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NEWTON POLYHEDRA AND THE GENUS OF COMPLETE INTERSECTIONS

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This paper is a direct continuation of [1] and adopts its constructions and notation. Here we continue the study of an algebraic variety X defined in $(\mathbf{C}\setminus 0)^n$ by a nondegenerate system of polynomial equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$.

The space $(\mathbf{C}\setminus 0)^n$ is compactified by means of an imbedding into a compact nonsingular toral variety M^n, $(\mathbf{C}\setminus 0)^n \subset M^n$. Using the polyhedra Δ, a compactification M^n is chosen in [1] which is such that the closure $\overline{X}$ of the variety $X \subset M^n$ is a compact nonsingular variety transverse to all the orbits of the variety M^n. Such a compactification will be said to be sufficiently complete for the polyhedra Δ.

The variety $\overline{X}$ is a complete intersection in M^n. In [1], results are cited of a calculation of the cohomology groups of M^n with coefficients in certain T^n-invariant sheaves with the aid of an algorithm in [2]. These results make it possible for us to calculate the arithmetic genus $\chi(\overline{X})$ of $\overline{X}$ and (if the polyhedra Δ have a complete intersection) describe all the holomorphic differential forms on $\overline{X}$.

The calculation of the Chern classes of M^n carried out in [3] allows us to calculate the Euler characteristic $E(\overline{X})$ of $\overline{X}$.

The variety $\overline{X}$ is closely related to the initial variety X. Consider, e.g., the holomorphic differential forms on X which extend holomorphically to some compactification on X. The set of such forms does not depend on the choice of the compactification. It therefore suffices to describe the holomorphic forms on the compactification $\overline{X}$. Analogously, $\chi(\overline{X}) = \chi(X)$.

In Sec. 3 the Euler characteristic E(X) is calculated. For this purpose the variety $\overline{X}$ is decomposed by the orbits of M^n into parts X_i, $E(\overline{X}) = \Sigma E(X_i)$. One of these parts is the variety X, while the other parts are varieties of the same type as X but of lower dimension.

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In Sec. 4 we consider the variety $\tilde{X}$ defined in C^n by a general system of polynomial equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$. Conditions on the polyhedra Δ are given under which a generic variety $\tilde{X}$ is nonsingular and transverse to the coordinate planes. The previous calculations are applied to the varieties $\tilde{X}$.

1. The Genus of Complete Intersections

1. Let M be a compact analytic projective variety. The number $h^{p,0}(M)$ is defined to be the dimension of the space of holomorphic differential forms on M of degree p. By the geometric genus p(M) of the variety M we mean the number $h^{n,0}(M)$ for $n = \dim M$. By the arithmetic genus $\chi(M)$ we mean the number $\chi(M) = \Sigma(-1)^p \times h^{p,0}(M)$. The numbers $h^{p,0}(M)$ are birational invariants. Therefore, we can define the numbers $h^{p,0}(Y)$, p(Y), and $\chi(Y)$ for singular and noncompact algebraic varieties Y: it is only necessary to put $h^{p,0}(Y) = h^{p,0}(M)$, where M is a compact analytic projective variety birationally equivalent to Y. For noncompact analytic varieties Y, the number $h^{p,0}(Y)$ is the dimension of the space of holomorphic forms of degree p on Y which extend to the compactification of Y.

The equality $h^{p,0}(M) = \dim H^p(M, \{0\})$ is valid, where $\{0\}$ is the trivial fibration on M (cf. [4]).

Now let M_k be a complete intersection, i.e., $M_k = D_1 \cap \ldots \cap D_k$, where $D_1, \ldots, D_k$ are nonsingular transversally intersecting divisors on M. The genus $\chi(M_k)$ and the numbers $h^{p,0}(M_k)$ can be calculated without dropping to the variety M_k provided the cohomology groups $H(M, \{-n_1D_1 - \ldots - n_kD_k\})$, $n_i = 0, 1$ of M are known. We recall how this is done.

2. Extract Sequences. As above, let $D_1, \ldots, D_k$ be transversely intersecting nonsingular divisors on M, and let $M_0 = M$, $M_1 = D_1, \ldots, M_k = D_1 \cap \cdots \cap D_k$ be a sequence of subvarieties. Assume that the divisor D consists of hypersurfaces transversely intersecting each other and the divisors D_i. Consider the exact sequence of sheaves

$$0 \to \Omega(M_{k-1}, \{-D_k - D\}) \xrightarrow{i} \Omega(M_{k-1}, \{-D\}) \xrightarrow{j} \hat{\Omega}(M_k, \{-D\}) \to 0. \tag{1}$$

Here $\Omega(M_{k-1}, \{-D_k - D\})$ is the sheaf of germs of regular functions on M_{k-1} which vanish on the intersection with the divisor $D_k + D$, $\Omega(M_{k-1}, \{-D\})$ the sheaf of germs of regular functions on M_{k-1} vanishing on the intersection with the divisor D, $\hat{\Omega}(M_k, \{-D\})$, the trivial extension on M_k of the sheaf of germs of regular functions on M_k vanishing on the intersection with D, i, the imbedding, and j, the homomorphism given by restricting functions to the subvariety M. The associated exact cohomology sequence has the form

$$0 \to H^0(M_{k-1}, \{-D_{k-1} - D\}) \to H^0(M_{k-1}, \{-D\}) \to H^0(M_k, \{-D\}) \to \ldots \tag{2}$$

Here we have used the isomorphism between the cohomology groups of the sheaf $\Omega(M_k, \{-D\})$ and its trivial extension $\hat{\Omega}(M_k, \{-D\})$.

