all tend to zero. The dimension n of the space of x coordinates must be at least three. These solutions are unique in the class of functions u(x, t) for which

$$\operatorname*{vrai\,max}_{x\in K_\rho,\, t\in[0,T]} |u(x,t)|, \quad \int_0^T \int_{K_\rho} (|u|^2 + |u_x|^2)\, dx dt$$

are finite for every T > 0.

$\rho$ may be taken equal to $\infty$ in the statement of this theorem. The solutions corresponding to this value are solutions of Cauchy's problem for (1.8), and $K_\infty = E^n$.

## LITERATURE CITED


1. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type, Amer. Math. Soc. (1968).
2. O. A. Ladyzhenskaya, Mathematical Theory of Viscous Incompressible Flow, Gordan and Breach (1969).


# PRINCIPLE OF DIFFERENTIAL OPTIMIZATION APPLIED TO A SINGLE-PRODUCT DYNAMICAL MODEL OF AN ECONOMIC STRUCTURE


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UDC 51:330.115


In this paper we discuss a general formulation of the principle of differential optimization. We consider an economic model in which the dynamics is subject to this principle. The system of equations of the model is a system of differential equations with lag in which the lag is not specified, but is itself determined by a differential equation. The equations are sufficient for the further development of the economic system to be determined by its present state. We consider some particular cases that are interesting from the economic point of view. In one of them the system of equations separates and lends itself to a complete solution. In other cases we can find characteristic exponential solutions.

Some words on the economic content: our model is a single-product dynamical one with stocks that are differentiated with respect to the moment of their creation. Scientific and technological progress and the growth of equipment stocks lead to a continuous increase in the productivity of labor on newly created stocks. The least effective stocks are withdrawn from production (and not used later), and the freed labor resources are directed to newly created stocks. In the model this process is subjected to the criterion of differential optimization, according to which the policy of withdrawal of virtually outdated stocks is optimal if it ensures at each moment a maximal rate of growth of the national income.

Principle of Differential Optimization. We consider a set A of vector-valued functions $\gamma$ of the variable t, $\gamma(t) = \{\gamma_0(t), \gamma_1(t), \ldots, \gamma_n(t)\}$. A vector-valued function $\gamma(t)$ in the set A is called a trajectory, and each trajectory describes one of the possible ways in which the system can develop. We assume that all the trajectories $\gamma(t)$ are smooth functions with a finite number of points of discontinuity. At a point $t_0$ of discontinuity of the trajectory $\gamma(t)$ we consider the two vectors

$$\gamma^+(t_0) = \lim_{t\to t_0^+} \gamma(t) \quad \text{and} \quad \gamma^-(t_0) = \lim_{t\to t_0^-} \gamma(t).$$

The component $\gamma_0(t)$ of the trajectory $\gamma(t)$ plays a special role. It is assumed that $\gamma_0(t)$ is a continuous function, and $\gamma_0(t) \le t$. The interval $[\gamma_0(t), t]$ will be called the "influence interval" of the trajectory $\gamma(t)$ at the moment of time t.

Suppose that a goal function $F_\gamma(t)$ is defined on the trajectories and depends on the trajectory $\gamma$ and the time t. We assume the following conditions for F:


Translated from Sibirskii Matematicheskii Zhurnal, Vol. 19, No. 5, pp. 1053-1064, September-October, 1978. Original article submitted March 2, 1978.

a) if the trajectory $\tilde{\gamma}$ coincides with the trajectory $\gamma$ on its influence interval $[\gamma_0(t), t]$ at the moment of time t, then $F_{\tilde{\gamma}}(t) = F_{\gamma}(t)$. The condition a) means that the functional F does not depend on the future development of the system, and is completely determined by its past;

b) for any trajectory $\gamma$ the function of the time $F_{\gamma}(t)$ has a right derivative $F_{\gamma}^{\prime +}(t)$.

Definition 1. The trajectory $\gamma$ is said to be differentially optimal at the point $t_0$ with respect to the functional F if for any other trajectory $\tilde{\gamma}$ such that $\gamma(t) = \tilde{\gamma}(t)$ for $\gamma(t_0) \leq t \leq t_0$ we have the condition:

$$F_{\gamma}^{\prime +}(t_0) \geqslant F_{\tilde{\gamma}}^{\prime +}(t).$$

Definition 2. The trajectory $\gamma(t)$ is said to be differentially optimal on the interval $[a, b]$ with respect to the functional F if it is differentially optimal at each point of this interval $[a, b]$. We say that the differentially optimal trajectories satisfy the criterion of differential optimality.

Figuratively speaking, a differentially optimal trajectory always moves in the direction of greatest increase of the functional F. We now give an illustrative geometric example.

Example. As the trajectories $\gamma(t)$ we consider trajectories with influence interval of zero length ($\gamma_0(t) = t$) that determine continuous piecewise-smooth motion of the point $x(t)$ in the n-dimensional space $R^n$, $\gamma(t) = (t, x(t))$, $x(t) = x_1(t), x_2(t), \ldots, x_n(t)$, with speed not exceeding 1, i.e.,

$$\| x'(t) \| = \sqrt{x_1'^2(t) + x_2'^2(t) + \ldots + x_n'^2(t)} \leqslant 1.$$

Let the goal functional $F\gamma(t)$ be the value of a smooth function $G(x)$ at the point $x(t)$. In this example a trajectory $\gamma(t) = \{t, x(t)\}$ is differentially optimal if the point $x(t)$ "rises along the gradient" of the function G with speed 1, i.e., if

$$x'(t) = \frac{\operatorname{grad} G(t)}{\| \operatorname{grad} G(t) \|}.$$

Indeed, we have $F_{\gamma}'(t) = (d/dt)G(x(t)) = \langle \operatorname{grad} G, x' \rangle$. For the trajectory $x(t)$ we have $\| x'(t) \| \leq 1$, therefore, the scalar product $\langle \operatorname{grad} G, x' \rangle$ is greatest if the vector $x'(t)$ is collinear to the vector grad G and equal to 1 in length.

