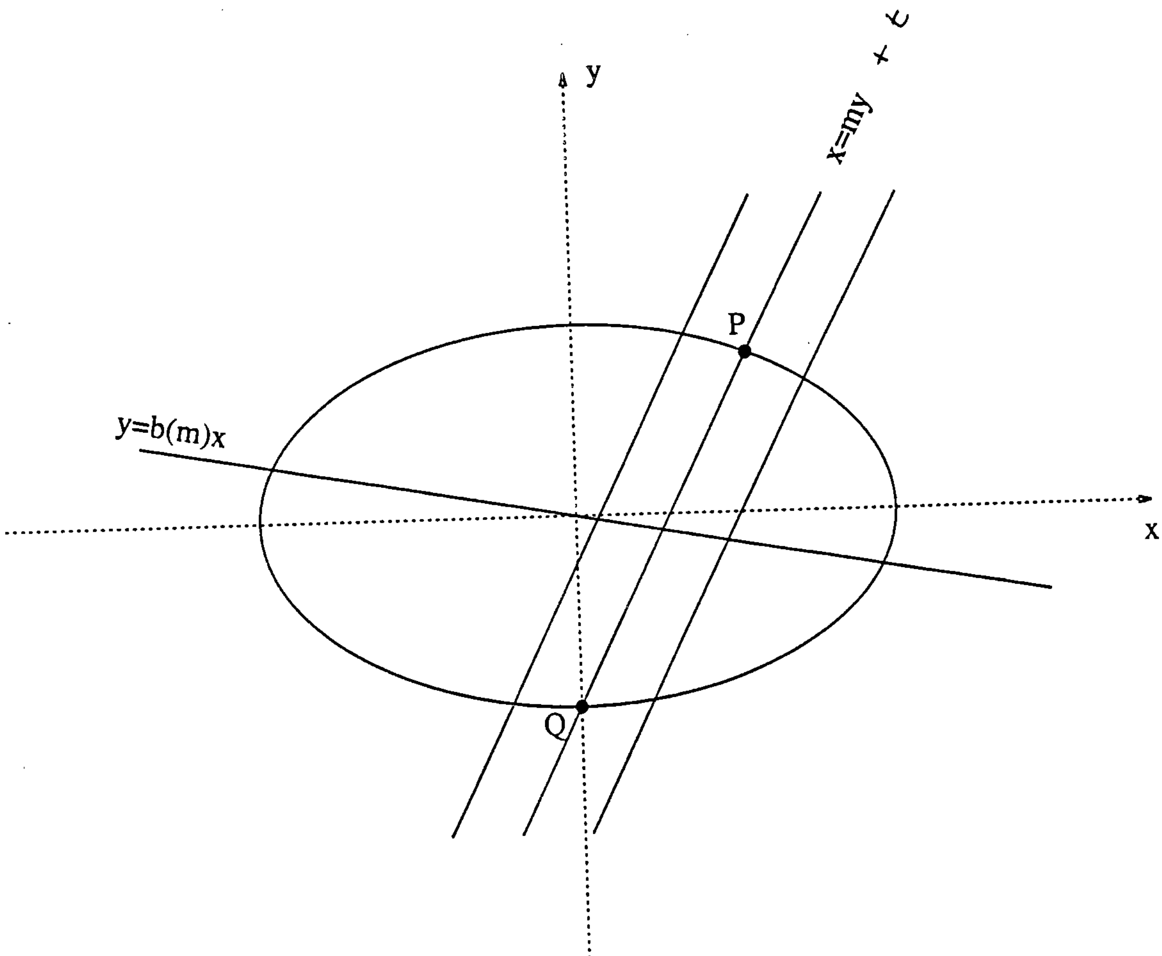

Figure 2.4: A property characterizing ellipses

a line (even if it is not unique) will be called a **tangent** of A at the given boundary point.

Let $Q = (0, -\alpha(0))$ be the point of the boundary of A with the smallest y-coordinate. Clearly, A does not intersect the lower of the two half-spaces defined by the line $y = \alpha(0)$ through Q. Consider a point of the form $P = (x, \alpha(x))$ in the boundary of A. The line joining P and Q is given by equation (2.41) with

$$m = \frac{x}{\alpha(x) + \alpha(0)} , \quad t = -m\alpha(0) . \tag{2.45}$$

If x is close enough to zero, then $|t| < \varepsilon$, $|m| < \varepsilon$.

Fix m as in (2.45). Consider the linear transformation $\mathcal{L}$ which fixes the line $y = b(m)x$ pointwise, and maps all points on the line $x = my$ to their negatives. Then $\mathcal{L}$ is conjugate to a reflection at a line. Its matrix is given by

$$\mathcal{L}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1 - m\,b(m)} \begin{pmatrix} 1 + m\,b(m) & -2m \\ 2b(m) & -1 - m\,b(m) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.46}$$

By assumption, for all t with $|t| < \varepsilon$, the intersection of A with the lines (2.41) differs only by a set of one-dimensional measure zero from a line segment centered on the line $y = b(m)x$. In other words, the intersections of A and $\mathcal{L}A$ with the strip

$$\{(x, y) \mid |my - x| < \varepsilon\}$$

differ by a set of measure zero. By definition, A, and consequently $\mathcal{L}A$ consist exactly of their Lebesgue points, so the two sets must coincide. In particular, $\mathcal{L}$ maps Q to P, and the half-space below the line $y = \alpha(0)$ to a half-space containing P in its boundary that does not meet A in a neighborhood of P. It follows that A_δ is convex provided δ is chosen small enough.

By construction, $\mathcal{L}$ maps tangents at P into tangents at Q, so there is a unique tangent at P if and only if there is a unique tangent at Q. Since P was an arbitrary point on the upper arc of the boundary of A_δ, and since, by convexity, all but countably many points on this arc have a unique tangent, every such point has a unique tangent. Hence, α is differentiable everywhere on the interval $(-\delta, \delta)$.

We can now derive a differential equation for α. Consider a pair of points $P = (x, \alpha(x))$ on the upper arc, and $Q = (z, -\alpha(z))$ on the lower arc of the boundary of A_δ. By assumption (2.42), the slope of the line though the midpoint of the line segment PQ,

$$b = \frac{\alpha(x) - \alpha(z)}{x + z}, \tag{2.47}$$

is a function of the slope of the line segment,

$$m = \frac{x - z}{\alpha(x) + \alpha(z)}.$$

Both the midpoint and the slope are differentiable functions of (x, z) for $x + z \neq 0$. Since b is a function of m, the gradients of $b(x, z)$ and $m(x, z)$ must be linearly dependent. We calculate these gradients at $(x, 0)$ $(x \neq 0)$

$$\begin{aligned} \nabla b(x,0) &= x^{-2} \begin{pmatrix} x\alpha'(x) - \alpha(x) + \alpha(0) \\ -\alpha(x) + \alpha(0) \end{pmatrix} \\ \nabla m(x,0) &= (\alpha(x) + \alpha(0))^{-2} \begin{pmatrix} \alpha(x) + \alpha(0) - x\alpha'(x) \\ -\alpha(x) - \alpha(0) \end{pmatrix}, \end{aligned}$$

where we have used that $\alpha'(0) = 0$ since α is even. They are linearly dependent, if

$$\big(x\alpha'(x) - \alpha(x) + \alpha(0)\big)\big(\alpha(x) + \alpha(0)\big) = \big(\alpha(x) - \alpha(0)\big)\big(\alpha(x) + \alpha(0) - x\alpha'(x)\big).$$