As in [1], we denote by $\chi(M, \{D\})$ the Euler characteristic of the sheaf of sections of the one-dimensional fibration over M corresponding to the divisor D.

Assertion. The equality $\chi(M_k, \{-D\}) = \chi(M, \{-D\}) - \sum_i \chi(M, \{D - D_i\}) + \sum_{i<j} \chi(M, \{-D - D_i - D_j\}) - \ldots + (-1)^k \chi(M, \{-D - D_1 - \ldots - D_k\})$ is valid. In particular, taking D to be the trivial divisor D = 0, we obtain a formula for $\chi(M_k)$.

Proof. The Euler characteristic of a sheaf is equal to the sum of the Euler characteristics of a subsheaf and the corresponding quotient sheaf. Therefore, by the exact sequence (2) we have

$$\chi(M_k, \{-D\}) = \chi(M_{k-1}, \{-D\}) - \chi(M_{k-1}, \{-D - D_k\}).$$

The assertion is now proved by induction on k.

The calculation of the numbers $h^{p,0}M_k$ requires a more careful study of the sequence (2).

The following sequence [related to sequence (1) by Serre duality] will also be of use to us

$$0 \to \Omega(M_{k-1}, \{D\} \otimes K_{k-1}) \xrightarrow{i} \Omega(M_{k-1}, \{D + D_k\} \otimes K_{k-1}) \xrightarrow{j} \hat{\Omega}(M_k, \{D\} \otimes K_k) \to 0. \tag{1'}$$

Here $\Omega(M_{k-1}, \{D + D_k\} \otimes K_{k-1})$ is the sheaf of germs of meromorphic forms of highest degree on M_{k-1} having a first-order pole on the intersection with the divisor $D + D_k$, $\Omega(M_{k-1}, \{D\} \otimes K_{k-1})$, the same sheaf for the divisor D, and $\hat{\Omega}(M_k, \{D\} \otimes K_k)$, the trivial extension on M_k of the sheaf of germs of meromorphic forms of highest degree having a pole on the intersection with D. Here i is the imbedding and j the Poincaré residue. The corresponding exact cohomology sequence looks like:

$$0 \to H^0(M_{k-1}, \{D\} \otimes K_{k-1}) \to H^0(M_{k-1}, \{D + D_k\} \otimes K_{k-1}) \to H^0(M_k, \{D\} \otimes K_k) \to \ldots \quad (2')$$

3. We recall some of the notation and results of [1]. Notation: $T(\Delta)$, the number of integral points lying in Δ; $B^+(\Delta)$, the number of integral points interior to Δ (in the topology of the smallest linear space containing Δ); $B(\Delta)$, the number $(-1)^{\dim \Delta} B^+(\Delta)$. Let X be a variety defined in $(C \setminus 0)^n$ by a nondegenerate system of equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$. Let M be a projective toral compactification of $(C \setminus 0)^n$ which is sufficiently complete for the polyhedra Δ. By the theorem on resolution of singularities (cf. [1, Sec. 2]) the closure $\overline{X}$ of the variety $X \subset M$ is nonsingular and transverse to the orbits of M. Let D_i be the divisors corresponding to the functions f_i with respect to this compactification (cf. [1, Sec. 3]). The cohomology groups

$$H(M, \{-n_1D_1 - \ldots - n_kD_k\}) \approx H(M, \{-n_1\Delta_1 - \ldots - n_k\Delta_k\}),\ n_i \geqslant 0,$$

are calculated in [1, Sec. 4]. In particular, $\chi(M, \{-n_1\Delta_1 - \ldots - n_k\Delta_k\}) = B(n_1\Delta_1 + \ldots + n_k\Delta_k)$. Substituting these values into the formula for the genus of an intersection, we obtain the following theorem.

THEOREM 1. The arithmetic genus $\chi(X)$ of the variety X defined in $(C \setminus 0)^n$ by a nondegenerate system of equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$ can be calculated by the formula

$$\chi(X) = 1 - \sum B(\Delta_i) + \sum_{i<j} B(\Delta_i + \Delta_j) - \ldots + (-1)^k B(\Delta_1 + \ldots + \Delta_k).$$

For $k = n$, the variety X consists of isolated points and $\chi(X)$ is equal to the number of them, i.e., to the number of solutions of the generic system of equations $f_1 = \ldots = f_k = 0$ in $(C \setminus 0)^n$.

The formula for the genus is somewhat cumbersome, and a little formalism is helpful here.

