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NEWTON POLYHEDRA AND TOROIDAL VARIETIES

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We consider an algebraic variety X defined in space $(C \setminus 0)^n$ by a nondegenerate system of polynomial equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$. The general problem consists of calculating the discrete invariants of variety X in terms of polyhedra Δ (see [1]). Here we carry out the preparation for such calculations.

Space $(C \setminus 0)^n$ is compactified by means of imbedding in a compact nonsingular toroidal variety M^n. Given polyhedra Δ, we choose a compactification M^n such that closure $\overline{X}$ of variety $X \subset M^n$ is a nonsingular variety manifold that is transversal to all the orbits of variety M^n.

The toroidal compactification $(C \setminus 0)^n \subset M^n$ plays the same role as the projective compactification $C^n \subset CP^n$ in the classical case. Toroidal varieties are well known [2, 3]. It is almost as easy to handle them as projective spaces. In a subsequent paper the geometry of toroidal varieties will be used for the calculation of the arithmetic genus and Euler characteristic of variety X. Here we discuss the connection of this geometry with the elementary geometry of integral polyhedra.

I have been influenced by [2-8] and by personal contact with V. I. Arnol'd and A. G. Kushnirenko. I especially thank D. N. Bernshtein and B. Ya. Kazarnovskii for useful discussions and V. L. Popov for a series of lectures on toroidal varieties.

§1. Toroidal Compactifications

In this section we discuss smooth compact toroidal varieties in a form that is necessary for us. We do not present proofs: they are either contained in Chap. 1 of [2] or can be easily derived from it.

1. The Torus, Its Characters and One-Parameter Groups. The space $(C \setminus 0)^n$ is the n-dimensional complex space C^n with coordinates $z = z_1, \ldots, z_n$ from which all coordinates planes have been removed, i.e., $z \in (C \setminus 0)^n$, if $z_1 \neq 0, \ldots, z_n \neq 0$. The space $(C \setminus 0)^n$ is an algebraic group with respect to componentwise multiplication. Such a group is called an n-dimensional torus and is denoted by T^n. In $(C \setminus 0)^n$ we fix a system of coordinates z. The group T^n with the fixed system of coordinates is called a standard torus.

We consider the group of algebraic characters, i.e., the algebraic homomorphisms of $(C \setminus 0)^n$ into the standard torus $(C \setminus 0)$ with coordinate t, $\chi: (C \setminus 0)^n \to (C \setminus 0)$. Each such character is a monomial, i.e., a function of form $t = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n}$, where α_i are any integers. We number the monomials (characters) by means of the integral vectors $\alpha = \alpha_1, \ldots, \alpha_n$ in the fixed n-dimensional real space R^n and use the brief notation $z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n} = z^\alpha$.

We consider the group of algebraic one-parameter groups, i.e., algebraic homomorphisms $\lambda: (C \setminus 0) \to (C \setminus 0)^n$. Each such homomorphism has the form $z_1 = t^{\xi_1}, \ldots, z_n = t^{\xi_n}$, where ξ_i are integers, or, briefly, $z = t^\xi$, $\xi = \xi_1, \ldots, \xi_n$. We number the one-parameter groups λ by integral points ξ of the fixed space R^{n*}. The fixing of a system of coordinates z in $(C \setminus 0)^n$ fixes systems of coordinates in R^n and R^{n*}. One-parameter groups λ_i of form $z_i = t^{\delta_{ij}}$ will be called basic one-parameter groups.

There is a natural scalar product $\langle \chi, \lambda \rangle$, between the one-parameter groups λ and the characters χ, viz., $\langle \chi, \lambda \rangle$ is equal to the degree of the composite homomorphism $\chi\lambda: (C \setminus 0) \to$

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$(C \setminus 0)$. In the coordinates this product has the standard form $\langle(\alpha_1, \ldots, \alpha_n), (\xi_1, \ldots, \xi_n)\rangle = \sum\alpha_i \xi_i$. Therefore, the scalar product $\langle\chi, \lambda\rangle$ can be extended to spaces R^n and R^{n*}. From this point the elements of space R^n will be called vectors and the elements of space R^{n*} covectors.

2. Isomorphisms of Standard Tori. Now let T^n be another standard torus, located in space C_1^n with coordinates $u = u_1, \ldots, u_n, u \in T^n$ for $u_1 \neq 0, \ldots, u_n \neq 0$.