Collecting terms, we see that α satisfies the differential equation

$$x(\alpha^2(x))' = -2(\alpha^2(0) - \alpha^2(x)).$$

The general solution of this differential equation with $\alpha(x) \leq \alpha(0)$ is given by (2.43). Inserting (2.43) into (2.47) gives the formula (2.44) for c. ■

Lemma 13 (Ellipses, continuation) *Let $A \subset \mathbb{R}^2$ be as in equation (2.40) Lemma 12. Assume that there exists a number $\delta > 0$ with $c\delta < \alpha(0)$, so that for $x \in [-\delta, \delta]$, the function α is given by formula (2.43). Let*

$$m_0 = \frac{\delta}{\sqrt{\alpha^2(0) - c^2\delta^2}}.$$

Assume moreover that there exist $\varepsilon > 0$ and a function $b(m)$ such for all m with $|m - m_0| < \varepsilon$ and for almost all t with $|t| < \varepsilon$ the intersection of A with the line given by (2.41) differs by a set of one-dimensional measure zero from a line segment whose midpoint lies on the line given by (2.42).

Then equation (2.43) holds on an open neighborhood of $[-\delta, \delta]$.

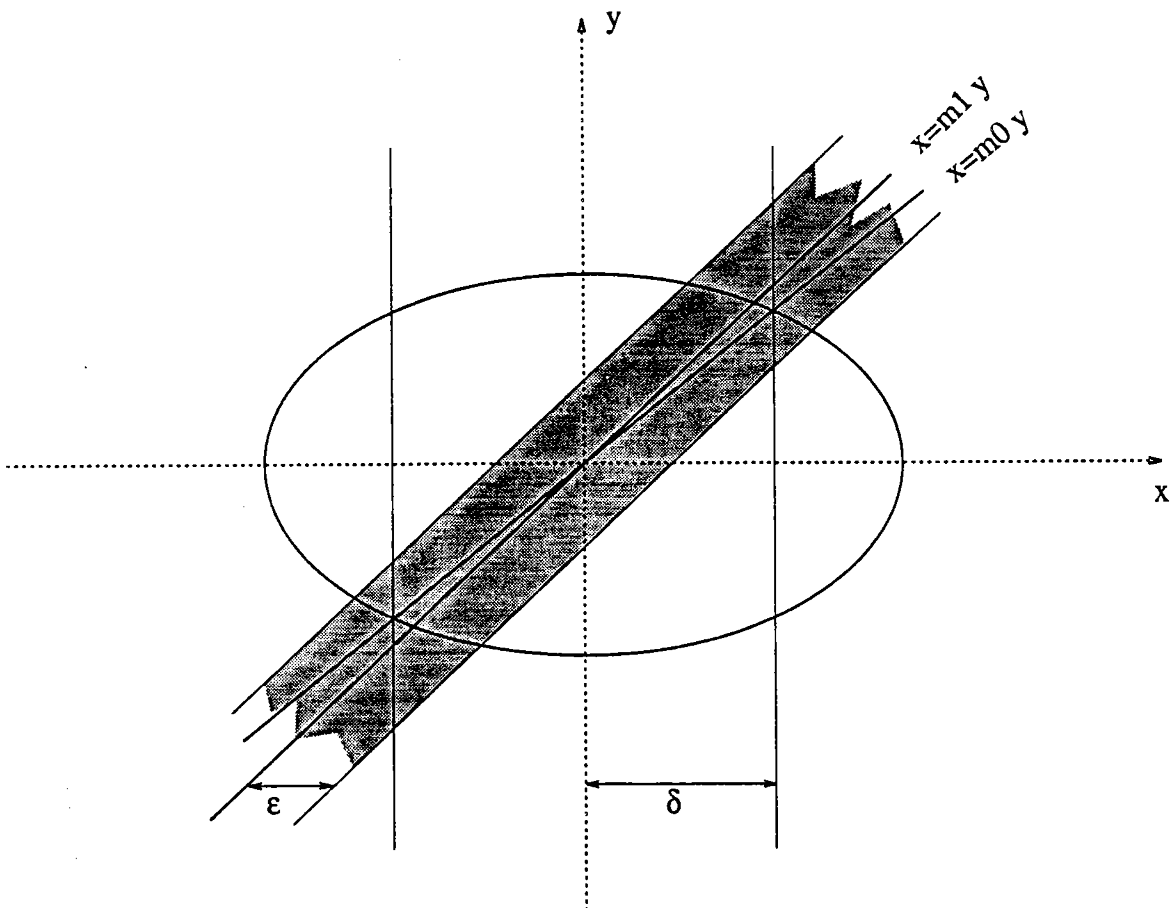

Figure 2.5: The continuation argument

Proof (see Figure 2.5) For $0 < m < m_0$ define

$$\varepsilon' = \delta - m\sqrt{\alpha^2(0) - c^2\delta^2} > 0$$

Choose m with $m_0 - \varepsilon < m < m_0$ close enough to m_0 that $\varepsilon' < \varepsilon$. By assumption, the intersection of A with lines with parameters (m, t), $|t| < \varepsilon$ consists of line segments whose midpoints are given by (2.42). In other words, the intersection of A with the strip

$$\left\{(x, y) \in \mathbb{R}^2 \mid |x - my| < \varepsilon\right\}$$

differs from its image under the skewed reflection $\mathcal{L}$ constructed in the proof of Lemma 12 by a set of measure zero. Since A consists exactly of its Lebesgue points, it follows that the strip is symmetric under $\mathcal{L}$. The upper arc of the strip with

$$-\varepsilon < x - my < \varepsilon'$$

and the lower arc with

$$-\varepsilon' < x - my < \varepsilon$$

consist of points $(x, \alpha(x))$, and $(x, -\alpha(x))$, respectively, where α is given by formula (2.43). Since the strip is symmetric under L, also the image of he lower arc under $\mathcal{L}$ is described by a quadratic equation. The image of the lower arc intersects the upper arc in an open arc since the line $x = my$ intersects both arcs. Consequently, (2.43) holds for the whole part of the boundary of A with $|x - my| < \varepsilon$. Repeating the argument for $m = -m_0$ proves the claim. ∎

2.4 Proof of Theorem 1 in higher dimensions

Following the outline at the beginning of the chapter, we will do a proof by induction. The base case was discussed in Section 2.2.

Suppose that Theorem 1 has been proven for dimensions up to n.

Proof of the inequality Let (A, B, C) be a triple of measurable sets of finite positive measure in $\mathbf{R}^{n+1}$. By Lemma 3,

$$\mathcal{J}(A,B,C) \leq \mathcal{J}\left(\mathcal{S}(A,B,C)\right) .$$

Since $\mathcal{J}$ is continuous in A, B, C with respect to symmetric difference, we have by Lemma 6, that

$$\begin{aligned} \mathcal{J}(A,B,C) \quad &\leq \quad \mathcal{J}\left(\mathcal{S}^k(A,B,C)\right) \\ &\longrightarrow \quad \mathcal{J}(A^*,B^*,C^*) \quad (k \to \infty). \end{aligned}$$

This shows the inequality in $\mathbf{R}^{n+1}$.

Identification of the cases of equality Let (A, B, C) be an optimizing triple of inequality (1.9) in $\mathbf{R}^{n+1}$, so that α, β, γ satisfy the strict triangle inequality.