4. Formalism. Let A be a commutative semigroup with zero and let F be a real-valued function on A. We are interested in the subgroup A_n of convex integral polyhedra on R^n with the operation of addition, and in the functions $T(\Delta)$, $B(\Delta)$, and $V(\Delta)$, where $V(\Delta)$ is the volume of Δ. Fix an element $h_1 \in A$. The function $L_{h_1}F(x) = F(x + h_1) - F(x)$ is called the first difference of $F(x)$ with respect to h_1. The function $L_{h_1,h_2}F(x) = L_{h_2,h_1}F(x) = F(x + h_1 + h_2) - F(x + h_1) - F(x + h_2) + F(x)$ is called the second difference of $F(x)$ with respect to h_1, h_2. Continuing in this way, we obtain the definition of the k-th difference $L_{h_1,\ldots,h_k}F(x)$ of the function $F(x)$ with respect to $h_1, h_2, \ldots, h_k$. In this notation the formula for the genus takes a simple form.

THEOREM 1'. $\chi(\overline{X}) = (-1)^k L_{\Delta_1,\ldots,\Delta_k} B(\cdot)$. Here $(\cdot)$ denotes the zero-dimensional polyhedron.

We remark that it is convenient to write down the formula for the arithmetic genus of the complete intersection $D_1 \cap \ldots \cap D_k$ in this notation for the general case also. Here one must take the semigroup A to be the group of divisors on the variety M and the function F to be the Euler characteristic $F(D) = \chi(M, \{-D\})$. Then $\chi(D_1 \cap \ldots \cap D_k) = (-1)^k L_{D_1,\ldots,D_k} \chi(M, \{0\})$.

A function F is called a polynomial (homogeneous polynomial) of degree m if for every fixed $x_1, \ldots, x_k \in A$ and nonnegative integers $n_1, \ldots, n_k$ the function $F(n_1x_1 + \ldots n_kx_k)$ is a polynomial (homogeneous polynomial) of degree m in $n_1, \ldots, n_k$. The functions $T(\Delta)$, $B(\Delta)$ are polynomials of degree n (cf. [1]), and $V(\Delta)$ is a homogeneous polynomial of degree n (cf. [5]). For a polynomial of degree m, $F(px) = p^m F_m(x) + \ldots + F_0(x)$. It is not hard to see that the coefficients $F_i(x)$ are homogeneous polynomials of degree i. We call them the homogeneous components of F. The homogeneous components of leading degree n of the polynomials $T(\Delta)$ and $(-1)^n B(\Delta)$ coincide with $V(\Delta)$: for polyhedra of larger dimension, the number of integral points belonging to the polyhedron and the number which lie inside the polyhedron coincide in the first approximation with the volume of the polyhedron.

For a polynomial of degree n, the k-th difference is a polynomial of degree $n - k$. For a homogeneous polynomial F of degree n, the number $(1/n!) L_{h_1,\ldots,h_n} F(0)$ is called the mixed value of F at $h_1, \ldots, h_n$ and is denoted by $F(h_1, \ldots, h_n)$. It is easy to see that $F(h, \ldots, h) = F(h)$. For the volume function V, the number $V(\Delta_1, \ldots, \Delta_n)$ is called the mixed volume of the polyhedra $\Delta_1, \ldots, \Delta_n$ (cf. [5]). For a polynomial of degree n, the number $L_{h_1,\ldots,h_n}F$ depends only on the homogeneous component of degree n. Therefore,

$$L_{\Delta_1,\ldots,\Delta_n}(-1)^n B(\cdot) = L_{\Delta_1,\ldots,\Delta_n} T(\cdot) = n!\, V(\Delta_1, \ldots, \Delta_n).$$

We come to the next theorem.

THEOREM. The intersection index of the n divisors $D_1, \ldots, D_n$ corresponding to the Newton polyhedra $\Delta_1, \ldots, \Delta_n$ on a sufficiently complete toral variety M^n, or the number of solutions in $(C \setminus 0)^n$ of a nondegenerate system of equations $f_1 = \ldots = f_n = 0$ with polyhedra $\Delta_1, \ldots, \Delta_n$, is equal to

$$(-1)^n L_{\Delta_1,\ldots,\Delta_n} B(\cdot) = L_{\Delta_1,\ldots,\Delta_n} T(\cdot) = n!\, V(\Delta_1, \ldots, \Delta_n).$$

Remark. The form of the answer $n!\,V(\Delta_1, \ldots, \Delta_n)$ in this theorem was known to Bernshtein [6], and the equality to $L_{\Delta_1,\ldots,\Delta_n}T(\cdot)$ was known to Kushnirenko [7]. In the form $(-1)^k L_{\Delta_1,\ldots,\Delta_k}B(\cdot)$, the answer makes sense even for $k \le n$ – it is the arithmetic genus of a complete intersection.*

5. Starting from the formula $n!\,V(\Delta_1, \ldots, \Delta_n)$ for the number of solutions of a system of n equations, Bernshtein found conditions on k of the polyhedra $\Delta_1, \ldots, \Delta_k$ under which the variety X is empty.

Definition. Polyhedra $\Delta_1, \ldots, \Delta_k$ lying in R^n are said to be dependent if there exists an l-dimensional plane, $0 \le l \le n$, which contains the affine translates of $l+1$ of the polyhedra Δ_i.

Assertion (D. N. Bernshtein). A nondegenerate system of equations $f_1 = \ldots = f_k = 0$ with polyhedra $\Delta_1, \ldots, \Delta_k$ is inconsistent in $(C \setminus 0)^n$ if and only if the polyhedra Δ_i are dependent.

The following geometric fact is used in the proof of this assertion: the mixed volume of the polyhedra is zero if and only if the polyhedra are dependent.