We consider an algebraic isomorphism φ of torus T^n into torus $(C \setminus 0)^n$, φ: $T^n \to (C \setminus 0)^n$. In the coordinates such an isomorphism is written by the equations $z_i = u_1^{a_{1i}} \cdot \ldots \cdot u_n^{a_{ni}}$, where $A = a_{ij}$ is an integral matrix and $|\det A| = 1$. Conversely, for each such matrix A we can write an isomorphism φ. With each isomorphism φ we associate a cone σ in space R^{n*}.

Definition. A simple cone in R^{n*} is a cone σ consisting of linear combinations $\sum c_i\xi_i$ of covectors $\xi_1, \ldots, \xi_n$ with nonnegative real coefficients $c_i \geqslant 0$, if the covectors ξ_i are integral and form a basis in the integral lattice of space R^{n*}.

Let covectors ξ_i number the images $\mu_i = \varphi\lambda_i$ of the basic one-parameter groups λ_i of the standard torus T^n under the isomorphism φ. We associate with isomorphism φ the simple cone generated by covectors ξ_i.

Conversely, for a given simple cone σ it is possible to generate an isomorphism φ: $T^n \to (C \setminus 0)^n$ (to within isomorphisms $\varkappa : T^n \to T^n$ that renumber the coordinates). For this, it is necessary simply to number the minimal integral covectors ξ_i on the edges of cone σ and form a matrix A from them. Integral points ξ of cone σ have a simple meaning. These points label precisely those one-parameter groups $\lambda(t)$ along which the coordinate functions u_i of point $\varphi^{-1}\lambda(t)$ remain bounded as $t \to 0$. A basic role is played by the one-parameter groups μ_i labeled by covectors ξ_i. Along such one-parameter groups $z = \mu_i(t)$ and along their shifts $z = z_0\,\mu_i(t)$, $z_0 \in (C \setminus 0)^n$, function u_i is proportional to t, while the remaining functions u_j, $j \neq i$, remain nonzero constants.

3. The Compactification Associated with the Isomorphisms. Suppose that we are given a finite family of isomorphisms φ_m of a standard torus $T^n \subset C_1^n$ into the torus $(C \setminus 0)^n$.

Definition. A completion of space $(C \setminus 0)^n$, compatible with the family φ_m, is an analytic variety M^n containing $(C \setminus 0)^n$, such that 1) the isomorphisms $\varphi_m : T^m \to (C \setminus 0)^n$ are extendible to regular imbeddings $\varphi_m : C_1^m \to M^n$, 2) the regions $U_m = \varphi_m(C_1^n)$ cover all of M^n.† A compact completion is called a compactification.

No more than one completion M^n is compatible with a family of isomorphisms φ_m. Frequently, there are no such completions at all.

A finite set of simple cones $\{\sigma_m\}$ is called admissible if distinct cones intersect only along the faces. An admissible set gives a regular decomposition of space R^{n*} if $\bigcup \sigma_m = R^{n*}$.

THEOREM. Completion M^n compatible with isomorphisms φ_m exists if and only if the corresponding set of cones $\{\sigma_m\}$ is admissible. The completion is compact if and only if $\bigcup \sigma_m = R^{n*}$.

This theorem makes it possible to associate with each regular decomposition of space R^{n*} a compact variety M^n. We describe its simplest properties. In regions U_m the mapping $\varphi_m : C_1^n \to M^n$ gives a local system of coordinates. The imbedding $(C \setminus 0)^n \to M^n$ is a birational isomorphism. The inverse mapping in the coordinates of region U_m coincides with rational mapping $\varphi_m : C_1^n \to (C \setminus 0)^n$. The action of torus $(C \setminus 0)^n$ on itself can be extended to manifold M^n (therefore, manifolds M^n are called toroidal). Under the action of the torus, M^n is decomposed into a finite number of orbits. Regions U_m are made up of orbits. Under the mapping $\varphi_m^{-1}: U_m \to C_1^n$ the orbits correspond to the coordinate planes in C_1^n, with the smallest coordinate planes omitted. Each orbit is a torus of dimension $\leqslant n$. The closure of each orbit is a smooth toroidal variety of the same dimension. The basic role for us is played by the orbits of dimension $n - 1$. There exists a one-to-one correspondence between orbits T_α^{n-1} of dimension $n - 1$ and minimal integral covectors ξ_α on the edges of a regular decomposition of R^{n*}. Closures O_α of orbits T_α^{n-1} are transversally intersecting hypersur-

†We frequently denote by one letter functions and their extensions and restrictions.

faces in $M^n, M^n = (C\setminus 0)^n \bigcup_\alpha O_\alpha$. The order of the zero of a meromorphic function $f: M^n \to C$ on O_α is equal to the order of the zero of the function $f(z_0\,\mu_\alpha(t))$ at the point $t = 0$. Here z_0 is a general point in $(C \setminus 0)^n$ and μ_α is the one-parameter group labeled by covector ξ_α.