Consider the regularized triple $\mathcal{S}^2(A, B, C)$, where $\mathcal{S}$ is the symmetrization operation defined by (2.14), which consists of a rotation followed by a Schwarz and a Steiner symmetrization operation. By Lemma 4, $\mathcal{S}^2(A, B, C)$ is again an optimizing triple.

We will find triples of cross sections satisfying the strict triangle inequality. By Lemma 8, $\mathcal{S}^2A$, $\mathcal{S}^2B$, and $\mathcal{S}^2C$ are rotational solids:

$$\begin{aligned} \mathcal{S}^2A \quad &= \quad \left\{\mathrm{x} \in \mathbf{R}^{n+1} \mid |\hat{x}| < \alpha(x^0)\right\} \\ \mathcal{S}^2B \quad &= \quad \left\{\mathrm{x} \in \mathbf{R}^{n+1} \mid |\hat{x}| < \beta(x^0)\right\} \\ \mathcal{S}^2C \quad &= \quad \left\{\mathrm{x} \in \mathbf{R}^{n+1} \mid |\hat{x}| < \gamma(x^0)\right\} \end{aligned}$$

The functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ are even, nonnegative, bounded, and nonincreasing for positive arguments, and continuous at 0.

Recall that Lemmas 1 and 2 imply that

$$\text{closure}\left(\mathcal{S}^2 C\right) \subset \mathcal{S}^2 A + \mathcal{S}^2 B$$

(see equation (2.7)). Hence,

$$\gamma(0) < \sup_{w \in \mathbb{R}} \left\{ \alpha(w) + \beta(-w) \right\} \leq \alpha(0) + \beta(0) .$$

By the symmetry (1.8), α, β, and γ may be permuted in this inequality to see that $\alpha(0)$, $\beta(0)$, $\gamma(0)$ satisfy the strict triangle inequality.

To show that the strict triangle inequality holds for cross sections nearby, we will use the continuity results from Lemma 7. Since by Lemma 7, the measures of the intersections of $\mathcal{S}^2 A$, $\mathcal{S}^2 B$ and $\mathcal{S}^2 C$ with hyperplanes of the form

$$x^0 = m x^1 + t \tag{2.48}$$

are jointly continuous in (m, t) at $(0, 0)$, there exists $\varepsilon > 0$ so that these intersections satisfy the strict triangle inequality for (m, t) with $|m| < \varepsilon$, and $|t| < \varepsilon$.

To apply the inductive hypothesis in form of Lemma 9, fix m with $|m| < \varepsilon$. Since Lemma 9 applies only to cross sections perpendicular to the x^0-axis, rotate the three sets simultaneously in the x^0-x^1-plane (fixing the other coordinates), so that the cross sections defined by (2.48) are rotated into cross sections of the rotated sets perpendicular to the x^0-axis. The rotated sets satisfy the assumptions of Lemma 9 with $I_A = I_B = I_C = (-\varepsilon', \varepsilon')$, where ε' can be chosen independently of m for m small. By Lemma 9, the intersections of $\mathcal{S}^2 A$, $\mathcal{S}^2 B$ and $\mathcal{S}^2 C$ with hyperplanes of the form (2.48) with $|m| < \varepsilon'$ and $|t| < \varepsilon'$ are ellipsoids. Since $\mathcal{S}^2 A$, $\mathcal{S}^2 B$, and $\mathcal{S}^2 C$ are symmetric about the origin, by equations (4.11) and (4.12), for every fixed m, the midpoints of these ellipsoids all lie on one line through the origin. Since by the rotational symmetry of the three sets, this line lies in the x^0-x^1-plane, the equation of this line can be written as

$$x^1 = b(m)x^0 \,, \quad x^i = 0 \; (i > 1) \,. \tag{2.49}$$

We have just seen that near the equatorial hyperplane $x^0 = 0$, the assumptions of Lemma 12 are satisfied for all three regularized sets with the same function $b(m)$ in (2.49). By Lemma 12, the parts of the three sets near the equatorial hyperplane $x^0 = 0$ look like similar ellipsoids. That is, there exists $\delta > 0$ so that

$$\begin{aligned} \alpha^2(x) &= \alpha^2(0) - c^2x^2 \,, \quad && \text{if } |x| < \alpha(0)\delta \\ \beta^2(x) &= \beta^2(0) - c^2x^2 \,, \quad && \text{if } |x| < \beta(0)\delta \\ \gamma^2(x) &= \gamma^2(0) - c^2x^2 \,, \quad && \text{if } |x| < \gamma(0)\delta \end{aligned} \tag{2.50}$$

where the constant c is the same for the three sets by (2.44). Note that the quadratic curves in (2.50) differ only by scale factors $\alpha(0) : \beta(0) : \gamma(0)$.

To show that equations (2.50) hold globally, we will make a continuation argument, using Lemma 13. Set

$$\delta_0 = \sup\Big\{\delta \;\mid\; (2.50 \text{ holds for } \delta\Big\} \,. \tag{2.51}$$

Assume that $c\delta_0 < 1$. Then $\sqrt{1 - c^2\delta_0^2} > 0$. We have just shown that $\delta_0 > 0$. Define as in Lemma 13

$$m_0 = \frac{\delta_0}{\sqrt{1 - c^2\delta_0^2}}$$

to be the slope of the hyperplane passing through the end points of the parts of the boundaries of the three regularized sets that are described by the quadratic equations (2.50). Since hyperplanes with $|m| < m_0$ meet only those parts of the three regularized sets that are described by (2.50), these intersections are in the fixed scaling proportions $\alpha(0) : \beta(0) : \gamma(0)$, so the intersections with the hyperplane $x^0 = mx^1$ satisfy the strict triangle inequality by the continuity results from Lemma 7. It follows moreover from Lemma 7 again that the sizes of cross sections with hyperplanes with m near m_0, and $|t|$ small enough satisfy the strict triangle inequality.

Now apply the inductive hypothesis in form of Lemma 9 again. Fix m near m_0, and rotate the three sets simultaneously in the x^0-x^1-plane so that the intersections of $\mathcal{S}^2(A, B, C)$ with hyperplanes (2.48) are mapped to cross sections of the rotated sets perpendicular to the x^0-axis. It follows from Lemma 9 applied to the rotated triple that all three sets satisfy the assumptions of Lemma 13. So (2.50) holds on a neighborhood of $[-\delta_0, \delta_0]$, which contradicts definition (2.51). It follows that either $c\delta_0 = 1$, or $c = 0$ and $\delta_0 = +\infty$. In the first case, $\mathcal{S}^2(A, B, C)$ is a triple of ellipsoids that are related by scale factors $\alpha(0) : \beta(0) : \gamma(0)$. Clearly, these factors must be in proportion $\alpha : \beta : \gamma$, which proves conclusion (1.10) in this case. In the second case, the three sets would have to be infinite strips, which contradicts the assumption that they have finite measure. This proves Theorem 1 for regularized sets.

So far we have shown that for any optimizing triple (A, B, C) where α, β, γ satisfy the strict triangle inequality, the regularized sets satisfy

$$\mathcal{S}^2 A = \alpha E \ , \quad \mathcal{S}^2 B = \beta E \ , \quad \mathcal{S}^2 C = \gamma E \ ,$$

where E is a centered ellipsoid. By Lemma 11, conclusion (1.10) also holds for the original sets (A, B, C). This completes the proof of Theorem 1. ∎

Chapter 3

Implications of Theorem 1

In this chapter, we will discuss various corollaries of Theorem 1. The first section contains everything that follows with the layer-cake principle from Theorem 1. We will prove that the Riesz rearrangement inequality (1.3) holds for any triple of nonnegative measurable functions for which the spherically decreasing rearrangement is defined (Theorem 1.3). We then characterize the optimizing triples of this inequality in the special cases given by Theorems 3 and 4.