2. The Numbers $h^{p,0}$ for Complete Intersections

1. In order to calculate the numbers $h^{p,0}(M_k)$ it is necessary to study the exact sequence (2) in Sec. 1 more carefully. Let $\Delta_1, \ldots, \Delta_k$, Δ be Newton polyhedra, M a projective compactification of $(C \setminus 0)^n$ which is sufficiently complete with respect to these polyhedra, and let $D_1, \ldots, D_k$ be divisors in general position corresponding to $\Delta_1, \ldots, \Delta_k$. Let the divisor D correspond to Δ and consist of hypersurfaces which intersect each other and the divisors D_i transversely.

LEMMA 1. If $\dim \Delta = n$ then the sheaf $\Omega(M_k, \{-D\})$ is acyclic in all dimensions except the leading dimension $n-k$ for any $k \le n$.

Proof. For $k = 0$ this assertion is contained in [1, Sec. 4]. We carry out the proof by induction on k. Since $\dim \Delta = n$, we have $\dim(\Delta_k + \Delta) = n$. The polyhedron $\Delta_k + \Delta$ corresponds to the divisor $D_k + D$. By induction hypothesis we may assume that for $i < n-k$

$$H^i(M_{k-1}, \{-D_k - D\}) = H^i(M_{k-1}, \{-D\}) = H^{i+1}(M_{k-1}, \{-D_k - D\}) = 0.$$

The required result is now obtained from the exact sequence (2): $H^i(M_k, \{-D\}) = H^i(M_{k-1}, \{-D\}) = 0$ for $i < n-k$.

LEMMA 2. If $\dim \Delta_1 = \ldots = \dim \Delta_k = n$ and $\dim \Delta = l$, the sheaf $\Omega(M_k, \{-D\})$ is acyclic in all dimensions except the l-th and the leading highest dimension $n-k$ for any $k \le n$. If $l < n-k$, then $\dim H^l(M_k, \{-D\}) = B^+(\Delta)$.

Proof. For $k = 0$ this assertion is contained in [1, Sec. 4]. We carry out the proof by induction on k. Since $\dim \Delta_k = n$, we have $\dim(\Delta_k + \Delta) = n$. Hence by Lemma 1, $H^i(M_{k-1}, \{-D_k - D\}) = H^{i+1}(M_{k-1}, \{-D_k - D\}) = 0$ for $i < n-k$. For $i < n-k$, we get from the exact sequence (2) that $H^i(M_k, \{-D\}) \approx H^i(M_{k-1}, \{-D\})$. This allows us to carry out the inductive step.

Let the variety X be defined in $(C \setminus 0)^n$ by a nondegenerate system of equations $f_1 = \ldots = f_k = 0$ with polyhedra $\Delta_1, \ldots, \Delta_k$, and let $\dim \Delta_i = n$. Let $\bar{X}$ be the closure of X in a sufficiently complete projective toral compactification.

THEOREM. The variety $\bar{X}$ is connected, i.e., $h^{0,0}(\bar{X}) = 1$, and it has no holomorphic forms of intermediate dimension, i.e., $h^{p,0}(\bar{X}) = 0$ for $1 < p < n-k$. The geometric genus $p(X) = h^{n-k,0}(\bar{X})$ can be calculated by the formula

$$p(\bar{X}) = B^+(\Delta_1 + \ldots + \Delta_k) - \sum_i B^+(\Delta_1 + \ldots + \hat{\Delta}_i + \ldots + \Delta_k) + \ldots + (-1)^{k-1} \sum B^+(\Delta_i).$$

Proof. Substitute into Lemma 2 the trivial divisor D corresponding to the polyhedron $(\cdot)$ consisting of a single point. We get $h^{0,0}(\bar{X}) = 1$ and $h^{p,0}(\bar{X}) = 0$ for $0 < p < n-k$. Taking into account the formula for the arithmetic genus $\chi(\bar{X})$ and the preceding equalities, we obtain that the geometric genus $p(\bar{X})$ has the required form.

2. Explicit Form of Forms of Highest Degree. The explicit form of forms of highest degree on $\bar{X}$ is obtained from the exact sequence (2') in Sec. 1 and the explicit description of the group $H^0(M, \{-D\} \otimes K)$ in [1, Sec. 4]. First consider the case of a hypersurface. Let X be a hypersurface in $(C \setminus 0)^n$ defined by a nondegenerate equation $f = 0$ with Newton polyhedron Δ and $\dim \Delta = n$, $n > 1$. Let $\bar{X}$ be the closure of X in a

*It turns out that the answer in the form $L_{\Delta_1,\ldots,\Delta_k}T(\cdot)$ is also meaningful for $k < n$; it is one of the genera of a (noncompact) complete intersection.

sufficiently complete projective compactification M.

Assertion. The geometric genus $p(\bar{X})$ is equal to $B^+(\Delta)$. Moreover, every holomorphic form ω on $\bar{X}$ is the Poincaré residue $\frac{P}{df}\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_n}{z_n}$ of a form $\frac{P}{f}\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_n}{z_n}$, where P is a Laurent polynomial and $\Delta(P) < \Delta$.