Besides the differentially optimal trajectories of motion of a system, we can also consider the trajectories on which the goal functional attains a maximum at the endpoints of time intervals of a fixed length $a$ ($a$ is the length of the planning period). Thus, we say that a trajectory $\gamma(t)$ is optimal with planning period of length $a$ (and origin at $t_0$) if for any natural number n and any trajectory $\tilde{\gamma}(t)$ such that $\tilde{\gamma}(t) = \gamma(t)$ for $t_0 \leq t \leq t_0 + (n-1)a$ we have the inequality $F(\gamma(t), t_0, na) \geq F(\tilde{\gamma}(t), t_0, na)$. It is natural to expect that as the length $a$ of the period converges to zero an optimal trajectory with planning period $a$ will converge to a differentially optimal one. For example, within the framework of the geometric example, we need to place only slight restrictions on the goal function G in order that this statement be true. Conditions for differential optimality frequently turn out to be significantly simpler and more natural than the conditions for optimality with a planning period of finite length.

Description of the Model. In an economic system that turns out a single product (a single-sector model) two main production factors stand out: the production stocks (realized capital), differentiated with respect to the moments of their creation and measured in product units, and the labor resources, measured in the number of labor units.

Let $T(t)$ be the labor resources at the moment of time t (a given function). We assume that at any moment of time t the effectiveness of production is characterized by a production function $U(x, y, \tau)$ that expresses the quantity of pure product created by the labor y (in a unit of time) with the use of production stocks (created at the moment $\tau$) of volume x product units ($\tau \leq t$). It is assumed that the function U increases monotonically with the argument $\tau$, which expresses the increased effectiveness of newer stocks under the effects of technological progress. This taking into account of technological progress in models has acquired the name "realized technological progress" [1-4]. It is assumed that the function U is positive-homogeneous and convex in the first two arguments. The first assumption reflects the absence of an effect from the scale of production, and the second the fact that the function U is based on optimal production methods.

The capital investments used for increasing stocks and replacing outgoing stocks are determined by the intensity of their input $\varkappa(t)$, i.e., it is assumed that the volume of stocks introduced into production in the time interval $[t, t + dt]$ is equal to $\varkappa(t)dt$. Various assumptions are made about the function $\varkappa(t)$ in various modifications (variants) of the model: the function $\varkappa(t)$ either is assumed to be an exogeneous variable of the model, i.e., a priori given, or it is assumed that it constitutes a constant part of the pure product produced at the moment t (more generally, it can be assumed that the function $\varkappa(t)$ is determined by the previous development of the economic structure).

The quantity of labor resources working on the newly released stocks is determined by the intensity of their introduction $\varphi(t)$, i.e., it is assumed that the labor resources introduced in the time interval $[t, t + dt]$ make up $\varphi(t)dt$ units. The function $\varphi(t)$ in the model is to be determined (an endogeneous variable of the model).

In the model it is assumed that the technological progress and the growth of equipment stocks lead to a continuous increase in the productivity of the labor on the newly created stocks. As a result the less effective stocks are continuously withdrawn from production (and not used later), while the freed labor resources are directed to the newly created stocks. Under the assumption of a monotonically growing labor productivity the oldest stocks (those having the earliest period of creation among the stocks taking part in the production process at the moment t) are withdrawn from production first, therefore, the policy of withdrawing stocks in the model is characterized by a function m(t) that expresses the moment of creation of stocks withdrawn from production at the moment of time t ($m(t) < t$). The function m(t) must be found.

The quantity of pure product produced from the stocks taking part in the production at the moment of time t (in a unit of time), i.e., the national income, is calculated in the model according to the following formula

$$P(t)=\int_{m(t)}^{t} U(\varkappa(\tau),\ \varphi(\tau),\ \tau)\,d\tau. \tag{1}$$

We write out the balance equations for the variables of the model.

<u>Balance Equation for Labor Resources:</u> $\int_{m(t)}^{t} \varphi(\tau)\,d\tau = T(t)$. Assuming full employment, the number of labor units $\int_{m(t)}^{t} \varphi(\tau)\,d\tau$ working with the stocks available at the moment t is equal to the size of the able-bodied population T(t). This relation can be written in the differential form

$$\varphi(t)=T'(t)+\varphi(m(t))\,m'(t), \tag{2}$$

in which this equation also has a simple economic meaning: the labor resources connected with the stocks newly created in the interval $[t, t + dt]$ are made up of the natural growth of labor resources dT(t) and the labor units taken from the stocks that are withdrawn in the interval $[m(t), m(t + dt)]$, i.e., $\varphi[m(t)]dm(t)$ units.