The second section is very short: It only contains an application of Theorem 4 to the problem of the optimizers of the weak Young inequality. In the third section, we discuss a dual version of inequality (1.3). Finally, we generalize Theorems 1–4 to the case of multiple convolutions.

Throughout this chapter, only sketches of proofs will be given, since all the results follows with standard techniques from Theorem 1.

3.1 Proof of Theorems 2–4

We begin by describing the **layer-cake principle**. Let f be a nonnegative measurable function. Then we can write

$$f(x) = \int_0^\infty \mathcal{X}_{f(x)>s}\, ds \tag{3.1}$$

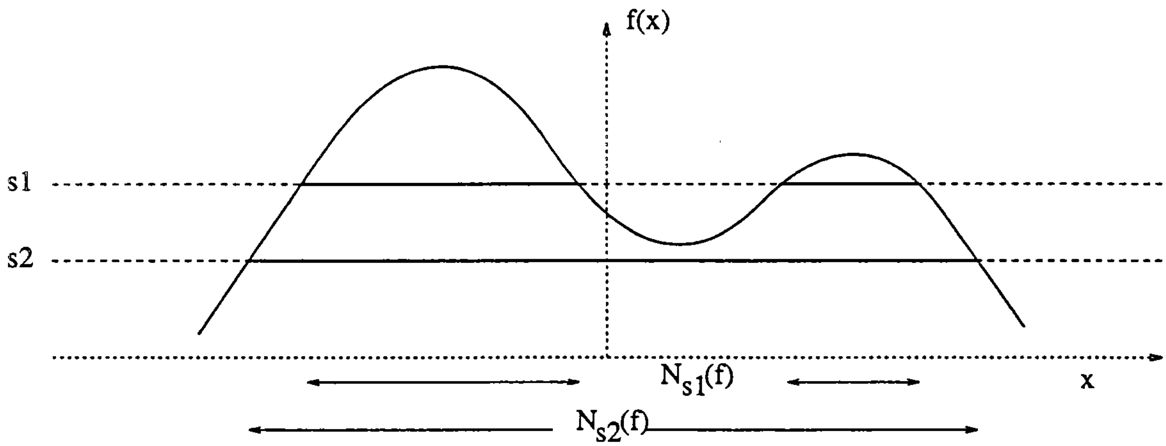

Figure 3.1: The layer-cake representation

(see Figure 3.1). Note that the integrand is jointly measurable in the two variables x and s.

Let f, g, and h be three nonnegative measurable functions on $\mathbf{R}^n$ so that all their level sets corresponding to positive values have finite measure. We can use the representation 3.1) and Fubini's theorem to write the functional $\mathcal{I}$ in terms of the level sets of the three functions:

$$\begin{aligned}
\mathcal{I}(f,g,h) &= \iint f(x)g(x-y)h(w)\,dydx \\
&= \iint \iiint \mathcal{X}_{f(x)>r}\mathcal{X}_{g(x-y)>s}\mathcal{X}_{h(z)>t}\,drdsdt\,dydx \\
&= \iiint \mathcal{J}\left(\mathcal{N}_r(f),\mathcal{N}_s(g),\mathcal{N}_t(h)\right)\,drdsdt\ .
\end{aligned}$$

Another useful notion is the distribution function of a nonnegative measurable function.

Definition Let f be a nonnegative measurable function on $\mathbf{R}^n$ whose level sets for positive values all have finite measure. The value of the **distribution function,** $\mathcal{D}_f$

of f at s is defined to be the Lebesgue measure of the level set of f at height s,

$$\mathcal{D}_f(s) := \mu(\mathcal{N}_s(f)) .$$

Definition Two measurable functions that give rise to the same distribution function are called **equimeasurable.**

It is easy to see that two equimeasurable functions will have the same L^p-norms.

Lemma 14 (Properties of the distribution function) *Let f be a nonnegative measurable function on $\mathbb{R}^n$, with the property, that all level set corresponding to positive values have finite measure. Then the distribution function is nonincreasing and continuous from the right.*

Proof The distribution function is nonincreasing, because $s_1 \leq s_2$ implies that $\mathcal{N}_{s_1}(g^*) \supset \mathcal{N}_{s_2}(g^*)$. To see that the function is semicontinuous, fix any value $s_0 > 0$. By definition of the level sets, we have

$$\mathcal{N}_{s_0}(f) = \bigcup_{s>s_0} \mathcal{N}_s(f)$$

and hence, that

$$\mathcal{D}_f(s_0) = \mu(\mathcal{N}_{s_0}(f)) = \lim_{s\downarrow s_0} \mu(\mathcal{N}_s(f)) = \lim s \downarrow s_0 \mathcal{D}_f(s) .$$

This shows that the distribution function is continuous from the right. ∎

Lemma 15 (Rearrangement preserves the distribution function) *Let f be a nonnegative measurable function on $\mathbb{R}^n$, with the property, that all level set corresponding to positive values of f have finite measure. Then f and f^* are equimeasurable, that is,*

$$\mathcal{D}_f(s) = \mathcal{D}_{f^*}(s) \quad \textit{for all } s > 0 .$$

Remark This is a general property of rearrangements. It holds in particular also for Steiner and Schwarz symmetrization of functions, which can be defined using the layer-cake representation.

Proof Since both the distribution functions of f and f^* are nonincreasing and lower semicontinuous, it is enough to show that they coincide on a dense set of values s. Let s_0 be a point where the distribution function of f^* is continuous.

Fix $\varepsilon > 0$. Choose x to be a point in the ball $\mathcal{N}_{s_0}(f^*)$ which is so close to the boundary that

$$\omega_n |x|^n > \mu(\mathcal{N}_{s_0}(f^*)) - \varepsilon .$$

By definition (1.2) of the spherically decreasing rearrangement, we have that

$$\begin{aligned} f^*(x) > s \implies & \exists s > s_0 : \mu(\mathcal{N}_s(f)) \geq \omega_n |x|^n \\ \implies & \mu(\mathcal{N}_{s_0}(f)) \geq \mu(\mathcal{N}_{s_0}(f^*)) - \varepsilon . \end{aligned}$$

Conversely, since the distribution function of f^* is continuous at s_0, we can find a point x so that

$$f^*(x) < s_0 \text{ and } \omega_n |x|^n < \mu(\mathcal{N}_{s_0}) + \varepsilon .$$

This implies that

$$\mu(\mathcal{N}_{s_0}(f)) < \omega_n |x|^n < \mu(\mathcal{N}_{s_0}(f^*)) + \varepsilon .$$

Since ε was arbitrary, this implies that the two distribution functions coincide at s_0.