4. Integral Polyhedra Δ, Their Support Functions l, and Sufficiently Full Varieties M^n. An integral polyhedron Δ is a convex polyhedron in R^n with vertices at integral points. In the course of this paper we encounter only such polyhedron. The support function l_Δ is the function on R^{n*} defined by $l_\Delta(\xi) = \min_{x\in\Delta} \langle \xi \cdot x\rangle$. Suppose that we are given a finite set of polyhedra Δ_i. We say that a regular decomposition of space R^{n*} into simple cones is sufficiently fine for the set of polyhedra if the support functions of all the polyhedra are linear on each cone of the decomposition. The corresponding toroidal variety M^n will be called sufficiently full for the polyhedra Δ_i. By the methods of Chap. 1 of [2], it is not hard to prove that for any finite set of polyhedra Δ_i there is a sufficiently fine decomposition. Moreover, this decomposition can be chosen so that the corresponding sufficiently full variety M^n is projective.

§2. Resolution of Singularities

1. Nonsingular Systems of Functions. For each polyhedron $\Delta \subset R^n$ and covector ξ we define the polyhedron Δ^ξ to be the face of the polyhedron Δ on which the function $\langle \xi \cdot x\rangle$ attains a minimum (in particular, $\Delta^0 = \Delta$).

A Laurent polynomial $f: (C\setminus 0)^n \to C$ is defined to be a finite linear combination of characters, i.e., $f(z) = \sum c_\alpha z^\alpha$.

The Newton polyhedron $\Delta(f)$ of a Laurent polynomial f is defined to be the convex hull of the points $\alpha \in R^n$, for which $c_\alpha \neq 0$. For each Laurent polynomial f and covector ξ we define the Laurent polynomial f^ξ, $f^\xi = \sum_{\alpha\in\Delta^\xi} c_\alpha z^\alpha$ (in particular, $f^0 = f$).

Definition. We say that a system of Laurent polynomials $f_1, \ldots, f_k$ is nonsingular for its Newton polyhedra $\Delta_1, \ldots, \Delta_k$, if for any covector $\xi \in R^{n*}$ the following condition (ξ) holds: for any solution z of system $f_1^\xi = \ldots = f_k^\xi = 0$, in $(C\setminus 0)^n$, differentials df_i^ξ are linearly independent in the tangent space to point z.

2. THEOREM (Resolution of Singularities). The condition of nonsingularity holds for almost all systems $f_1, \ldots, f_k$ with the polyhedra $\Delta_1, \ldots, \Delta_k$. If the condition of nonsingularity holds and the variety M^n is sufficiently full for the polyhedra $\Delta_1, \ldots, \Delta_k$, then closure $\overline{X}$ of variety $X \subset M^n$ is a nonsingular variety that is transversal to the orbits of variety M^n.

For the proof we need one auxiliary assertion. Let C_1^n be an n-dimensional complex space, C_I the set of its coordinate planes (of which there are 2^n), and $\pi_I: C_1^n \to C_I$ the set of projections.

Assertion. Let $g: C_1^n \to C^k$ be an analytic mapping. With it we associate the 2^n mappings $g\pi_I: (C \setminus 0)^n \to C^k$. It is asserted that almost every point $c \in C^k$ is noncritical for all mappings $g\pi_I$. For such a point c the variety $g^{-1}(c)$ is nonsingular in C_1^n and transversal to all the coordinate planes.

The first part of the assertion follows from Sard's lemma, and the second part is easily checked.

We proceed to the proof of the theorem. Let $\{\sigma_m\}$ be a sufficiently fine decomposition of space R^{n*} and M^n the variety corresponding to it. We take some cone σ_m of the decomposition. We show that conditions (ξ) hold almost always for $\xi \in \sigma_m$ and that their satisfaction guarantees the nonsingularity of variety X and the transversality of variety $\overline{X}$ to the orbits lying in chart $U_m \subset M^n$.