<u>Stocks Balance Equation.</u> This equation arises in the variant of the model in which $\varkappa(t)$ is equal to the constant part of the national income $\varkappa(t) = \gamma P(t)$ ($0 < \gamma < 1$ is the constant norm of accumulation). Using the formula (1), we get

$$\varkappa(t)=\gamma \int_{m(t)}^{t} U[\varkappa(\tau),\ \varphi(\tau),\ \tau]\,d\tau. \tag{3}$$

In addition to the balance equations, the model variables are subject to the differential optimization equation. This equation expresses the optimization criterion used in the model, according to which a policy of withdrawing virtually outdated stocks is optimal if it ensures a maximal rate of growth of the national income. This criterion consists in maximization of a certain functional (the national income) on an infinitely small (infinitesimal) interval. In the previous section we considered the mathematical problem of finding the differentially optimal trajectories of the development of the system (the principle of differential optimization), and now, using the concepts introduced in this section, we give a derivation of the equation of differential optimization for the economic model under discussion.

In our economic model the trajectories are the pairs of functions $\gamma(t) = \{m(t), \varphi(t)\}$ subject to Eq. (2) (for the variant of the model in which the function $\varkappa(t)$ is an exogeneous variable), and the triples of functions $\gamma(t) = \{m(t), \varphi(t), \varkappa(t)\}$ subject to Eqs. (2), (3) (for the variant of the model in which the function $\varkappa(t)$ is equal to a constant part of the national income). These trajectories give a balanced (with respect to labor resources and stocks) development of the economic structure. The influence interval for these trajectories of development of the economic system is the time interval $[m(t), t]$, and the stocks created in this period take part in the production at the moment t. The function $\varphi(t)$ is assumed to be a discontinuous piecewise-smooth function, and the functions m(t) and $\varkappa(t)$ are assumed to be continuous piecewise-smooth functions. On the trajectories we consider the functional $P_\gamma(t)$ calculated by the formula (1): the formula for computing the national income in the model at the moment of time t.

We consider the equation resulting from the principle of differential optimization with the functional $P_\gamma(t)$. We first analyze the model in which the function $\varkappa(t)$ is given exogeneously. We calculate the first derivative $P_\gamma'^{+}(t)$ for the trajectory $\gamma(t) = \{m(t), \varphi(t)\}$ at an arbitrary point $t_0$:

$$P_\gamma'^{+}(t_0) = U[\varkappa(t_0),\ \varphi(t_0),\ t_0] - U[\varkappa(m(t_0)),\ \varphi(m(t_0)),\ m(t_0)]\, m'(t_0).$$

If trajectories coincide up to the moment $t_0$, then for them the quantities $\varkappa(m(t_0))$, $\varphi(m(t_0))$, and $m(t_0)$ are the same. For the functions $\varkappa(t)$ and $\varphi(t)$ this follows from the fact that $m(t_0) < t_0$, and for the function $m(t)$ from its continuity. The labor resource balance [Eq. (2)] connects $\varphi^{+}(t_0)$ and $m'^{+}(t_0)$. Using this connection, $P_\gamma'^{+}(t_0)$ can be regarded as a uniqueness function of the argument $m'^{+}$:

$$P_\gamma'^{+}(t_0) = U[\varkappa(t_0),\ T'(t_0) + \varphi(m(t_0))m'^{+}(t_0),\ t_0] - U[\varkappa(m(t_0)),\ \varphi(m(t_0)),\ m(t_0)]\, m'(t_0). \qquad (4)$$

We see from Eq. (4) that the function $P_\gamma'^{+}$ is convex in $m'^{+}$. Therefore, the maximum of $P_\gamma'^{+}$ is attained at a point at which the derivative vanishes. From this, we get the following equation:

$$\frac{d(P_\gamma'^{+})}{dm'^{+}} = \frac{\partial U(\varkappa(t),\ \varphi^{+}(t),\ t)}{\partial\varphi}\,\varphi(m(t)) - U[\varkappa(m(t)),\ \varphi(m(t)),\ m(t)] = 0.$$

Thus, the **equation of differential optimization** has the form:

$$\frac{\partial U[\varkappa(t),\ \varphi^{+}(t),\ t]}{\partial\varphi} = \frac{U[\varkappa(m(t)),\ \varphi(m(t)),\ m(t)]}{\varphi[m(t)]}. \qquad (5)$$

In what follows we are interested only in continuous solutions. For such solutions $\varphi^{+}(t) = \varphi(t)$, and we can omit the symbol for passage to the limit from the right. For a production function of Cobb–Douglas type $U(\varkappa(t),\ \varphi(t),\ t) = f(t)\varkappa^{\alpha}(t)\varphi^{\beta}(t)$, $\alpha + \beta = 1$ [here the exogeneously given function $f(t)$ simulates the technological progress realized in the stocks of the period t] the equation of differential optimization has the following form:

$$\beta f(t)\varkappa^{\alpha}(t)\varphi^{-\alpha}(t) = f(m(t))\varkappa^{\alpha}(m(t))\varphi^{-\alpha}(m(t)).$$

In the variant of the model with endogeneously given function $\varkappa$ ($\varkappa(t) = \gamma P(t)$) the principle of differential optimization also leads to Eq. (5). Indeed, in this case the quantity $P_\gamma'^{+}$ is given on the trajectory $\gamma(t) = \{\varkappa(t), \varphi(t), m(t)\}$ by the formula:

$$P_\gamma'^{+} = U[\varkappa^{+}(t),\ \ \varphi^{+}(t),\ \ t] - U[\varkappa(m(t)),\ \varphi(m(t)),\ m(t)]\, m'^{+}(t).$$

The function $\varkappa$ depends continuously on the time, as is clear from Eq. (3). Consequently, the number $\varkappa^{+}(t_0) = \varkappa(t_0)$ is the same for all trajectories that coincide for $t < t_0$ and cannot change. We arrive at the same extremal problem and at the previous equation of differential optimization.