∎

Proof of Theorem 2 We use the layer-cake representation to decompose f, g, and h, and their spherically decreasing rearrangements into the characteristic functions of their level sets. By Lemma 15, inequality (1.3) is equivalent to

$$\iiint \mathcal{J}\left(\mathcal{N}_r(f), \mathcal{N}_s(g), \mathcal{N}_t(h)\right) \;\leq\; \iiint \mathcal{J}\left(\mathcal{N}_r(f)^*, \mathcal{N}_s(g)^*, \mathcal{N}_t(h)^*\right)\, drdsdt\ ,$$

which follows directly from inequality (1.9). For equality we need that for (Lebesgue-) almost all triples (r, s, t) of positive numbers, the level sets $\mathcal{N}_r(f)$, $\mathcal{N}_s(g)$, and $\mathcal{N}_t(h)$ produce equality in inequality (1.9). ∎

Proof of Theorem 3 We will use an argument that is very similar to the inductive step in the proof of Theorem 1.9 (see the introduction of Chapter 2 and Section 2.4).

By Theorem 2, equality in inequality (1.3) implies that almost all triples of level sets of f, g, and h produce equality in inequality (1.9). We want to apply conclusion (1.10) of Theorem 1 to these triples of level sets. Thus, we need to find triples of level sets for which the strict triangle inequality holds.

Consider the distribution function of g^*. From

$$\bigcap_{s>0} \mathcal{N}_s(g^*) = \emptyset$$

it follows that

$$\lim_{s\to\infty} \mu(\mathcal{N}_s(g^*)) = 0\ . \tag{3.2}$$

Similarly, since g^* is strictly spherically decreasing, we have

$$\bigcup_{s>0} \mathcal{N}_s(s^*) = \mathbf{R}^n$$

and hence, that

$$\lim_{s\to 0} \mu(\mathcal{N}_s(g^*)) = \infty\ . \tag{3.3}$$

Moreover, the distribution function is continuous because g^* is strictly decreasing by assumption. By the intermediate value theorem, (3.2) and (3.3) show that g^* has level sets of any size.

It follows that for any $r, t > 0$ such that the level sets $\mathcal{N}_r(f)$ and $\mathcal{N}_t(h)$ have positive measure, there exists a nonempty open interval of values s such that $\mathcal{N}_r(f)$, $\mathcal{N}_s(g)$, $\mathcal{N}_t(h)$ satisfy the strict triangle inequality. By Theorem 1, these level sets can be written as

$$\begin{aligned} \mathcal{N}_r(f) &= a(r) &+\mathcal{L}(r,s,t)\mathcal{N}_r(f)^* \ , \\ \mathcal{N}_s(g) &= b(s) &+\mathcal{L}(r,s,t)\mathcal{N}_s(g)^* \ , \\ \mathcal{N}_t(h) &= c(t) &+\mathcal{L}(r,s,t)\mathcal{N}_t(h)^* \ , \end{aligned} \tag{3.4}$$

and the centers satisfy

$$a(r) + b(s) = c(t) \ . \tag{3.5}$$

Since we assumed that $g = g^*$, we may take $\mathcal{L}(r,s,t) = I$, and $b(s) = 0$ in (3.4). It follows from (3.5) that $a(r)$ and $c(t)$ must be constant. This shows the claim (1.16).

■

Proof of Theorem 4 Again, we can apply Theorem 2 to see that almost all triples of level sets of f, g, and h produce equality in (1.9).

To find triples of level sets satisfying the strict triangle inequality, we use that by assumption, f^* and h^* are strictly spherically decreasing. By the proof of Theorem 3, they have level sets of all sizes. By Lemma 15, the distribution functions of f and h coincide with the distribution functions of f^* and h^* .

Fix $r_0 > 0$ and $s_0 > 0$ such that $\mathcal{N}_{r_0}(f)$ and $\mathcal{N}_{s_0}(g)$ have positive measure. Since h^* is strictly spherically decreasing and has level sets of all sizes, there exists t_0 such that the radii of the level sets of f^*, g^*, and h^* at heights r_0, s_0, and t_0, respectively, satisfy the strict triangle inequality. It follows from the continuity of the distribution functions of f^* and h^* that the strict triangle inequality holds for level sets corresponding to values r, s_0, t with r and t in an open neighborhood of r_0, t_0. By conclusion (1.15) of Theorem 1, these level sets can be written as

$$\begin{aligned} \mathcal{N}_r(f) &= a(r) &+\mathcal{L}(r,s_0,t)\mathcal{N}_r(f)^* , \\ \mathcal{N}_{s_0}(g) &= b(s_0) &+\mathcal{L}(r,s_0,t)\mathcal{N}_{s_0}(g)^* , \\ \mathcal{N}_t(h) &= c(t) &+\mathcal{L}(r,s_0,t)\mathcal{N}_t(h)^* , \end{aligned} \tag{3.6}$$

where for each value of r and t, $\mathcal{L}$ is a linear map of determinant one on $\mathbf{R}^n$, and a and c are vectors in $\mathbf{R}^n$ with

$$a(r) + b(s_0) = c(t) \tag{3.7}$$

In relation (3.6), the image of a centered ball under $\mathcal{L}(r, s_0, t)$ cannot depend on r and t near r_0 and t_0. Similarly, by equation (3.7), a and c are locally constant. Since the set

$$\Big\{(r,t) \ \big|\ \text{the strict triangle inequality holds for } \mathcal{N}_r(f), \mathcal{N}_{s_0}(g), \mathcal{N}_t(h)\Big\}$$

is connected, a and c must be constant, and L can be chosen constant, too. Since formulas (3.6) and (3.7) hold for any other level set of g, neither L nor b can depend on s. This shows the claim (1.17). ∎

3.2 An application: The weak Young inequality

The weak Young inequality states that for any three measurable functions f, g, and h, for which the right hand side is finite,

$$\left|\iint f(y)g(x-y)h(x)\,dydx\right| \leq C(p,\lambda,n)\,\|f\|_p\,\|g\|_{w,q}\,\|h\|_r \ , \tag{3.8}$$

where $1 < p, q, r < \infty$, $1/p + 1/q + 1/r = 2$, and $\lambda = n/q$, and and $C(p, \lambda, n)$ is the best constant. The weak q-norm of g is given by

$$\|g\|_{w,q} := \sup_{s>0} s \left(\frac{\mu(\mathcal{N}_s(|g|))}{\omega_n} \right)^{1/q} .$$

In [20], Lieb proved the weak Young inequality as a corollary of the Hardy-Littlewood-Sobolev inequality

$$\left| \iint f(y)\, |x-y|^{-\lambda}\, h(x)\, dy dx \right| \leq C(p, \lambda, n)\, \|f\|_p\, \|h\|_r . \tag{3.9}$$

The best constant $C(p, \lambda, n)$ is the same as in (3.8). In the proof, he also showed that the spherical rearrangements of all optimizers of the Hardy-Littlewood-Sobolev inequality are *strictly* decreasing functions. We will prove the following statement, using Theorem 4.

Theorem 5 *There is equality in the weak Young inequality (3.8), if and only if (f^*, h^*) is an optimizing pair of the Hardy-Littlewood-Sobolev inequality, and, moreover, there exist a linear map of determinant one, $\mathcal{L}$, and vectors a, b, and $c = a + b$ such that*

$$\begin{aligned} g(x) &= const. \left|\mathcal{L}^{-1}x - b\right|^{-\lambda} , \\ f(x) &= f^*(\mathcal{L}^{-1}x - a) , \\ h(x) &= h^*(\mathcal{L}^{-1}x - c) . \end{aligned}$$

In particular, if $p = r = 2n/(2n - \lambda)$, then, up to multiplication by constants and the symmetries (1.7), the optimizers are given by

$$g(x) = |x|^{-\lambda} ,$$

$$
\begin{aligned}
f(x) &= h(x) = (1+|x|^2)^{-n/p} , \\
C(p,\lambda,n) &= \pi^{\lambda/2}\frac{\Gamma(n/2-\lambda/2)}{\Gamma(n-\lambda/2)}\left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{-1+\lambda/n} .
\end{aligned}
$$

Proof Clearly, the left hand side can only increase if the functions f, g, and h are replaced by $|f|$, $|g|$, and $|h|$, while the right hand side stays the same. If at least one of the three functions changes signs, then the left hand side increases strictly. Hence we may assume that the three functions are nonnegative.