Let $\tilde{f}_i(u)$ be the Laurent polynomial obtained from $f_i(z)$ by the substitution $z = \varphi_m(u)$, and $\tilde{\Delta}_i$ its Newton polyhedron. The polyhedron $\tilde{\Delta}_i$ has a vertex α_i on which all the coordinate functions attain a minimum: this follows from the condition of linearity of support function l_{Δ_i} on cone σ_m. Function $\tilde{f}_i(u)\cdot u^{-\alpha_i} = \tilde{g}_i(u)$ is thus a polynomial with nonzero free term. Let $g_i(u) = \tilde{g}_i(u) - c_i$, where c_i is the free term of polynomial $\tilde{g}_i(u)$. On the torus

$u_1 \neq 0, \ldots, u_n \neq 0$ the system of equations $\tilde{f}_1 = \ldots = \tilde{f}_k = 0$ can be written in the form $g_i(u) = c_i$. The theorem now reduces to the auxiliary assertion for the mapping $g: C_1^n \to C^k$, defined by $c_i = g_i(u)$. Condition (ξ) coincides for $\xi \in \sigma_m$ with the condition that the point c is not critical for one of the mappings $g\pi_I : (C \setminus 0)^n \to C^k$.

Remarks. 1. It is clear from the theorem that the variety X is nonsingular in the general case. The singularity of variety X lies in its noncompactness, and we resolve this singularity.

2. It is clear from the proof that we can attain nonsingularity of the system by changing the coefficients of only the monomials corresponding to the vertices of the Newton polyhedra.

3. Singular systems have real codimension not less than 2. Therefore, it is possible to pass continuously from any nonsingular system to any other in such a way that the variety $\overline{X}$ always remains smooth and transversal to the orbits of the variety M^n. Many invariants of the variety X do not change under such a deformation: e.g., its differential type and the Euler characteristics with coefficients in certain sheaves do not change. Therefore, all the invariants of this kind depend on the Newton polyhedra $\Delta_1, \ldots, \Delta_k$ and do not depend on the specific choice of a nonsingular system of polynomials $f_1, \ldots, f_k$.

4. The local behavior of the system of functions $f_1, \ldots, f_k$ around the point 0 in C^n is determined on the whole by the parts of the Newton polyhedra $\Delta_1, \ldots, \Delta_k$ that are turned to the point zero. These parts are called Newton diagrams (see [9, 10] for a similar definition). The definition of nonsingularity can be carried over directly to systems of functions with given Newton diagrams. Here it is necessary to require that the condition (ξ) holds for all covectors ξ with positive coordinates. The theorem on resolution of singularities also carries over to the local case with the help of a suitable toroidal variety. Here it is necessary to take a sufficiently fine decomposition of the positive octant in R^{n*}. We note that the subject of Newton polyhedra began precisely from local problems: from the rich empirical material and conjectures of V. I. Arnol'd and the first results of A. G. Kushnirenko.

§3. Objects Associated with a Laurent Polynomial f and a Polyhedron Δ under a Toroidal Compactification

1. General Notation. Let M be a compact analytic variety and D a divisor of it. Associated with the divisor D we have the sheaf $\Omega\{D\}$ of germs of meromorphic functions on M: a germ $g \in \Omega\{D\}$, if the germ $g\varphi$ is holomorphic, where $\varphi = 0$ is the local equation of the divisor D. For the divisor D a one-dimensional analytic fibering V is constructed in the standard way. The fibering V with the indicated divisor D is denoted by $\{D\}$. The sheaf of holomorphic sections of the fibering V is denoted by ΩV. The sheaf of germs of sections ΩV and the sheaf of germs of meromorphic functions $\Omega\{D\}$ are isomorphic. The one-dimensional Chern class $c_1 V$ of fibering $\{D\}$ is realized by the divisor D (we denote by one letter the divisor D and the class of two-dimensional cohomologies that is dual to it). We denote by K the one-dimensional canonical fibering on M. The sheaf $\Omega\{D\} \otimes K$ of germs of sections of the fibering $\{D\} \otimes K$ is isomorphic to the sheaf of germs of meromorphic differential forms of highest degree on M with coefficients in the sheaf $\Omega\{D\}$.

2. Now let $f : (C \setminus 0)^n \to C$ be an arbitrary Laurent polynomial. For this f we construct the Newton polyhedron $\Delta(f)$ and its support function $l_\Delta : R^{n*} \to R$. We now fix an arbitrary toroidal compactification M^n of space $(C \setminus 0)^n$. The function f can be extended meromorphically to space M^n, since imbedding $(C \setminus 0)^n \to M^n$ is a birational equivalence.

In the first place we are interested in the divisor D that is the closure in M^n of the divisor in $(C \setminus 0)^n$, defined by the equation f = 0, and the sheaves $\Omega\{mD\}$, $\Omega\{mD\} \otimes K$ associated with it.