We turn to the economic meaning of the equation of differential optimization. The left-hand side of Eq. (5) is the limiting productivity of labor at the moment of time m(t), or, in other words, the norm of effectivity with respect to one of the production factors: the labor resources. The right-hand side of Eq. (5) is the productivity of labor on the stocks created at the moment of time t. For the differentially optimal development of the economical structure these quantities must be equal, in view of the following qualitative considerations. We consider the influence on the production of output when a small number of labor units (we denote it by $\Delta T$) is transferred from stocks created at the moment m(t) (taken out of production at the moment of time t) to stocks created at the moment of time t. On the stocks of time t there will be produced an additional $\frac{\partial U[\varkappa(t),\ \varphi(t),\ t]}{\partial\varphi}\Delta T$ product units, and on the stocks created at the moment m(t) the production is decreased by $\frac{U[\varkappa(m(t)),\ \varphi(m(t)),\ m(t)]}{\varphi[m(t)]}\Delta T$ product units. If the losses exceed the additional production, i.e., $\frac{\partial U[\varkappa(t),\ \varphi(t),\ t]}{\partial\varphi}\Delta T < \frac{U[\varkappa(m(t)),\ \varphi(m(t)),\ m(t)]}{\varphi[m(t)]}\Delta T$, then the transfer of labor resources to the more modern stocks is economically unjustifiable. But if the transfer of a small number of labor resources to more modern stocks enables us to increase the total production of output at the moment of time t, i.e., $\frac{\partial U[\varkappa(t),\ \varphi(t),\ t]}{\partial\varphi}\Delta T < \frac{U[\varkappa(m(t)),\ \varphi(m(t)),\ m(t)]}{\varphi[m(t)]}\Delta T$, then this policy of closing the virtually outdated stocks is not differentially optimal. Consequently, the differentially optimal development of the economic structure requires the equality of the limiting productivity of labor on the newly created stocks and on the stocks that are virtually outdated (least economical) at this moment. The appropriate equation is Eq. (5), derived mathematically from the principle of differential optimization.

The System of Equations of the Model. Suppose that at the moment of time $t_0$ we know all the parameters of the model. The problem is to calculate their subsequent behavior. We first dwell on the more complicated variant of the model, in which the strength of the input of capital investments makes up a constant part of the national income.

At the moment of time $t_0$ the stocks created in the course of the time interval $m(t_0) \leq t \leq t_0$ take part in the production. Therefore, information on the initial state of the economic structure implies the presence of the following data:

a) the numbers $t_0$ and $m(t_0)$ [moreover, $m(t_0) < t_0$];

b) the functions $\varkappa(t)$ and $\varphi(t)$ defined on the interval $m(t_0) \leq t \leq t_0$.

Moreover, for the initial data we must have the compatibility condition:

c) $$\frac{U[\varkappa(m(t_0)), \varphi(m(t_0)), m(t_0)]}{\varphi(m(t_0))} = \frac{\partial U[\varkappa(t_0), \varphi(t_0), t_0]}{\partial \varphi(t_0)},$$

which means that the equation of differential optimization holds at the initial moment of time $t_0$.

In differential form the equation of the model has the form:

$$\varphi(t) - \varphi(m(t)) = T'(t), \tag{i}$$

$$\varkappa'(t) = \gamma[U(\varkappa(t), \varphi(t), t)] - U[\varkappa(m(t)), \varphi(m(t)), m(t)]m'(t), \tag{ii}$$

$$\frac{U(\varkappa(m(t)), \varphi(m(t)), m(t))}{\varphi(m(t))} = \frac{\partial U(\varkappa(t), \varphi(t), t)}{\partial \varphi(t)}. \tag{iii}$$

We consider the more general system

$$m'(t) = \Phi_1[\varkappa(t), m(t), t, \varphi(t), \varkappa(m(t)), \varphi(m(t))], \tag{6}$$

$$\varkappa'(t) = \Phi_2[\varkappa(t), m(t), t, \varphi(t), \varkappa(m(t)), \varphi(m(t))], \tag{7}$$

$$\varphi(t) = \Phi_3[\varkappa(t), m(t), t, \varkappa(m(t)), \varphi(m(t))], \tag{8}$$

where $\Phi_1$, $\Phi_2$, $\Phi_3$ are known functions. To reduce the original system to this form it is necessary to solve Eq. (i) optimally in $m'(t)$ and substitute the resulting expression for $m'(t)$ in the right-hand side of Eq. (ii). Then we must solve Eq. (iii) with respect to $\varphi(t)$. We consider the solution of the system of equations (6)-(8) with the following initial data:

1) the numbers $t_0$ and $m(t_0)$ are given [with $m(t_0) < t_0$);

2) the functions $\varkappa(t)$ and $\varphi(t)$ are given on the interval $m(t_0) \leq t \leq t_0$.

Moreover, we assume the compatibility condition

3) $\varphi(t_0) = \Phi_3[\varkappa(t_0), m(t_0), t_0, \varkappa(m(t_0)), \varphi(m(t_0))]$.

