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STRAIGHTENING OF PARALLEL CURVES

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To each plane curve Γ we adjoin the two-parameter family of curves Γ_q obtained by parallel-translating Γ by all possible vectors $q \in R^2$. For which curves Γ does there exist a (local) diffeomorphism φ of the plane which maps all curves Γ_q into straight lines? This question is answered in this paper. Moreover, all straightening diffeomorphisms are described. This gives a solution to a well-known problem of theoretical nomography (cf. [1], p. 132). In nomographic terminology, this problem consists in describing all transformations of nomographs made of aligned points into nomograms with oriented guide-grid, carrying a curvilinear scale.

Let us formulate our results.

THEOREM 1. *A family of curves Γ_q can be straightened locally if and only if the curve Γ is affinely equivalent to one of the following five curves*:

1) $e^x + e^y = 1$;
2) $e^x \cos y = 1$;
3) $y = e^x$;
4) $y = x^2$;
5) $y = 0$.

THEOREM 2. *A local diffeomorphism straightens some family of curves Γ_q, not consisting of straight lines, if and only if it can be transformed into one of the following four forms by means of a linear transformation of the domain space (x, y) and a projection transformation of the domain space (u, v)*:

1) $u = e^x,\ v = e^y$;
2) $u = e^x \cos y,\ v = e^x \sin y$;
3) $u = e^x,\ v = ye^x$;
4) $u = x^2 + y,\ v = x$.

Let us begin by proving Theorem 2. We shall classify the local diffeomorphisms φ: $R_1^2 \to R_2^2$ which straighten some family of curves Γ_q. Take the mapping in R_1^2 defined by parallel transport by some vector r. In the plane R_2^2 there is a corresponding mapping π_r, defined by $\pi_r(p) = \varphi[\varphi^{-1}(p) + r]$.

LEMMA 1. *If the curve Γ is not a straight line segment, then the mapping π_r: $R_2^2 \to R_2^2$ must be projective.*

In fact, it is clear from the definitions that the mapping π_r sends the two-parameter family of lines $\varphi\Gamma_q$ into itself (more precisely, π_r maps a line $\varphi\Gamma_q$ onto the line $\varphi\Gamma_{q+r}$). We say that a family of lines L is representative in a region U if for each point $p \in U$ there is a line $l(p) \in L$ and an angle $\alpha(p) > 0$, depending continuously on p, such that the family L

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contains all lines through p making an angle not exceeding $\alpha(p)$ with the line $l(p)$. We need the following fact.

LEMMA 2. *Let π be a homeomorphism of a connected region U_1 into a region U_2. If π maps the lines of a family L which is representative for U_1 into lines of the region U_2, then π is a projective transformation.*

We sketch the proof of Lemma 2. It is well known that a homeomorphism which sends all lines into lines is a projective transformation. The proof of this fact is based on constructing an everywhere flat Möbius net (cf. [2], p. 134). The proof of Lemma 2 is based on the same arguments. In the neighborhood of each point $p \in U_1$ we can construct a flat Möbius net, all of whose lines lie in the family L. This construction shows that the mapping π is locally projective. The connectedness of the region U_1 now implies that π is projective.

Lemma 1 follows from Lemma 2. Indeed, it is easy to see that if the curve Γ is not a straight line, then the family of lines $\varphi\Gamma_q$ is representative.

Let us continue the proof of Theorem 2. The transformations π_r: $R_2^2 \longrightarrow R_2^2$ corresponding to translations of R_1^2 form a commutative local group of transformations of the plane R_2^2. This group is locally transitive, i.e. it has a two-dimensional orbit. Lemma 1 reduces the original task to the problem of describing all commutative two-dimensional groups of projective transformations of the plane.

To a local group of projective transformations of a plane there corresponds a local group of linear transformations of three-space with determinant one. If we add the scalar matrices λE to the algebra of this group, we arrive at the problem of classifying the three-dimensional commutative Lie aglebras which act on R^3.

LEMMA 3. *Each three-dimensional commutative subalgebra of the Lie algebra of all linear transformations of R^3 can be put into one of the following six forms by a suitable change of basis*:

$$1)\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}; \quad 2)\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & -c \\ 0 & c & b \end{pmatrix}; \quad 3)\ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & c & b \end{pmatrix};$$

$$4)\ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & b & a \end{pmatrix}; \quad 5)\ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ c & b & a \end{pmatrix}; \quad 6)\ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix}.$$

In each of these forms, the parameters a, b and c denote arbitrary real numbers.

The full proof of this lemma is based on some straightforward computations and is not given here. The proof uses one general argument. We say that an $n \times n$ matrix is a *Sylvester matrix* if each eigenvalue has exactly one Jordan block. It is easy to show that all matrices commuting with a Sylvester matrix commute with each other and form an n-dimensional commutative algebra. Algebras 1)–4) have the following types of Sylvester matrices: 1) three distinct real eigenvalues; 2) one real and two conjugate complex eigenvalues; 3) a one-dimensional Jordan cell—and they consist of all matrices commuting with these, respectively. Algebras not having any Sylvester matrices are analyzed separately. We note that commutative algebras of transformations of higher-dimensional space have continuous moduli and resist tractable classification.

LEMMA 4. *A two-parameter commutative group of projective transformations of the plane can be transformed into one of the following six groups G by means of a linear coordinate change in the parameter plane (x, y) and a projective coordinate change in the plane (u, v). A mapping $\pi(x, y)$ of the group G maps a point (u, v) into one of the following points, respectively*:

1) $(e^x u, e^y v)$;
2) $e^x(u \cos y - v \sin y,\ u \sin y - v \cos y)$;
3) $e^x(u, yu + v)$;
4) $(u + xv + y + x^2/2,\ v + x)$;
5) $(u + x,\ v + y)$;
6) $(u + xv + y, v)$.

To prove Lemma 4 it suffices to combine the algebras given in Lemma 3 into groups of linear transformations and to consider the corresponding groups of projective transformations.

Let us now conclude the proof of Theorem 2. Let φ: $R_1^2 \longrightarrow R_2^2$ be a diffeomorphism that straightens some family Γ_q not consisting of lines. To each point $r \in R_1^2$ there corresponds a projective transformation π_r of the plane R_2^2 such that $\varphi(r) = \pi r \circ \varphi(0)$. These groups of transformations π_r are described in Lemma 4. Hence, in order to define the mapping φ it is only necessary to fix the point $\varphi(0)$ in a two-dimensional orbit of one of the groups 1)–6). For groups of form 1)–4), the mapping φ does not depend on the choice of the two-dimensional orbit or of the fixed point $\varphi(0)$, neglecting projective transformations of the image space. Accordingly, the mappings φ are the diffeomorphisms 1)–4) of Theorem 2. A group of type 6) generally does not have a two-parameter orbit. For groups of type 5) the mapping φ is a translation. A translation sends lines into lines. The proof of Theorem 2 is complete.

Theorem 1 follows from Theorem 2: nonlinear curves Γ with straightenable families Γ_q are preimages of straight lines under the mappings 1)–4) described in Theorem 2.

REMARK. In the hypotheses of Theorems 1 and 2 the smoothness requirements on the curve Γ and the mapping φ were made only for convenience of presentation. Theorems 1 and 2 are true also for continuous curves Γ and homeomorphisms φ; the fact is that continuous homomorphisms of Lie groups are automatically smooth mappings (cf. for example [3], Chapter V, §3, Theorem 3.2).

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