We define the divisor D_∞ characterizing the behavior of the function f "at infinity." Let O_α be the closure in M^n of the (n − 1)-dimensional orbit T_α^{n-1}, $M^n = (C \setminus 0)^n \bigcup\limits_\alpha O_\alpha$. Divisor D_∞ is the sum of the hypersurfaces O_α. The hypersurface O_α appears in divisor D_∞ with multiplicity k if the function f has a pole of multiplicity k on this hypersurface, and with multiplicity −k if f has a zero of multiplicity k.

The Divisors D and D_∞ are Linearly Equivalent. Indeed, divisor $D - D_\infty$ is the divisor of the meromorphic function f. The passage from the "curvilinear" divisor D to the "rectilinear" (i.e., consisting of orbits) divisor D_∞ is basic for what follows.

Since the divisors D and D_∞ are linearly equivalent, the same fibering V corresponds to them. The one-dimensional Chern class of the fibering V is realized by the divisor D or the divisor D_∞. The sheaf $\Omega\{D\}$ is isomorphic to the sheaf ΩV and to the sheaf $\Omega\{D_\infty\}$.

Let ξ_α be covectors corresponding to the $(n-1)$-dimensional orbits T_α^{n-1} of the compactification $M^n, O_\alpha = \bar{T}_\alpha^{n-1}$, and l the support function of the polyhedron $\Delta(f)$.

Assertion 1. $D_\infty = -\sum l(\xi_\alpha) \cdot O_\alpha$.

Assertion 1 is obtained by considering the asymptotic behavior of the function $f(z_0\mu_\alpha(t))$ as $t \to 0$, where $z_0\mu_\alpha$ is the shifted one-parameter group μ_α corresponding to the covector ξ_α.

It follows from Assertion 1 that the class of equivalence of the divisor D depends only on the Newton polyhedron Δ (for a fixed compactification M^n) and does not depend on the concrete choice of the function f. Therefore, we can introduce the following notation:

$[\Delta]$ is the class of linearly equivalent divisors D corresponding to any Laurent polynomial f with $\Delta(f) = \Delta$;

$\{\Delta\}$ is the one-dimensional fibering corresponding to the class $[\Delta]$.

It is convenient to carry out cohomological calculations with the sheaf $\Omega\{D_\infty\}$, which is isomorphic to the sheaf $\Omega\{D\}$. Divisor D_∞ is made up of orbits and is invariant with respect to the action of torus T^n. The sheaf $\Omega\{D_\infty\}$ is hence also T^n-invariant. With T^n-invariant sheaves there is associated an order function j: a piecewise-linear function on covectors ξ. The definition of this function can be found in [2, pp. 26-27]. The role of the order function j lies in the fact that the cohomologies of variety M^n with coefficients in a T^n-invariant sheaf can be calculated only with regard to the order function j. A purely geometric algorithm for such a calculation is given in [2, pp. 42-43].

Assertion 2. On covectors ξ_α corresponding to the $(n-1)$-dimensional orbits T_α^{n-1} functions j and l coincide, $j(\xi_\alpha) = l(\xi_\alpha)$. On the remaining covectors the function j is regenerated by piecewise linearity, i.e., by linearity inside each cone σ_m of the decomposition corresponding to the toroidal closure M^n.

Assertion 2 follows from Assertion 1 and the definition of the order function j. For sufficiently full compactifications, Assertion 2 takes a particularly simple form.

Assertion 2'. If variety M^n is sufficiently full for variety Δ, then the functions j and l coincide.

3. Preservation of Structure. The juxtapositions described preserve the natural structures: To the product of Laurent polynomials there correspond the sum of divisors D, the tensor product of fiberings V, the sum of divisors D_∞, the sum of order functions j, the sum of Newton polyhedra Δ, and the sum of their support functions l.

We consider the commutative semigroup A_n of convex integral polyhedra in R^n with respect to addition. This is a semigroup with cancellation, i.e., if $\Delta_1 + \Delta = \Delta_2 + \Delta$, then $\Delta_1 = \Delta_2$. Therefore, the semigroup A_n can be extended to a group $\bar{A}_n$. The mapping $\Delta \to [\Delta]$ is a homomorphism of the semigroup A_n into the group of classes of linearly equivalent divisors on M^n with respect to addition, and the mapping $\Delta \to \{\Delta\}$ is a homomorphism of A_n into the group of one-dimensional vector fiberings on M^n with respect to tensor multiplication. Both of these mappings can be extended to the group $\bar{A}_n$. For example, the class of the divisor $[-\Delta]$ is defined to be the class $-[\Delta]$, and the fibering $\{-\Delta\}$ is defined to be $\{\Delta\}^{-1}$.