Proof. Consider a piece of the exact sequence (2'):

$$0 \to H^0(M, K) \xrightarrow{i} H^0(M, \{D\} \otimes K) \xrightarrow{j} H^0(M_1, K_1) \to H^1(M, K) \to \dots$$

By the theorem in [1, Sec. 4], $H^0(M, K) = H^1(M, K) = 0$ for $n > 1$. Therefore all the holomorphic forms on $\bar{X} = M_1$ are the Poincaré residues of forms in $H^0(M, \{D\} \otimes K)$. We can now refer to the theorem in [1, Sec. 4].

Remark. The formula $p(X) = p(\bar{X}) = B^+(\Delta)$ for the geometric genus of a hypersurface X was discovered by Hodge [8]. However, the arguments of Hodge do not suffice for the proof (he lacked an important technical device, the theorem on resolution of singularities [1]). We observe that the formula for p(X) can be obtained without cohomological calculation by manipulating the polyhedra.

Example.* Let $\bar{X}$ be a smooth compactification of the curve X defined in $(C \setminus 0)^2$ by a nondegenerate equation $P(x, y) = 0$ with Newton polyhedron Δ. Assume that the polynomial P is not divisible by x and y, and that Δ has dimension two. Then $\bar{X}$ is a sphere with handles, the number of handles being equal to the number of interior points of Δ. All the regular differentials on $\bar{X}$ can be written down explicitly. If the integral point (k, l) lies inside the Newton polyhedron Δ, then the form $\frac{x^k \cdot y^l}{dP}\frac{dx}{x}\wedge\frac{dy}{y}$ is a regular differential on $\bar{X}$. These differentials are independent and their linear combinations span the set of all regular differentials on the curve $\bar{X}$. The equation $p(\bar{X}) = B^+(\Delta)$ in the case of a general polynomial $P(x, y)$ of degree n gives $p(\bar{X}) = (n-1)(n-2)/2$, and in the hyperelliptic case $y^2 = P_n(x)$ gives $p(\bar{X}) = (n-1)/2$ for n odd and $p(\bar{X}) = (n-2)/2$ for n even. The cases are well known. We remark that the number of integral points on the boundary of Δ also has a simple geometric meaning: the curve X is obtained from the compact curve $\bar{X}$ by blowing down as many points as there are integral points lying on the boundary of the Newton polyhedron.

An explicit form for the forms of highest degree can also be obtained for complete intersections. We formulate the answer for $k = 2$ (assuming that $n > 2$). Let the variety X be given in $(C \setminus 0)^n$ by a nondegenerate system of equations $f_1 = f_2 = 0$ with polyhedra Δ_1 and Δ_2, with $\dim \Delta_1 = \dim \Delta_2 = n$. Then forms of the type $\frac{P}{df_1 \wedge df_2}\frac{dz_1}{z_1}\wedge\dots\frac{dz_n}{z_n}$ are regular on the compactification $\bar{X}$ of X, provided $\Delta(P) < \Delta_1 + \Delta_2$. Their "number" is equal to $B^+(\Delta_1 + \Delta_2)$. However, these forms are dependent: if the polynomial P is divisible by f_1, i.e., $P = Q_1 f_1, \Delta(Q_1) < \Delta_2$, or if it is divisible by f_2, i.e., $P = Q_2 f_2$, $\Delta(Q_2) < \Delta_1$, then the corresponding forms are equal to zero. The "number" of such zero forms is equal to $B^+(\Delta_1) + B^+(\Delta_2)$. These relations are independent, and there are no others. Altogether, $p(\bar{X}) = B^+(\Delta_1 + \Delta_2) - B^+(\Delta_1) - B^+(\Delta_2)$, which agrees with the general formula for the geometric genus.

For $k > 2$, one writes out the forms explicitly, then the relations among them, then the relations among the relations, etc.

3. The Euler Characteristic

Assume N transversely intersecting hypersurfaces $V_1, \dots, V_N$ are given on an analytic variety Y. A stratification of Y is associated with these hypersurfaces: the stratum Y_0 of greatest dimension in this stratification is $Y \setminus \bigcup V_i$, the strata Y_i of the next dimension are given by $V_i \setminus \bigcup_{i \neq j} V_j$, etc. Each stratum of this stratification is the intersection of several surfaces V_i from which all smaller intersections of the V_i have been thrown out, $Y = \bigcup Y_I$, $Y_I \cap Y_L = \phi$ for $I \neq L$.

LEMMA (on the Additivity of the Euler Characteristic). $E(Y) = \Sigma E(Y_I)$.

We give a detailed proof for the case of a single hypersurface V ($N = 1$). In this case we are dealing with the stratification $Y = V \cup Y \setminus V$ and are interested in the equality $E(Y) = E(V) + E(Y \setminus V)$. Let V_u denote a tubular neighborhood of V, $\bar{V}_u$ its closure, and ∂V_u its boundary. The variety Y has the form of a sum of the closed subsets $\bar{V}_u$ and $Y \setminus \bar{V}_u$ which intersect along ∂V_u. Therefore, $E(Y) = E(V_u) + E(Y \setminus V_u) - E(\partial V_u)$. Further,

*This example apparently forms the content of an inaccessible paper of H. F. Baker, "Examples of application of Newton's polygon applied to the theory of singular points of algebraic functions," Trans. Cambridge Phil. Soc., 15, 403-450 (1893).