Following the argument in [20], we have that (3.8) can be obtained from the chain of inequalities

$$
\begin{aligned}
\iint f(y)g(x-y)h(x)\,dy dx &\le \iint f^*(y)g^*(x-y)h^*(x)\,dy dx \\
&\le \|g\|_{w,q} \iint f^*(y)\,|x-y|^{-\lambda}\,h^*(x)\,dy dx \\
&\le C(p,\lambda,n)\,\|f\|_p\,\|g\|_{w,q}\,\|h\|_r ,
\end{aligned}
$$

where the first line is the Riesz rearrangement inequality (1.3), the second follows from the definition of the weak norm, and the third is the Hardy-Littlewood-Sobolev inequality (3.9). For equality in (3.8), there has to be equality in all three lines. By Theorem 2.3(ii) from [20], equality in the Hardy-Littlewood-Sobolev inequality implies that f^* and h^* are strictly spherically decreasing. Equality in the second line implies that $g^*(x) = \|g\|_{w,p}\,|x|^{-\lambda}$. By Theorem 4, equality in the first line implies the claim. ■

3.3 The dual inequality

In this section, we discuss the inequality

$$
\int \left(f * g(x) - s\right)_+ dx \le \int \left(f^* * g^*(x) - s\right)_+ dx , \tag{3.10}
$$

which is related to inequality (1.3) in the following way: We can write

$$(y-s)_+ = \sup_{0\le x\le 1} (y-s)x \ ,$$

and consequently

$$(f(x)-s)_+ = \sup_{0\le h\le 1} (f(x)-s)h(x) \ . \tag{3.11}$$

The supremum is achieved by choosing h to be the characteristic function of the level set $\mathcal{N}_s(f)$. Note, that the convex function $(y-s)_+$ corresponds to

$$x \longmapsto \begin{cases} sx & \text{if } 0 \le x \le 1 \\ +\infty & \text{if } x > 1 \end{cases}$$

under Legendre transformation.

Inequality (3.10) follows by inserting expression (3.11) into inequality (1.3). We call the inequality a dual of inequality (1.3), because, by Corollary 2, which will be proved at the end of the section, and the layer-cake principle, it implies inequality (1.3). Moreover, Theorems 1–4 can be recovered from Theorems 1′–4′ with Corollary 2, using Lemma 2, and, in case of Theorem 1′, the Brunn-Minkowski inequality.

The functional in inequality (3.10) has the following symmetries.

$$\int \left(\tilde{f} * \tilde{g}(x) - s\right)_+ dx \ = \ \int \left(f * g(x) - s\right)_+ dx \ .$$

whenever $\tilde{f}$ and $\tilde{g}$ are given by

$$\tilde{f}(x) = f(\mathcal{L}^{-1}(x-a)) \ , \quad \tilde{g}(x) = g(\mathcal{L}^{-1}(x-b)) \ ,$$

where $\mathcal{L}$ is a linear map on $\mathbf{R}^n$, and a and b are vectors in $\mathbf{R}^n$. We can also permute f and g without changing the value of $\mathcal{I}'$.

We now state the dual version of Theorem 1.

Theorem 1′ (Dual rearrangement inequality for characteristic functions) *For any two measurable sets A and B of finite positive measure in $\mathbf{R}^n$, the following inequality holds.*

$$\int \left(\mathcal{X}_A * \mathcal{X}_B(x) - s\right)_+ dx \leq \int \left(\mathcal{X}_{A^*} * \mathcal{X}_{B^*}(x) - s\right)_+ dx \tag{3.12}$$

There is equality if and only if either $s = 0$, or $s \geq min\{\mu(A), \mu(B)\}$, or there exist vectors a and b in $\mathbf{R}^n$, and a linear map $\mathcal{L}$ of determinant one, so that

$$A = a + \mathcal{L}A^* , \qquad B = b + \mathcal{L}A^* .$$

We delay the proof until after the proof of the dual version of Theorem 2.

Theorem 2′ (Dual rearrangement inequality) *Inequality (3.10) holds for any pair of nonnegative measurable functions f and g with spherical rearrangements f^* and g^*, and any number $s > 0$.*

Let h be the characteristic function of the level set $\mathcal{N}_s(f * g)$. There is equality, if and only if (f, g, h) is an optimizing triple of inequality (1.3).

Proof Let h be as in the assumptions. Then

$$\begin{aligned} \int \left(f * g(x) - s\right)_+ dx &= \int f * g(x)\, h(x)\, dx - s \int h(x)\, dx && \text{by definition of } h \\ &\leq \int f^* * g^*(x)\, h^*(x)\, dx - s \int h(x)\, dx && \text{by inequality (1.3)} \\ &\leq \int (f^* * g^*(x) - s)_+\, dx && \text{by (3.11)}\ . \end{aligned}$$

Clearly, equality implies that there is equality in the second line. ■

Proof of Theorem 1′ Inequality (3.12) is a special case of inequality (3.10), which we just proved. To identify the cases of equality, note that if

$$0 < s < \max_x \mathcal{X}_{A^*} * \mathcal{X}_{B^*} = min\{\mu(A), \mu(B)\}$$

then $\mathcal{N}_s(f^* * g^*)$ satisfies the strict triangle inequality with A and B. Theorem 1 implies the claim. ∎

The following two theorems are the dual versions of Theorems 3 and 4.

Theorem 3′ (Cases of equality, one strictly spherically decreasing function) *If $f = f^*$ is strictly spherically decreasing and $s > 0$, then equality in (3.10) implies that g is a translate of a spherically decreasing function.*

Theorem 4′ (Cases of equality, two strictly decreasing rearrangements) *If both f^* and g^* are strictly decreasing, then equality in inequality (3.10) implies that there exists $\mathcal{L}$ so that $f \circ \mathcal{L}^{-1}$ and $g \circ \mathcal{L}^{-1}$ are translates of spherically decreasing functions.*

Corollary 1 *Let G be a nonnegative convex function with $G(0) = 0$. Then, for any pair of measurable nonnegative functions f and g with spherical rearrangements f^* and g^*, the following inequality holds.*

$$\int G \circ (f * g)\, dx \;\leq\; \int G \circ (f^* * g^*)\, dx\ . \tag{3.13}$$

If G is strictly convex, then equality implies that there exists a linear map $\mathcal{L}$ so that all level sets of $f \circ \mathcal{L}^{-1}$, $g \circ \mathcal{L}^{-1}$, and $(f * g) \circ \mathcal{L}^{-1}$ are balls. If moreover f^* is strictly decreasing, then equality implies that $f \circ \mathcal{L}^{-1}$ and $g \circ \mathcal{L}^{-1}$ are translates of f^* and g^*.