The system of equations (6)-(8) with the initial data 1)-3) can be solved as follows: we substitute the expression for $\varphi(t)$ from Eq. (8) into the right-hand sides of Eqs. (6), (7). We get equations of the form

$$m'(t) = G_1[\varkappa(t), m(t), t, \varkappa(m(t)), \varphi(m(t))], \tag{9}$$

$$\varkappa'(t) = G_2[\varkappa(t), m(t), t, \varkappa(m(t)), \varphi(m(t))]. \tag{10}$$

If the values of the function $m(t)$ are contained in the interval $[m(t_0), t_0]$, i.e., if $m(t_0) \leq m(t) \leq t_0$, then the functions $\varkappa(m(t))$ and $\varphi(m(t))$ are known from the initial data. Therefore, under these assumptions the system (9)-(10) is a system of ordinary differential equations for determining $m(t)$ and $\varkappa(t)$ [with the initial data $m(t_0)$ and $\varkappa(t_0)$]. We assume that for the solution of the system (9)-(10) the function $m(t)$ is monotonically increasing (such an assumption was made for the simulation). The right-hand sides of the system (9), (10) are determined as long as $m(t)$ is less than $t_0$. This system can be solved on a digital computer (by approximation methods) up to the critical moment $t_1$ for which $m(t_1) = t_0$. After this, we know the functions $m(t)$ and $\varkappa(t)$ on the interval $t_0 \leq t \leq t_1$. Equation (8) now makes it possible to determine the function $\varphi(t)$ on this interval. Thus, the functions $\varkappa(t)$ and $\varphi(t)$ are now known on the interval $t_0 \leq t \leq t_1$. Substituting them in the right-hand sides of the system (9), (10), we get a new system of differential equations for $m(t)$, $\varkappa(t)$ on the interval $t_0 \leq m(t) \leq t_1$. It is determined up to the critical moment $t_2$ for which $m(t_2) = t_1$. Repeating this construction, we calculate successively the unknown functions on the interval $[t_2, t_3]$, where $m(t_3) = t_2$, and on the interval $[t_3, t_4]$, where $m(t_4) = t_3$, etc.

The variant of the model for which the function $\varkappa(t)$ is given exogeneously leads to an analogous but simpler system of equations. The initial data here are:

a) the numbers $t_0$ and $m(t_0)$ [with $m(t_0) < t_0$];

b) the function $\varphi(t)$ defined on the interval $[m(t_0), t_0]$.

The compatibility condition 3) must also be satisfied for the initial data.

These considerations are sufficient for the solution of the system of equations on a digital computer. We remark that in this way it is also possible to solve the analogous system of equations for vector-valued functions $\varkappa(t)$, $\varphi(t)$, i.e., a system in which $\vec{\varkappa}(t) = \varkappa_1(t), \ldots, \varkappa_k(t)$ is a k-dimensional vector-valued function, $\vec{\varphi}(t) = \varphi_1(t), \ldots, \varphi_n(t)$ is an n-dimensional vector-valued function, and Eqs. (7), (8) are also vector equations:

$$\vec{\varkappa}'(t) = \vec{\Phi}_2(\vec{\varkappa}(t), \vec{m}(t), \ldots),$$
$$\vec{\varphi}(t) = \vec{\Phi}_3(\vec{\varkappa}(t), \vec{m}(t), \ldots).$$

More generally, we can consider a system in which $\vec{m}(t) = m_1(t), \ldots, m_N(t)$ is also a vector-valued function. The right-hand sides of the equations here can be regarded as depending on the values $\vec{\varkappa}(t)$, $\vec{\varphi}(t)$ at the points $t, m_1(t), \ldots, m_N(t)$.

An investigation of such systems is fairly difficult. We mention that the theory of equations with deflected argument treats similar systems, but in it the deflection functions $m_1(t), \ldots, m_N(t)$ are assumed to be given (and not determined from equations, as in our system). The method of solution considered here is analogous to the method of steps in the theory of equations with deflected argument. Apparently, systems with unknown deflection have not been considered before.

Variant of Exogeneous Capital Investments and Constant Labor Resources. In this section we obtain an explicit solution of the system of equations of the model under the following assumptions:

1) the production function $U(\varkappa, \varphi, t)$ is a Cobb–Douglas function, i.e., $U(\varkappa, \varphi, t) = f(t)\varkappa^{\alpha}\varphi^{\beta}$, $\alpha + \beta = 1$;

2) the labor resources do not change with time, $T(t) = \text{const} = T_0$;

3) the function $\varkappa(t)$ is given on the ray $[t_0, \infty\rangle$ (it is assumed to be positive and continuous).

It will be convenient for us to use the function $\Pi(t)$ equal to the productivity of the labor on the stocks created at the moment t. For $\Pi(t)$ we obviously have the formula $\Pi(t) = \frac{U(\varkappa(t), \varphi(t), t)}{\varphi(t)}$. Under the above assumptions this formula takes the form $\Pi(t) = f(t)\varkappa^{\alpha}(t)\varphi^{-\alpha}(t)$. We rewrite the system of equations (2), (5) for the variant of the model under discussion in this section:

$$\varphi(t) = \varphi(m(t))m'(t), \tag{11}$$

$$f(m(t))\varkappa^{\alpha}(m(t))\varphi^{-\alpha}(m(t)) = \beta f(t)\varkappa^{\alpha}(t)\varphi^{-\alpha}(t). \tag{12}$$

In our notation Eq. (12) can be written in the form

$$\Pi(m(t)) = \beta\Pi(t). \tag{13}$$

The initial data in our variant of the model are: the initial interval $[m(t_0), t_0]$ (the "influence interval") and the function $\varphi(t)$ on this initial interval.