4. We need some definitions connected with convex integral polyhedra. Let Δ be a k-dimensional integral polyhedron lying in the n-dimensional space R^n. We construct the k-dimensional plane R^k in which the polyhedron Δ lies. Points that are limit points both for the polyhedron and for its complement $R^k \setminus \Delta$ are called boundary points of the polyhedron. The remaining points are called interior. We mention that for a zero-dimensional polyhedron (•) the unique point belonging to it is interior. We say that the polyhedron Δ_1 is strictly less than polyhedron Δ_2, and write $\Delta_1 < \Delta_2$, if all the points of polyhedron Δ_1 are interior

for Δ_2. We introduce more notation: $T(\Delta)$ is the number of integral points belonging to the polyhedron Δ, $B^+(\Delta)$ is the number of interior integral points of the polyhedron Δ, $B(\Delta) = (-1)^k B^+(\Delta)$, where $k = \dim \Delta$.

§4. Results of Calculation of Cohomologies and Their Geometric Consequences

1. Let Δ be a convex integral polyhedron, M^n a toroidal compactification of $(C \setminus 0)^n$, that is sufficiently full for Δ, and K the canonical fibering on M^n. Further, let f be any Laurent polynomial with Newton polyhedron Δ, and D, D_∞ the corresponding divisors.

Assertion 1.

1. $\dim H^i(M^n, \{D_\infty\}) = \begin{cases} T(\Delta) & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$

2. The sections in $H^0(M^n, \{D_\infty\})$ are precisely the Laurent polynomials P for which $\Delta(P) \subseteq \Delta$.

Assertion 2.

1. $\dim H^i(M^n, \{D_\infty\} \otimes K) = \begin{cases} 0 & \text{for } i \neq \operatorname{codim} \Delta = n - \dim \Delta, \\ B^+(\Delta) & \text{for } i = \operatorname{codim} \Delta. \end{cases}$

2. For polyhedra Δ of full dimension n(codim Δ = 0) the sections in $H^0(M^n, \{D_\infty\} \otimes K)$ are precisely the differential forms ω of the form $\omega = P \cdot \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}$ with Laurent polynomials P for which $\Delta(P) < \Delta$.

Assertions 1 and 2 are not hard to get from the geometric algorithm for calculation of cohomologies with coefficients in a T^n-invariant sheaf and in the sheaf of its differentials according to the order function j (see [2]). In our case we are dealing with the sheaf $\Omega\{D_\infty\}$, the sheaf of its differentials $\Omega\{D_\infty\} \otimes K$, and the order function j that is equal to the support function l of the polyhedron Δ.

We are interested also in the sheaves $\Omega\{-D_\infty\}$ and $\Omega\{-D_\infty\} \otimes K$. The cohomologies with coefficients in these sheaves can be calculated at once from the Assertions 1 and 2 by the Serre duality (see [8]). In our case the Serre duality is expressed in the isomorphisms

$$H^i(M^n, \{-D_\infty\}) \approx H^{n-i}(M^n, \{D_\infty\} \otimes K)$$

and

$$H^i(M^n, \{-D_\infty\} \otimes K) \approx H^{n-i}(M^n, \{D_\infty\}).$$

Remark. The cohomologies $H(M^n, \{-D_\infty\})$ and $H(M^n, \{-D_\infty\} \otimes K)$ can also be calculated directly from the order function $-l$. We mention that in the geometric algorithm for calculation of the cohomologies of T^n-invariant sheaves from the order function the Alexander duality corresponds to the Serre duality.

The given calculations contain complete information on the cohomology groups $H(M^n, \{m\Delta\}) \approx H(M^n, \{mD\}) \approx H(M^n, \{mD_\infty\})$ and $H(M^n, \{m\Delta\} \otimes K) \approx H(M^n, \{mD\} \otimes K) \approx H(M^n, \{mD_\infty\} \otimes K)$ for all integers m. We write the part of this information that is useful to us in the necessary form.