$E(V_u) = E(V)$ and $E(Y \setminus V_u) = E(Y \setminus V)$. We now observe that the variety ∂V_u is fibered over V with fiber S^1 so that $E(\partial V_u) = E(V) \cdot E(S^1) = 0$. The general case ($N > 1$) reduces to the one considered by induction on the number of hypersurfaces.

THEOREM 1. Let $D_1, \ldots, D_k$ be analytic hypersurfaces of a toral variety M^n which are transverse to one another and to all the orbits of M^n. Then

$$E(D_1 \cap \ldots \cap D_k \cap T^n) = \prod D_i (1 + D_i)^{-1}.$$

The right-hand side of the equality must be understood as follows: in the Taylor series at the point 0 of the analytic function $F(x_1, \ldots, x_k) = \prod x_i(1 + x_i)^{-1}$ take the homogeneous component F_n of degree n. The number $F(D_1, \ldots, D_k)$ is defined as follows. For a monomial $x_1^{n_1} \ldots x_k^{n_k}$ of degree n, $n_1 + \ldots + n_k = n$, it is equal to the intersection index of the divisors $\underbrace{D_1, \ldots, D_1}_{n_1 \text{ times}}, \ldots, \underbrace{D_k, \ldots, D_k}_{n_k \text{ times}}$. For a homogeneous function F_n of degree n, $F(D_1, \ldots, D_k)$ is defined by extending linearly, and for a function F, the number $F(D_1, \ldots, D_k) = F_n(D_1, \ldots, D_k)$.

The proof rests on a theorem of Ehlers (cf. [3]).

THEOREM (Ehlers). The Chern class $c_m(M^n) \in H^{2m}(M^n)$ of a smooth toral variety M^n is Poincaré dual to the sum ΣT_α^{n-m} of all orbits T_α^{n-m} of dimension $n - m$ in M^n. (The orbit T_α^{n-m} is not a closed variety, but nevertheless the dual cohomology class is well defined, since $\bar{T}_\alpha^{n-m}$ is a smooth compact variety and the boundary $\bar{T}_\alpha^{n-m} \setminus T_\alpha^{n-m}$ has smaller dimension.)

If the Chern classes of M^n are known, the Chern classes of a complete intersection can be calculated [4]. We give the answer for the Euler class:

$$E(D_1 \cap \ldots \cap D_k) = \prod D_i (1 + D_i)^{-1} + \sum_{m>0} c_m \left[\prod D_i (1 + D_i)^{-1}\right]_{n-m}.$$

The right-hand side of this formula must be understood as follows: take the homogeneous component F_{n-m} of degree $n - m$ in the analytic function F and replace $x_1, \ldots, x_k$ in F_{n-m} by the varieties $D_1, \ldots, D_k$. Here the product of the variables must be interpreted as intersection of varieties (brought if necessary into general position). The number $c_m[\prod D_i(1 + D_i)^{-1}]_{n-m}$ is the value of the Chern class c_m on the cycle $[\prod D_i(1 + D_i)^{-1}]_{n-m}$.

In the case of a toral variety M^n, we obtain upon applying Ehlers' theorem

$$E(D_1 \cap \ldots \cap D_k) = \prod D_i (1 + D_i)^{-1} + \sum_{m>0} \sum_\alpha T_\alpha^{n-m} \left[\prod D_i (1 + D_i)^{-1}\right]_{n-m}, \tag{1}$$

where the inner sum is carried out over all $(n - m)$-dimensional orbits T_α^{n-m}. Let D_i^α denote the intersection of the divisor D_i with the orbit T_α^{n-m}. The divisors $D_1^\alpha, \ldots, D_k^\alpha$ in the toral variety $\bar{T}_\alpha^{n-m}$, the closure of the orbit T_α^{n-m}, are transverse to one another and to the orbits of the variety $\bar{T}_\alpha^{n-m}$. It is clear that

$$T_\alpha^{n-m} [\prod D_i (1 + D_i)^{-1}]_{n-m} = \prod D_i^\alpha (1 + D_i^\alpha)^{-1}.$$

We stratify the variety $D_1 \cap \cdots \cap D_k$ by the intersections with the orbits T_α^{n-m} of all dimensions, $m = 0, 1, \ldots,$

$$D_1 \cap \ldots \cap D_k = (D_1 \cap \ldots \cap D_k \cap T^n) \cup \bigcup_{m>0} \bigcup_\alpha D_1 \cap \ldots \cap D_k \cap T_\alpha^{n-m}.$$

From the additivity of the Euler characteristic, we obtain

$$E(D_1 \cap \ldots \cap D_k) = E(D_1 \cap \ldots \cap D_k \cap T^n) + \sum_{m>0} \sum_\alpha E(D_1 \cap \ldots \cap D_k \cap T_\alpha^{n-m}).$$

By induction, we may assume that for $m > 0$

$$E(D_1 \cap \ldots \cap D_k \cap T_\alpha^{n-m}) = \prod D_i^\alpha (1 + D_i^\alpha)^{-1} = T_\alpha^{n-m} [\prod D_i (1 + D_i)^{-1}]_{n-m}.$$

Finally,

$$E(D_1 \cap \ldots \cap D_k) = E(D_1 \cap \ldots \cap D_k \cap T^n) + \sum_{m>0} \sum_\alpha T_\alpha^{n-m} [\prod D_i (1 + D_i)^{-1}]_{n-m}. \tag{2}$$

Comparing Eqs. (1) and (2), we get that $E(D_1 \cap \cdots \cap D_k \cap T^n) = \prod D_i(1 + D_i)^{-1}$. Theorem 1 is proved.