Assertion. Suppose that the initial data are such that the function $\Pi(t)$ is continuous and monotonically increasing on the initial interval $[m(t_0), t_0]$. For the initial data suppose that the compatibility condition $\Pi(m(t_0)) = \beta\Pi(t_0)$ holds. For such initial data there exists a unique solution of the system of equations (11), (12). This solution is defined on the ray $[t_0, \infty\rangle$. For this solution the function $\Pi(t)$ is monotonically increasing and the function $\varphi(t)$ is positive. The function $m(t)$ is monotonically increasing and for any t remains less than t [$m(t) < t$]. The function $m(t)$ can be found from the equation

$$\int_{m(t_0)}^{m(t)} f^{\frac{1}{\alpha}}(\tau)\varkappa(\tau)\,d\tau = \beta^{\frac{1}{\alpha}} \int_{t_0}^{t} f^{\frac{1}{\alpha}}(\tau)\varkappa(\tau)\,d\tau.$$

Proof. We assume that the functions $\varphi(t)$, $m(t)$ satisfy the system (11), (12), and the initial conditions. Raising both sides of Eq. (12) to the power $1/\alpha$, we get

$$f^{1/\alpha}(m(t))\varkappa(m(t))\varphi^{-1}(m(t)) = \beta^{1/\alpha}f^{1/\alpha}(t)\varkappa(t)\varphi^{-1}(t).$$

Multiplying this equation by Eq. (11), we get

$$f^{1/\alpha}(m(t))\varkappa(m(t))m'(t)=\beta^{1/\alpha}f^{1/\alpha}(t)\varkappa(t). \tag{14}$$

Equation (14) is a differential equation with separable variables for the determination of m(t). Integrating this equation, we get

$$\int_{m(t_0)}^{m(t)} f^{1/\alpha}(\tau)\varkappa(\tau)\,d\tau=\beta^{1/\alpha}\int_{t_0}^{t} f^{1/\alpha}(\tau)\varkappa(\tau)\,d\tau. \tag{15}$$

It is not difficult to see that Eq. (15) has a monotonically increasing solution m(t) defined on the ray $[t_0, \infty\rangle$. From the positivity of the function $f^{1/\alpha}(\tau)\varkappa(\tau)$ and from the inequality $0 < \beta < 1$ we get automatically the inequality $m(t) < t$.

Let m(t) be the solution of Eq. (15). To determine the functions $\varphi(t)$ and $\Pi(t)$ Eqs. (11) and (13) remain. We show that these equations make it possible to uniquely determine the functions $\varphi(t)$ and $\Pi(t)$. Indeed, we consider the time interval $[t_0, T]$ (T is an arbitrary number larger than $t_0$). As shown above, the function $t - m(t)$ is positive, therefore, on the interval $[t_0, T]$ it exceeds some positive constant $\varepsilon$. The critical points $t_1, t_2, \ldots, t_k, \ldots$ are determined recursively from the equations $m(t_1) = t_0, \ldots, m(t_k) = t_{k-1}, \ldots$. Since the function $t - m(t) > \varepsilon$, the critical points $t_1, t_2, \ldots$ do not accumulate on the interval $[t_0, T]$, and hence, for some N we have the inequality $t_{N+1} > T$. We denote by $I_k$ the interval $t_{k-1} \le t \le t_k$ and by $I_0$ the initial interval $m(t_0) \le t \le t_0$. It is clear that if $t \in I_k$, then $m(t) \in I_{k-1}$, and if $t \in I_1$, then $m(t) \in I_0$. Therefore, Eqs. (11) and (12) make it possible to determine the functions $\varphi(t)$ and $\Pi(t)$ recursively, first on the interval $I_1$, then on $I_2$, and so on. By assumption, on the initial interval $I_0$ the function $\varphi(t)$ is positive and the function $\Pi(t)$ is monotonically increasing, the recursively determined functions $\varphi(t)$ and $\Pi(t)$ obviously have the same properties. The assertion is proved.

In the variant just analyzed the natural assumptions of the simulation [monotone growth of the productivity of the labor on the stocks of the period t, the inequality $\varphi(t) > 0$, the monotone growth of the function m(t), and the inequality $m(t) < t$] are automatically satisfied for the solution if they are satisfied on the initial interval. Furthermore, the stability of the function m(t) under a change in the initial data is obvious. In particular, the function m(t) in the variant being considered does not generally depend on the distribution of the labor resources with respect to the stocks of various moments of creation on the initial interval [on the function $\varphi(t)$ for $m(t_0) \le t \le t_0$]. For a rapidly growing function $f^{1/\alpha}(t)\varkappa(t)$ the function m(t) rapidly assumes a behavior that does not depend on the length of the initial period. Indeed, the equation for m(t) can (for large t) be written in the form

$$\int_{m(t_0)}^{t_0} f^{1/\alpha}(\tau)\varkappa(\tau)\,d\tau+\int_{t_0}^{m(t)} f^{1/\alpha}(\tau)\varkappa(\tau)\,d\tau=\beta^{1/\alpha}\int_{t_0}^{t} f^{1/\alpha}(\tau)\,d\tau.$$

The initial data $m(t_0)$, which gives the length of the "influence interval" $[m(t_0), t_0]$, affects only the first term $\int_{m(t_0)}^{t_0} f^{1/\alpha}(\tau)\varkappa(\tau)\,d\tau$, which for large t is small in comparison with the remaining terms.

We mention that the stability of function m(t) with respect to a change in the initial data was first discovered in numerical experiments (the approximate solution of the system of equations of the model on a digital computer for various variants). These experiments indicate the high stability of the function m(t) also in other variants of the model [in the case of exponential growth of the labor resources and in the case of an endogeneous character for the function $\varkappa(t)$].