THEOREM 1. We have

$$\begin{aligned} \dim H^i(M^n, \{-\Delta\}) &= \dim H^i(M^n, \{-D\}) = \dim H^{n-i}(M^n, \{\Delta\} \otimes K) = \\ &= \dim H^{n-i}(M^n, \{D\} \otimes K) = \begin{cases} 0 & \text{for } i \neq \dim \Delta, \\ B^+(\Delta) & \text{for } i = \dim \Delta. \end{cases} \end{aligned}$$

2. For $\dim \Delta = n$ the group of global sections of the sheaf $\Omega\{D\} \otimes K$ consists of the meromorphic differential forms ω of the form $\omega = \frac{P}{f} \frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}$ with Laurent polynomial P, $\Delta(P) < \Delta$.

3. $\dim H^i(M^n, \{0\}) = \dim H^i(M^n, \{0\} \otimes K) = \begin{cases} 1 & \text{for } i = 0, \\ 0 & \text{for } i > 0. \end{cases}$

4. $\chi(M^n, \{-\Delta\}) = B(\Delta)$; $\chi(M^n, \{\Delta\}) = T(\Delta)$; $\chi(M^n, \{0\}) = 1$.

Proof. Part 1 follows from Assertion 2 and the Serre duality. Part 2 follows from Assertion 2 and the explicit specification of an isomorphism of the sheaves $\Omega\{D_\infty\}\otimes K$ and $\Omega\{D\}\otimes K$. Part 3 is a particular case of Part 1 for the trivial divisor $\{0\}$. In Part 4 the Euler characteristics of the sheaves $\Omega\{\Delta\}$, $\Omega\{-\Delta\}$, and $\Omega\{0\}$ are calculated. The first of them is obtained from Assertion 1. The two others are obtained from Parts 1 and 3.

2. Corollaries from Elementary Geometry. COROLLARY 1. Let $\Delta_1, \ldots, \Delta_k$ be fixed convex integral polyhedra and $n_1, \ldots, n_k$ nonnegative integers. Then the number $T(n_1\Delta_1+\ldots+n_k\Delta_k)$ of integral points in the polyhedron $\Delta=n_1\Delta_1+\ldots+n_k\Delta_k$ depends polynomially on $n_1, \ldots, n_k$.

Proof. Let M^n be a toroidal compactification of $(C\setminus 0)^n$, that is sufficiently full for the polyhedra $\Delta_1, \ldots, \Delta_k$. By the theorem (Part 4)

$$T(n_1\Delta_1+\ldots+n_k\Delta_k)=\chi(M^n, \{n_1\Delta_1+\ldots+n_k\Delta_k\})=\chi(M^n, \{\Delta_1\}^{n_1}\otimes\ldots\otimes\{\Delta_k\}^{n_k}).$$

By the Riemann–Roch theorem (see [8]), the Euler characteristic of the sheaf of sections of the one-dimensional fibering depends polynomially on its Chern class (and on the Chern classes of the variety M^n). It remains to observe that the Chern class of the fibering $\{\Delta_1\}^{n_1}\otimes\ldots\otimes\{\Delta_k\}^{n_k}$ is equal to $n_1[\Delta_1]+\ldots+n_k[\Delta_k]$ and depends linearly on the numbers $n_1, \ldots, n_k$.

COROLLARY 2. Under the conditions of Corollary 1 the number $B(\Delta)$, which is equal to the number of interior integral points of the polyhedron $\Delta=n_1\Delta_1+\ldots+n_k\Delta_k$, multiplied by $(-1)^{\dim\Delta}$, depends polynomially on $n_1, \ldots, n_k$.

Proof. By the theorem (Part 4), $B(\Delta)=\chi(M^n, \{-\Delta_1\}^{n_1}\otimes\ldots\otimes\{-\Delta_k\}^{n_k})$. Corollary 2 now follows from the Riemann–Roch theorem.

From Corollaries 1 and 2 it follows, in particular, that for a fixed polyhedron Δ the functions $T(m\Delta)$ and $B(m\Delta)$ are polynomials for nonnegative integers m. We extend the definitions of these polynomials $T(m\Delta)$ and $B(m\Delta)$ to any integer m.

COROLLARY 3. The polynomials $T(m\Delta)$ and $B(m\Delta)$ are interchanged under the involution $m\to -m$, i.e., $T(m\Delta)=B(-m\Delta)$.

Proof. We consider the function $\chi(m)$ of an integer m defined by the formula $\chi(m)=\chi(M^n, \{\Delta\}^m)$. By the Riemann–Roch theorem this function is a polynomial. For $m\geqslant 0$, $\chi(m)=\chi(M^n, \{\Delta\}^m)=T(m\Delta)$. For $m\leqslant 0$, $\chi(m)=\chi(M^n, \{-\Delta\}^{-m})=B(-m\Delta)$. The corollary is proved. We mention that the proof of this corollary is implicitly based on the Serre duality.