Proof The one-sided derivative

$$G'(0) := \lim_{y\downarrow 0} \frac{G(y) - G(0)}{y}$$

is finite, since it is the limit of a nonnegative nonincreasing sequence. We can write

$$G(y) = G'(0)y \; + \int_0^\infty (y - s)_+ \, d\mu(s)$$

where μ is the measure associated with the nondecreasing function G', which is defined everywhere except for at most countably many points. Inequality (3.13) follows with Fubini's theorem and Theorem 2′ from

$$\int G \circ (f * g)\, dx \;=\; G'(0) \int f * g \, dx \;+\; \int\int_0^\infty (f * g - s)_+ \, dx \, d\mu \,.$$

To verify the claims about the cases of equality, note that, if G is strictly convex, then the measure μ is positive. Equality in the inequality implies that there is equality in (3.10) for almost all s.

We want to show that for any pair of level sets $A = \mathcal{N}_r(f)$ and $B = \mathcal{N}_s(g)$, there exists level sets $\mathcal{N}_t(f * g)$ whose sizes satisfy the strict triangle inequality with the two given level sets. Assume that r_0 and s_0 have been chosen so that the distribution functions of f and g are continuous at r and t, respectively (this is satisfied for all but a countable number of choices, since the distribution functions are nonincreasing).

We can write $f^* * g^*$ as the sum of nonincreasing functions

$$f^* * g^*(x) = \int\int\int \mathcal{X}_{f(y)>s} \mathcal{X}_{g(x-y)>t} \, dy \, ds dt$$

with the layer-cake representation. Fix now two level sets $A = \mathcal{N}_s(f)$ and $B = \mathcal{N}_t(g)$, and denote by α and β the radii of A^* and B^*, respectively. Since $\mathcal{X}_{A^*} * \mathcal{X}_{B^*}$ is strictly decreasing between $|\alpha - \beta|$ and $\alpha + \beta$, it follows that also $f^* * g^*$ is strictly decreasing in that interval. It follows that it has level sets that satisfy the strict triangle inequality with A and B. By Theorem 1, there exists $\mathcal{L}$ such that $A = a + \mathcal{L}A^*$, $B = b + \mathcal{L}B^*$. Since this must hold for any pair of level sets, $\mathcal{L}$ cannot depend on the level. This shows the claim in the first case.

If moreover f^* is strictly decreasing, then also $f^* * g^*$ is strictly decreasing. The argument in the proof of Theorem 3 with the level sets of h replaced by the level sets of $f * g$ gives the result. ∎

The next corollary of Theorem 2′ will be useful in the following section. It is a slight generalization of the Riesz rearrangement inequality (1.3).

Corollary 2 *For any three nonnegative measurable functions f, g, and h for which the right hand side is defined and nonzero, the following inequality holds.*

$$\int (f * g)^* \, h^* \, dx \leq \int f^* * g^* \, h^* \, dx \; .$$

Proof By the layer-cake principle, it is enough to prove the inequality in case h is the characteristic function of a measurable set C. Set

$$s := \inf_{x \in C^*} f^* * g^* \, (x) \; .$$

Then

$$\begin{aligned} \int (f * g)^* h^* \, dx &= \int ((f * g)^* - s) h^*(x) \, ds \; + \; s \int h(x), dx \\ &\leq \int (f * g - s)_+ \, dx \; + \; s \int h(x) \, dx \qquad \text{by (3.11)} \\ &\leq \int (f^* * g^* - s)_+ \, dx \; + \; s \int h^*(x) \, dx \qquad \text{by Theorem 2} \\ &\leq \int f^* * g^* h^* \, dx \qquad \text{by definition of } s \; . \end{aligned}$$

∎

3.4 An inequality for multiple convolutions

Consider the following generalization of inequality (1.3).

$$\begin{aligned}\mathcal{I}_k(f_1\dots,f_k,g,h) := \mathcal{I}(f_1*\dots*k_k,g,h) &= \int f_1*\dots*f_k*g\,h\,dx \\ &\le \mathcal{I}_k(f_1^*,\dots,f_k^*,g^*,h^*)\,, \qquad (3.14)\end{aligned}$$

where $f_1,\dots,f_k,g,h$ are nonnegative measurable functions on $\mathbf{R}^n$ with spherical rearrangements $f_1^*,\dots f_k^*,g^*,h^*$, and $k\ge 1$. For $k=1$, the inequality reduces to (1.3). We will show that Theorems 1–4 hold for this inequality, with some minor changes. We will only sketch the proofs, since the results are straightforward generalizations of Theorems 1–4

Note that in case $k=0$, this inequality reduces to the well known inequality

$$\int g(x)h(x)\,dx \le \int g^*(x)h^*(x)\,dx\,.$$

We do not include this inequality in our discussion, because it is not really a geometric inequality. First, it holds for any rearrangement, not just for the spherically decreasing rearrangement, of the two functions. Secondly, there is equality, if the two functions have essentially the same level sets (more precisely, if for any pair of level sets of the two functions, one of them is contained in the other one). This is satisfied, for example, if $g(x)$ is a nondecreasing function of $h(x)$. In other words, there are no restrictions on the shape of level sets of optimizers.

It follows directly from the definition of $\mathcal{I}_k$ and the symmetries (1.4) and (1.5) of $\mathcal{I}$, that the functional $\mathcal{I}_k$ has the following symmetries. Let $\mathcal{L}$ be a linear map on $\mathbf{R}^n$ of determinant ±1, and $a_1,\dots,a_k,b,c$ be vectors in $\mathbf{R}^n$ with

$$\sum_{i=1}^k a_i + b \;=\; c\,.$$

Then, for $\tilde{f}_i$, $\tilde{g}$, and $\tilde{h}$ defined by

$$\begin{aligned} \tilde{f}_i(x) &= f(\mathcal{L}^{-1}(x-a_i)) \quad (i=1,\dots k)\ , \\ \tilde{g}(x) &= = g(\mathcal{L}^{-1}(x-a_i)), \end{aligned}$$

we have that

$$\mathcal{I}_k(\tilde{f}_1\dots,\tilde{f}_k,\tilde{f}g,\tilde{h}) = \mathcal{I}_k(f_1\dots,f_k,g,h) \tag{3.15}$$

(see (1.7)). Similarly, the functional is symmetric under permutation of the first k indices, and

$$\mathcal{I}_k(f_1,f_2\dots,f_k,g,h) = \mathcal{I}_k(h^-,f_2\dots,f_k,g^-) \tag{3.16}$$

(see (1.8)). Moreover, by the associativity of the convolution operation,

$$\mathcal{I}_{k+1}(f_1,f_2\dots,f_{k+1},g,h) = \mathcal{I}_k(f_1,\dots,f_k,f_{k+1}*g,h) \tag{3.17}$$

We first consider the analogue of Theorem 1. We define $\mathcal{J}_k$ as a functional of $k+2$ measurable sets by setting

$$\mathcal{J}_k(A_1\dots,A_k,B,C) := \mathcal{I}_k(\mathcal{X}_{A_1}\dots,\mathcal{X}_{A_k},\mathcal{X}_B,\mathcal{X}_C)\ . \tag{3.18}$$

Clearly, $\mathcal{J}$ inherits the symmetries of $\mathcal{I}$. We will assume as before that $k\geq 1$.

The strict triangle inequality generalizes in the following way. We will say that a set of (at least three) positive numbers satisfies the **strict polygon inequality**, if

$$\alpha_i < \sum_{j\neq i}\alpha_j \qquad \text{for all } i\ , \tag{3.19}$$

that is, if these numbers could form the side lengths of a polygon with interior.