We dwell now on the asymptotic behavior of the solutions of the model as $t \to \infty$. As already mentioned, the asymptotic behavior of the function m(t) depends first of all on the behavior of the function $f^{1/\alpha}(t)\varkappa(t)$ as $t \to \infty$. It can be shown that if the function $f^{1/\alpha}(t)\varkappa(t)$ has power growth, then the function m(t) is approximately linear: $m(t) \approx at + b$, $a < 1$. More interesting is the case when the function $f^{1/\alpha}(t)\varkappa(t)$ is of exponential form, i.e., the case $f^{1/\alpha}(t)\varkappa(t) = Ce^{\rho t}$. We dwell on it in detail. In this case Eq. (12) takes the form

$$\int_{m(t_0)}^{m(t)} Ce^{\rho\tau}\,d\tau=\beta^{1/\alpha}\int_{t_0}^{t} Ce^{\rho\tau}\,d\tau$$

or

$$e^{\rho m(t)}-e^{\rho m(t_0)}=\beta^{1/\alpha}(e^{\rho t}-e^{\rho t_0}), \tag{16}$$

from which

$$m(t) = \frac{1}{\rho\alpha}\ln\beta + \frac{1}{\rho}\ln\left[e^{\rho t} + \beta^{-1/\alpha}e^{\rho m(t_0)} - e^{\rho t_0}\right].$$

We set $A = -(1/\rho\alpha)\ln\beta$ (the number A is positive, since $\ln\beta < 0$) and $B = \beta^{-1/\alpha}e^{\rho m(t_0)} - e^{\rho t_0}$. Then $m(t) = (1/\rho)\ln[e^{\rho t} + B] - A$. The function m(t) will have an especially simple form if $B = \beta^{-1/\alpha}e^{\rho m(t_0)} - e^{\rho t_0}$ is equal to zero. In this case $m(t) = t - A$. For the function $\varphi(t)$ we get the equation $\varphi(t) = \varphi(t - A)$, which shows in a simple way that the function $\varphi(t)$ extends from the initial interval $I_0$ as a periodic function. We show that the system has an analogous behavior as $t \to \infty$ also for $B \neq 0$. We rewrite the formula for m(t) in the form $m(t) = t - A + (1/\rho)\ln[1 + Be^{-\rho t}]$. With growing t the term $Be^{-\rho t}$ becomes small, and the function m(t) rapidly goes to a stationary mode

$$m(t) \approx t - A,\ A = -\frac{1}{\rho\alpha}\ln\beta.$$

We now consider the behavior of the function $\varphi(t)$. For large t the equation $\varphi(t) = \varphi(m(t))m'(t)$ coincides more and more precisely with the equation $\varphi(t) = \varphi(t - A)$, therefore, it is natural to expect that the function $\varphi(t)$ passes to a periodic mode of behavior (with period A) in the course of time.

We perform a calculation proving this assertion and enabling us to compute the limiting periodic function in terms of the initially given $\varphi(t)$ for $m(t_0) \le t \le t_0$. The function m(t) maps the interval $I_k$ into $I_{k-1}$, in particular, $I_1$ into $I_0$. The inverse function t(m) realizes the inverse mapping, and its k-th iterate $\underbrace{t(t(\ldots(t(m))\ldots))}_{k\text{ times}} = t_{(k)}(m)$ realizes in particular, the mapping of the initial interval $I_0$ into the k-th interval $I_k$ that is of interest to us.

From Eq. (16), regarding t as a function of m, we get $e^{\rho t(m)} = qe^{\rho m} + p$, where $q = \beta^{-1/\alpha}$ and $p = q(e^{\rho t_0} - e^{\rho m(t_0)})$. Substituting the function t(m) instead of m in this equation, we get

$$e^{\rho t_{(2)}(m)} = e^{\rho t(t(m))} = qe^{\rho t(m)} + p = q^2e^{\rho m} + qp + p.$$

Carrying out this substitution k times, we get

$$e^{\rho t_{(k)}(m)} = q^k e^{\rho m} + (q^{k-1} + q^{k-2} + \ldots + 1)\,p = q^k\left(e^{\rho m} + \frac{1 - 1/q^k}{1 - q}\,p\right).$$

From this, we have an explicit formula for the k-th iterate of the function t(m)

$$t_{(k)}(m) = kA + \frac{1}{\rho}\ln\left[e^{\rho m} + \frac{1 - 1/q^k}{1 - q}\,p\right].$$

For large k we get $t_{(k)}(m) \approx kA + \frac{1}{\rho}\ln\left[e^{\rho m} + \frac{p}{1-q}\right]$. Next, inverting the equations $\varphi(t) = \varphi(m(t))m'(t)$ and $\beta\Pi(t) = \Pi(m(t))$ and iterating k times, we arrive at the equations

$$\begin{cases}\varphi(t_{(k)}(m)) \cdot t'_{(k)}(m) = \varphi(m),\\ \beta^k\Pi(t_{(k)}(m)) = \Pi(m).\end{cases}$$

These formulas, together with the explicit and asymptotic formula for $t_{(k)}(m)$, make it possible to obtain explicit and asymptotic formulas for $\varphi(t_{(k)}(m))$ and $\Pi(t_{(k)}(m))$. From the asymptotic formula it is not hard to see that for large t the function $\varphi(t)$ really does pass to a periodic mode of behavior that can be explicitly determined.