It is not crucial in Theorem 1 that M^n be a toral variety. Let M^n be a compact analytic variety and $O_1, \ldots, O_N$, transversely intersecting divisors. Assume that the tangent bundle TM^n has the same Chern classes as the fibration $\{O_1\} + \ldots + \{O_N\}$. By Ehlers' theorem, this is the case if M^n is a toral variety and the $O_1, \ldots, O_N$ are the closures of $(n-1)$-dimensional orbits of M^n. Let $D_1, \ldots, D_k$ be smooth divisors which are transverse to one another and to the divisors $O_1, \ldots, O_N$. Let X denote the part of the intersection $D_1 \cap \ldots \cap D_k$ lying in the "finite part" of M^n, i.e., $X = D_1 \cap \cdots \cap D_k \setminus \bigcup O_\alpha$. Then the Euler characteristic of X can be expressed in terms of the intersection indices of the divisors D_i alone.

THEOREM 1'. $E(X) = \Pi D_i(1 + D_i)^{-1}$.

The proof of Theorem 1' is an almost verbatim repetition of the proof of Theorem 1.

We will need a bit a formalism. Let $\Delta_1, \ldots, \Delta_n$ be n-polyhedra in an n-dimensional space. The mixed volume of these polyhedra is denoted by $V(\Delta_1, \ldots, \Delta_n)$ as before. Now let $F(x_1, \ldots, x_k)$ be the Taylor series of an analytic function of the k variables $x_1, \ldots, x_k$ at the point 0. We wish to determine the number $F(\Delta_1, \ldots, \Delta_k)$. If F is a monomial of degree n, $x = x_1^{n_1} \ldots x_k^{n_k}$, $n_1 + \ldots + n_k = n$, then we put

$$F(\Delta_1, \ldots, \Delta_k) = n!\, V(\underbrace{\Delta_1, \ldots, \Delta_1}_{n_1 \text{ times}}, \ldots, \underbrace{\Delta_k, \ldots, \Delta_k}_{n_k \text{ times}}).$$

For homogeneous functions F_n of degree n, we complete the definition of $F(\Delta_1, \ldots, \Delta_k)$ by extending it linearly, and for arbitrary F, we put $F(\Delta_1, \ldots, \Delta_k) = F_n(\Delta_1, \ldots, \Delta_k)$, where F_n is the homogeneous component of F of degree n.

THEOREM 2. Let X be a variety defined in $(C \setminus 0)^n$ by a nondegenerate system of equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$. Then $E(X) = \Pi\Delta_i(1 + \Delta_i)^{-1}$. For example, for a hypersurface $(k = 1)$, $E(X) = (-1)^{n-1} n!\, V(\Delta_1)$, and for a curve $(k = n-1)$, $E(X) = -n!\, V(\Delta_1, \ldots, \Delta_{n-1}, \Delta_1 + \ldots + \Delta_{n-1})$.

Proof. The theorem on the resolution of singularities (cf. [1]) reduces Theorem 2 to Theorem 1, since the intersection index of divisors in our case can be expressed in the required way in terms of the mixed Minkowski volume (cf. Sec. 1, Paragraph 4).

Remark. Theorem 2 is not new. It was recently announced by Bernshtein (cf. [9], where the history of the problem is also discussed). His proof is based on Morse theory. It is more elementary, but our proof gives a better explanation of the form of the formula for E(X).

4. Complete Intersections in C^n

1. Consider a variety $\tilde{X}$ defined in C^n by a nondegenerate system of polynomial equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$. We first assume that all the polynomials f_i have zero constant term, or, in other words, that all the polyhedra Δ_i contain the point zero. We introduce some notation. Let R_I denote the coordinate planes in the space R^n, $R_\phi = R^n$. Let Δ_i^I denote the intersection $R_I \cap \Delta_i$, $\Delta_i^\phi = \Delta_i$. The number $\prod_i \Delta_i^I (1 + \Delta_i^I)^{-1}$ is defined as the result of substituting the polyhedra Δ_i^I into the analytic function $\Pi x_i(1 + x_i)^{-1}$. Here we take into account only monomials of degree $\dim R_I$. They are interpreted with the aid of the mixed volumes of the corresponding polyhedra Δ_i^I in R^n (cf. Sec. 3).

THEOREM. Let the system of polynomials $f_1, \ldots, f_k$ be nondegenerate (cf. [1]) for the Newton polyhedra $\Delta_1, \ldots, \Delta_k$, and let all the f_i have zero constant term. Then:

1) The variety $\tilde{X}$ is smooth and transverse to all the coordinate planes of the space C^n.

2) The variety $\tilde{X}$ is birationally equivalent to the part X lying in $(C \setminus 0)^n$. In particular, $h^{p,0}(\tilde{X}) = h^{p,0}(X)$ and $\chi(\tilde{X}) = \chi(X) = (-1)^k \cdot L_{\Delta_1, \ldots, \Delta_k} B(\cdot)$.