Remark. The assertions of Corollaries 1 to 3 are not new (see [11], and also [12]). The previous proofs of these assertions were geometric. The connection with algebra (with the Riemann–Roch theorem and the Serre duality) was unknown.

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SERIES IN THE ROOT VECTORS OF OPERATORS THAT ARE VERY CLOSE TO BEING SELF-ADJOINT

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This note is related to [1-3]. The meaning of the theorems given here is that the convergence of the series mentioned in the title is all the better, the closer the operator is to being self-adjoint and the more "smooth" the vector being expanded is. Applications that have determined the content of these theorems are given in 6.

1. Let H be a separable complex Hilbert space, and A_0 a self-adjoint lower semibounded operator in H with discrete spectrum. Let $\{f_j\}$ ($j = 1, 2, \ldots$) be an orthonormal basis in H formed from the eigenvectors of the operator A_0, $A_0 f_j = \nu_j f_j$, $\nu_j \leqslant \nu_{j+1}$ and $\nu_j \geqslant C_0 j^p$ for all j ($C_0 > 0$, $p > 0$). In particular, we assume, for simplicity, that $\nu_j > 0$ for all j; it is not hard to get rid of this restriction. We consider the operator $A = A_0 + A_1$ in H under the assumption that the operator $A_1 A_0^{-q}$ is bounded for some $q < 1$. The spectrum of such an operator A is discrete, and the system of its root vectors is complete in H (see [4]). We set $R_A(\mu) = (A - \mu I)^{-1}$, $R_{A_0}(\mu) = (A_0 - \mu I)^{-1}$. We mention that all the characteristic values μ_k of the operator A lie in the union of the disks $O_{j,C} = \{\mu : |\mu - \nu_j| \leqslant C\nu_j^q\}$, $C = \|A_1 A_0^{-q}\|$. (This was shown in [2, Chap. III, §1] with the aid of the equation $\|A_0^q R_{A_0}(\mu)\| = \sup \nu_j^q |\nu_j - \mu|^{-1}$ for $q \geqslant 0$, but the proof remains valid for $q < 0$ (see [3, §1]).) We can assume that operator $A_0^{-q} A_1$ is bounded instead of $A_1 A_0^{-q}$ (with the condition $\mathscr{D}(A_1) \supset \mathscr{D}(A_0)$ for $q > 0$); then $C = \|A_0^{-q} A_1\|$.

Let $\{\alpha_l\}$ ($l = 0, 1, \ldots$) be an increasing sequence of real numbers. We call it admissible if each of the numbers ν_j, $\operatorname{Re} \mu_k$ is contained in one of the intervals (α_l, α_{l+1}) ($l = 0, 1, \ldots$). For an admissible sequence $\{\alpha_l\}$, we denote by Γ_l a closed contour lying in the strip $\{\mu; \alpha_l \leqslant \operatorname{Re} \mu \leqslant \alpha_{l+1}\}$ and enclosing all ν_j, μ_k in this strip. We introduce the Riesz projections

$$P_l = -(2\pi i)^{-1} \int_{\Gamma_l} R_A(\mu)\, d\mu, \qquad Q_l = -(2\pi i)^{-1} \int_{\Gamma_l} R_{A_0}(\mu)\, d\mu. \tag{1}$$

Operators Q_l are orthogonal projections, and $f = \sum Q_l f$ in H for any $f \in H$ and any admissible system $\{\alpha_l\}$. By a theorem in [1] (see also [5]), for $p(1 - q) > 1$ there exists an admissible system $\{\alpha_l\}$, such that $\sum \|P_l - Q_l\|^2 < \infty$, so that $\{P_l H\}$ is a basis of subspaces in H that is quadratically close to orthonormal. Theorems 1 and 2 strengthen this theorem under additional assumptions.

First of all, we assume that

$$\nu_j = a j^p + O(j^r) \text{ for } j \to \infty, \tag{2}$$

where $a > 0$, $p > 0$, $r < p$. Equation (2) is imitative of the asymptotic behavior of the eigenvalues of elliptic operators (see survey [6]).

Let $b > 0$, $\rho > 0$. We subject $\{\alpha_l\}$ to the inequalities

$$b l^{\rho p} < \alpha_l < b(l+1)^{\rho p} \qquad (l \geqslant l_0), \tag{3}$$

which allows the possibility that P_l and Q_l vanish for some l.

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