Exponential Solutions. In the previous section the labor resources were assumed to be constant. The assumption that the labor resources grow exponentially is more realistic. Let $T(t) = T_0e^{pt}$ and $f^{1/\alpha}(t)\varkappa(t) = Ge^{\rho t}$ (here $T_0$, p, G, and $\rho$ are given constants). In this variant the model equations do not lend themselves to the same complete investigation, but the characteristic exponential solutions can be found also here. Namely: it is not hard to show that the following functions satisfy the system of equations of the model: $\varphi(t) = \varphi_0e^{pt}$ and $m(t) = t - A$, where

$$\varphi_0 = \frac{pT_0}{1 - e^{-A\beta}} \text{ and } A = -\frac{\ln\beta}{\rho - p}.$$

We now return to the variant of the model in which the capital investments constitute a constant part of the national income, i.e., in which $\varkappa(t) = \gamma\int\limits_{m(t)}^{t} U(\varkappa(\tau),\ \varphi(\tau),\ \tau)\,d\tau$. We assume that $U(\varkappa, \varphi, \tau)$ is an arbitrary Cobb–Douglas function $U(\varkappa(t), \varphi(t), t) = e^{\delta t}\varkappa^{\alpha}(t)\varphi^{\beta}(t)$ and that $T(t) = T_0e^{pt}$. There are characteristic exponential solutions also in this variant. There exists a (unique) set of parameters $\varphi_0$, $l$, $\varkappa_0$, $\mu$, and A such that the functions $\varphi(t) = \varphi_0 e^{lt}$, $\varkappa(t) = \varkappa_0 e^{\mu t}$, and $m(t) = t - A$ satisfy the system of model equations. In this set $l = p$ and for the form of the function $m(t) = t - A$ follows from the exponential form of the functions $\varphi(t)$ and $\varkappa(t)$. We write out the resulting exponential solution:

$$m(t) = t - A, \quad A = \frac{-\beta \ln \beta}{\delta};$$

$$\varphi(t) = \varphi_0 e^{pt}, \quad \varphi_0 = \frac{pT_0}{1 - \beta^{p\beta/\delta}};$$

$$\varkappa(t) = \varkappa_0 e^{(p+\delta/\beta)t}, \quad \varkappa_0 = \gamma^{1/\beta}\varphi_0 \left[\frac{1 - \beta^{1+p\beta/\delta}}{p + \delta/\beta}\right]^{1/\beta}.$$

For this solution the quantity A indicates the period of virtual deterioration of the stocks, which does not depend on the moment of creation of the stocks and is inversely proportional to the coefficient $\delta$ (the coefficient characterizing the rate of scientific and technological progress).

This exponential solution enables us to get analytic expressions (in the framework of the modification of the economic model considered here) for the most important macroeconomic characteristics: the national income and the norm of effectivity of capital investments. The formulas that are obtained for calculating these economic characteristics make it possible to estimate the influence of technological progress and other parameters of the model on these economic characteristics.

## LITERATURE CITED


1. L. V. Kantorovich and L. I. Gor'kov, "The functional equations of a single-product model," Dokl. Akad. Nauk SSSR, 129, No. 4, 732 (1959).
2. L. V. Kantorovich and V. I. Zhiyanov, "A single-product dynamical model of an economy that takes into account the structure of the stocks with technological progress," Dokl. Akad. Nauk SSSR, 211, No. 6, 1280 (1973).
3. R. Solow, "Investment and technical progress," in: Mathematical Methods in the Social Sciences, K. J. Arrow et al. (editors), Stanford (1960), p. 89.
4. L. Johansen, "Substitution versus fixed production coefficients in the theory of economic growth; a synthesis," Econometrica, 27, 157 (1959).


# ESTIMATES OF GREEN'S FUNCTIONS AND SCHAUDER ESTIMATES FOR SOLUTIONS OF ELLIPTIC BOUNDARY PROBLEMS IN A DIHEDRAL ANGLE


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UDC 517.948

In recent years a number of papers have appeared which are devoted to studying elliptic boundary problems in domains with edges on the boundary ([1-11], etc.). In these papers, conditions were studied for boundary problems to be Noetherian in weighted Sobolev spaces generated by norms in $L_2$ and in $L_p$, $1 < p < \infty$; the behavior of solutions near an edge was also studied.

In this article, we consider a boundary problem for a homogeneous elliptic operator of order 2m with constant coefficients in an n-dimensional dihedral angle. We obtain here precise estimates for Green's functions and Poisson kernels, which are then applied to prove coercive estimates for solutions in weighted Hölder classes. Applications of these results to elliptic problems in bounded domains with edges will be given in another article.

## 1. Boundary Problem in a Dihedral Angle

1. Function Spaces. Let $(r, \omega)$ be polar coordinates in the Euclidean plane $\{y\}$, $y = (y_1, y_2)$, $\alpha \in (0, 2\pi]$ and K be the sector $\{y : 2|\omega| < \alpha\}$ with edges $\gamma^{\pm} = \{y : 2\omega = \pm\alpha, |y| > 0\}$. We let $\mathscr{D}$ denote the dihedral angle $K \times R^{n-2}$, $n \geq 3$, with faces $\Gamma^{\pm} = \gamma^{\pm} \times R^{n-2}$ and edge $M = \{x = (y, z) : y = 0, z \in R^{n-2}\}$.


Translated from Sibirskii Matematicheskii Zhurnal, Vol. 19, No. 5, pp. 1065-1082, September-October, 1978. Original article submitted November 28, 1977.




0037-4466/78/1905-0752$07.50