Theorem 1″ (Characteristic functions) *Let $A_1,\dots,A_k,B,C$ be measurable sets of finite positive measure in $\mathbf{R}^n$. Then the following inequality holds for the functional*

$\mathcal{J}_k$ defined by (3.18).

$$\mathcal{J}_k(A_1,\ldots,A_k,B,C) \le \mathcal{J}_k(A_1^*,\ldots,A_k^*,B^*,C^*)\ . \tag{3.20}$$

Denote by $\alpha_1,\ldots,\alpha_k,\beta,\gamma$ the radii of the balls $A_1^*,\ldots,A_k^*,B^*,C^*$. If these radii satisfy the strict polygon inequality (3.19) then there is equality in (3.20) if and only if, up to sets of measure zero,

$$\begin{aligned} A_i &= a_i + \alpha_i E \quad (i=1,\ldots k)\ , \\ B &= b + \beta E\ , \\ C &= c + \gamma E\ , \end{aligned} \tag{3.21}$$

where E is a centered ellipsoid, and $a_1,\ldots,a_k,b,c$ are vectors in $\mathbf{R}^n$.

Otherwise, permute the three sets so that $\gamma \ge \sum_{i=1}^k \alpha_i + \beta$, using (3.16). Then there is equality in (3.20) if and only if the sets $A_1,\ldots,A_k,B,C$ can be changed by sets of measure zero so that

$$C \supset A_1 + \cdots A_k + B\ . \tag{3.22}$$

where "+" denotes is the Minkowski sum. In particular, for $\gamma = \sum_{i=1}^k \alpha_i + \beta$, there is equality in (3.20) if and only if, up to sets of measure zero,

$$\begin{aligned} A_i &= a_i + \alpha_i M \quad (i=1,\ldots k)\ , \\ B &= b + \beta M\ , \\ C &= c + \gamma M\ , \end{aligned} \tag{3.23}$$

where $M \subset \mathbf{R}^n$ is convex and open, and $a_i,\ldots,a_k,b,c$ are vectors in $\mathbf{R}^n$.

As for Theorem 1, if the radii $\alpha_1, \ldots, \alpha_k, \beta, \gamma$ satisfy the strict polygon inequality, then equality in inequality (3.20) implies that the sets $A_1, \ldots, A_k, B, C$ are equivalent to centered balls under the symmetries (3.15). The following result generalizes Theorem 2.

Theorem 2″ (Multiple convolution inequality) *Inequality (3.14) holds for any $(k+2)$ nonnegative measurable functions $f_1, \ldots f_k, g, h$ on $\mathbb{R}^n$ with spherically decreasing rearrangements $f_1^*, \ldots, f_k^*, g^*, h^*$.*

There is equality in inequality (3.14), if and only if for (Lebesgue-) almost all $k+2$-tuples $r_1, \ldots, r_k, s, t$ of positive numbers, the level sets $\mathcal{N}_{r_1}(f_1), , \ldots \mathcal{N}_{r_k}(f_k), \mathcal{N}_s(g)$, $\mathcal{N}_t(h)$ produce equality in inequality (3.20).

Proof By induction over k. We already mentioned that the inequality reduces to inequality (1.3) for $k = 1$. Assume the inequality has been proven for some $k \geq 1$. Then

$$\begin{aligned}
\mathcal{I}_{k+1}(f_1, \ldots, f_{k+1}, g, h) &= \mathcal{I}_k(f_1, \ldots f_k, f_{k+1} * g, h) \\
&\leq \mathcal{I}_k(f_1^*, \ldots, f_k^*, (f_{k+1} * g)^*, h^*) \\
&\qquad \text{by the inductive assumption} \\
&= \mathcal{I}(f_1^* \cdots * f_k^*, (f_{k+1} * g)^*, h^*) \\
&\qquad \text{by definition of } \mathcal{I}_k \\
&= \int (f_1^* * \cdots f_k^* \, h^*)(f_{k+1} * g)^* \, , dx \\
&\leq \mathcal{I}_{k+1}(f_1^*, \ldots, f_{k+1}^*, g^*, h^*) \\
&\qquad \text{by Corollary 2 .}
\end{aligned}$$

The statement about the cases of equality follows with the layer-cake principle. ∎

Proof of Theorem 1″ Inequality (3.20) is a special case of inequality (3.14), which we just proved.

To identify the cases of equality when the radii satisfy the strict polygon inequality, repeat the induction over k from the proof of Theorem 3.14″. Suppose conclusion (3.21) has been proved for some fixed $k \geq 1$. If $A_1, \ldots A_{k+1}, B, C$ produce equality in (3.20), then $\mathcal{X}_{A_1} * \ldots * \mathcal{X}_{A_k}$ together with $\mathcal{X}_{A_k+1} * \mathcal{X}_B$ and $\mathcal{X}_C$ produce equality in inequality (1.3). The strict polygon inequality for $\alpha_1, \ldots, \alpha_{k+1}, \beta, \gamma$ guarantees that convolution $\mathcal{X}_{A_k} * \mathcal{X}_B$ has level sets satisfying the strict polygon inequality with $\alpha_1, \ldots, \alpha_k$ and γ (recall that the convolution of the characteristic functions of A^*_{k+1} and B^* is strictly spherically decreasing for radii between $\alpha_{k+1} + \beta$ and $|\alpha_{k+1} - \beta|$). By inductive assumption, there exist a linear map, $\mathcal{L}$, and vectors $a_1 \ldots, a_k, c$, so that $A_1 \ldots, A_k, C$ satisfy conclusion (3.21). Permuting the sets gives the result.

In case the radii do not satisfy the polygon inequality, we use that the support of the convolution $\mathcal{X}_{A_1} * \cdots * \mathcal{X}_{A_{k+1}}$ is the (essential) Minkowski sum of the sets. This shows (3.22). The Brunn-Minkowski inequality gives the claim (3.23). ■

Since the convolution of a strictly spherically decreasing function with a spherically decreasing function is always strictly spherically decreasing, the following two statements can be obtained easily from Theorems 3 and 4 with the inductive argument from Theorem 2″.

Theorem 3″ (Cases of equality, one strictly spherically decreasing function) *Let $f_1 \ldots, f_k, g, h$ be nonnegative measurable functions on $\mathbf{R}^n$, with spherically decreasing rearrangements $f_1^*, \ldots f_k^*, g^*, h^*$. Assume that one of these functions is strictly spherically decreasing. Then $f_1, \ldots, f_k, g, h$ produce equality in inequality (3.14), if* *and only if they are equivalent to their rearrangements* $f_1^*, \ldots, f_k^*, g^*, h^*$ *under the symmetries (3.16) with* $= \mathcal{L} = 1$.

Theorem 4″ (Cases of equality, two strictly decreasing rearrangements)
Let $f_1 \ldots, f_k, g, h$ *be nonnegative measurable functions on* $\mathbf{R}^n$*, with spherically decreasing rearrangements* $f_1^*, \ldots f_k^*, g^*, h^*$*. Assume that at least two of the rearrangements are strictly spherically decreasing. Then* $f_1, \ldots, f_k, g, h$ *produce equality in inequality (3.14), if and only if the functions are equivalent to their rearrangements under the symmetries (3.15).*

Corollary 3 *The dual versions, Theorems 1′–4′, and Corollaries 1 and 2 from Section 3.3 also hold for multiple convolutions.*

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