3) $E(\tilde{X}) = \sum_I \prod_i \Delta_i^I (1 + \Delta_i^I)^{-1}$.

Proof. Part 1) follows at once from the nondegeneracy condition of the system f_i (cf. [1]). The imbedding $(C \setminus 0)^n \to C^n$ is a birational isomorphism. The imbedding $X \to \tilde{X}$ is also a birational isomorphism, since by part 1) the variety $\tilde{X}$ contains no irreducible components in the coordinate planes. Further, $\tilde{X}$ is stratified by its intersections with the coordinate planes. According to part 1), this stratification satisfies the hypotheses of the Lemma on the additivity of the Euler characteristic. It remains to use Theorem 2 of Sec. 3 and sum the Euler characteristics $\Pi(\Delta_i^I(1 + \Delta_i^I))^{-1}$ of all the strata. The theorem is proved.

2. We now weaken the condition on the Newton polyhedra Δ_i. It is not possible to remove this condition entirely: if all the polyhedra Δ_i are located far from the zero point, then all the functions f_i will vanish at the point 0 of C^n with a large multiplicity, and the point 0 will be very singular for $\tilde{X}$ (but if at least one of the

polynomials f_i has a nonzero constant term, the point 0 does not lie in $\tilde{X}$ and the high multiplicity of the zero of the remaining functions at 0 plays no role).

We introduce some definitions. We say that the coordinate plane R_I is attached to the polyhedra Δ_i if not all the polyhedra $\Delta_i^I = R_I \cap \Delta_i$ are empty; an attached plane is called weakly attached if some of the Δ_i^I are empty, and strongly attached otherwise. We say that the system of polyhedra $\Delta_1, \ldots, \Delta_k$ is regularly attached to the coordinate cross if all the coordinate planes are attached to the polyhedra Δ_i and for each weakly attached plane R_I the polyhedra Δ_i^I are dependent.

Assertion 1. If a generic variety $\tilde{X}$ with polyhedra $\Delta_1, \ldots, \Delta_k$ is transverse to all the coordinate planes, then the polyhedra Δ_i are regularly attached to the coordinate cross in R^n.

Indeed, if at least one of the functions f_i becomes identically zero on a coordinate plane C_I, the transversality of $\tilde{X}$ with C_I means that there are no points of intersection. By Bernshtein's condition (Sec. 1, Paragraph 5), this means in the general case that the polyhedra among the Δ_i^I which are nonempty are dependent.

Assertion 2. Let the polyhedra $\Delta_1, \ldots, \Delta_k$ be regularly attached to the coordinate cross. Assume in addition that: 0) the functions f_i are nondegenerate for the Δ_i (cf. [1]); 1) for weakly attached planes R_I, the nonzero functions among the $f_i^I = f_i|_{C_I}$ are nondegenerate for the polyhedra among the Δ_i^I which are nonempty. Then the variety $\tilde{X}$ is nonsingular and transverse to all the coordinate planes.

Proof. Let $\overset{0}{C}_I$ denote the coordinate plane C_I from which all smaller coordinate planes have been removed. If R_I is a strongly attached plane, condition 0) guarantees transversality of $\tilde{X}$ to $\overset{0}{C}_I$. If R_I is a weakly attached plane, condition 1) guarantees that there are no points of intersection of $\tilde{X}$ with $\overset{0}{C}_I$.

All the assertions of Theorem 1 carry over verbatim to systems $f_1 = \ldots = f_k = 0$ for which the polyhedra $\Delta_1, \ldots, \Delta_k$ are regularly attached to the coordinate cross and for which the functions $f_1, \ldots, f_k$ satisfy the hypotheses of Assertion 2. [We remark that in the formula for the Euler characteristic $E(\tilde{X}) = \sum_I \prod_i \Delta_i^I (1 + \Delta_i^I)^{-1}$, the summation must be performed only over the strongly attached planes R_I.]

5. Remarks

1. A part of the calculations of this paper can be carried over to the local case. For example, the formula for the arithmetic genus of a hypersurface takes the same form here.

THEOREM. The genus of singularity for a nondegenerate function f with a favorable Newton diagram (cf. [10]) is equal to $(-1)^n$ times the number of integral points with positive coordinates lying on and below the Newton diagram.

2. The following problem seems to be very attractive: describe the mixed Hodge structure on a variety X defined in $(C \setminus 0)^n$ by a nondegenerate system of equations $f_1 = \ldots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$. The same problem can be posed for a variety $\tilde{X}$ defined by an analogous system in C^n. Here it is natural to limit oneself to the case when the polynomials f_i have a constant term [or to the case when the polyhedra Δ_i are regularly attached to the coordinate cross (cf. Sec. 4)]. These problems are related to the description of complete intersections in smooth compact toral varieties. In projective spaces, complete intersections have been thoroughly studied [4]. In the projective case, one in fact deals with general systems of equations with special Newton polyhedra (simplexes defined in R^n by inequalities $x_i \geq 0$, $\Sigma x_i \leq m$). In the toral case, no restrictions are made on the form of the polyhedra.

Recently, V. I. Danilov and I solved all the above problems under certain (insignificant) restrictions on the polyhedra $\Delta_1, \ldots, \Delta_k$. The solution will be discussed in a joint paper presently being prepared for publication.

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