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REAL ANALYTIC VARIETIES WITH THE FINITENESS PROPERTY AND COMPLEX ABELIAN INTEGRALS

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It is known from algebra that the number of components of a level surface of a polynomial is finite and can be explicitly estimated in terms of its degree. A similar situation is known in analysis [1]: The number of components in a cube of a level surface of a real real analytic function is finite. There is no explicit estimate in this case; however, there is uniform boundedness with respect to a parameter: For a function which depends analytically on a parameter running through another cube, the number of components of a level surface is uniformly bounded [2].

One is able to find an explicit estimate for the number of components for certain integral manifolds (called separating solutions) of algebraic distributions of hyperplanes. A separating solution of a Pfaffian equation (cf. [3, 7] and Sec. 1) is a bounding domain of an integral manifold of a distribution of cooriented hyperplanes, whose coorientation coincides with the coorientation of the boundary of the domain. The properties of separating solutions recall the properties of level surfaces. Thus, the number of components of a separating solution of a Pfaffian equation with polynomial coefficients is finite and can be estimated in terms of the degrees of the coefficients. One can read about theorems of this kind and their applications to algebra and the theory of elementary functions in [3-7] (cf. also Sec. 1).

Separating solutions can also be considered for Pfaffian equations with analytic coefficients. With the help of such solutions we construct a class of real analytic varieties in Secs. 2 and 3. For the varieties constructed, finiteness theorems are valid (cf. Sec. 4). There are no explicit estimates here, but on the other hand there is uniform boundedness with respect to parameters.

Example. We consider in an open cube of the space $\mathbf{C}^n$ a single-valued branch I of an Abelian integral. It follows from the finiteness theorems of Sec. 4 that the number of components of the complex level surface I = c of this function is finite and bounded by one constant, independent of the choice of level c. The theorem on complex Abelian integrals is given in Sec. 5. This theorem arose as a result of thinking about the analogous result of Varchenko on real Abelian integrals [8]. Reference [8] impelled me to repeat the construction of Pfaffian varieties, using Pfaffian equations not only with polynomial but also with analytic coefficients.

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1. Separating Solutions of Pfaffian Equations

Let M be a smooth manifold (possibly not connected and nonorientable) and α be a 1-form on it. The following definition plays an important role in what follows.

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Definition. A submanifold of codimension one in M will be called a separating solution of the Pfaffian equation $\alpha = 0$, if: a) the restriction of the form α to the submanifold is identically equal to zero; b) the submanifold does not pass through singular points of the equation (i.e., at each point of the submanifold the form α does not vanish in the tangent space to M); c) the submanifold is the boundary of a domain in M, while the coorientation defined by the form coincides with the coorientation of the boundary of the domain (i.e., on vectors, applied at points of the submanifold and emanating from the domain, the form α is positive).

Example. A nonsingular level surface H = c of the function H is a separating solution of the Pfaffian equation dH = 0 (it bounds the domain H < c).

For separating solutions the following version of Rolle's theorem holds.

Assertion 1. Between two points of intersection of a connected smooth curve with a separating solution of a Pfaffian equation, there is a point of contact, i.e., a point at which the tangent vector to the curve lies in the hyperplane $\alpha = 0$.

We do not need the full extent of this assertion. We shall prove only the following lemma.

LEMMA 1. Assertion 1 is valid if the curve intersects the separating solution transversely.

Proof. At neighboring points of intersection the values of the form α on the tangent vectors to the curve, directed according to its orientation, have different signs. Hence the form α vanishes at some intermediate point.

Remark. If under the hypotheses of Assertion 1 the Pfaffian equation is completely integrable, then one can eliminate the requirement of smoothness of the curve. It suffices merely to require that the curve have a tangent vector at each point. Rolle's, Lagrange's and Cauchy's theorems are obtained from this assertion if the manifold is the plane and the Pfaffian equation defines a family of parallel lines on it.

COROLLARY 1. Under the hypotheses of Lemma 1, let the curve have B noncompact connected components and N points of contact with the distribution $\alpha = 0$. Then the number of points of intersection of the curve with a separating solution does not exceed N + B.

(Under the hypotheses of Corollary 1 the number of compact components of the curve is in no way restricted and can be infinite.)

Corollary 1 is basic for all the estimates of this section. The condition of contact of the curve with the distribution can turn out to be very degenerate. Assertion 2, formulated below, allows us to perturb the contact condition. It furnishes a version of Corollary 1 which is more convenient for applications.

First some definitions and lemmas. Let $f: M^n \to N^n$ be a smooth map of manifolds of the same dimension. By the upper number of preimages (u.n.p.) of a point $a \in N^n$ is meant the smallest number C, for which one can find a neighborhood of the point a, in which all regular values of the map f have no more than C preimages.

LEMMA 2. At the point $a \in N^n$ let there exist an u.n.p. equal to C; then: 1) at all points sufficiently close to a there exists a u.n.p. and it is $\leqslant C$; 2) the number of nondegenerate preimages of the point a does not exceed C.

Proof. 1) is obvious. 2) Let there be C + 1 nondegenerate preimages among the preimages of a. Then by the implicit function theorem, points close to a have at least C + 1 preimages.

We call a smooth positive function ρ on M *covering*, if it tends to zero at "infinity" in M, i.e., if the map $\rho^{-1}: M \to \mathbf{R}^1$ is proper. We associate with a smooth map F of an n-dimensional manifold M^n into an (n − 1)-dimensional manifold N^{n-1} the map $F_\infty: M^n \to N^{n-1} \times \mathbf{R}^1$, defined by the formula $F_\infty = (F, \rho)$.

LEMMA 3. Let the u.n.p. of the point $(a, 0) \in N^{n-1} \times \mathbf{R}^1$ for the map F_∞ exist and not exceed 2B. Then the preimage of any regular value of the map F, sufficiently close to a, is a smooth curve, which has no more than B noncompact components.

Proof. On each noncompact component of the curve the function $\rho - \varepsilon$, where ε is a sufficiently small positive number, has not less than two zeros. One finishes the rest of the proof just as in Lemma 2.

On the manifolds M^n and N^{n-1} let there be fixed volume forms v^n and w^{n-1}. We denote by $*$ the map of n-forms on M into functions which is defined by the relation $(*\tau^n)v^n = \tau^n$. Let $\alpha = 0$ be a Pfaffian equation on M^n and let Γ be a separating solution of it. We associate with a smooth map $F:M^n \to N^{n-1}$ two mappings of M^n into $N^{n-1} \times \mathbf{R}^1$: the first map is $F_\infty = (F, \rho)$ and the second map is $F_\alpha = (F, *(\alpha \wedge F^*w^{n-1}))$.

Assertion 2. Let the u.n.p. of the point $(a, 0) \in N^{n-1} \times \mathbf{R}^1$ for the maps F_∞ and F_α exist and not exceed 2B and N, respectively. Then the u.n.p. of the point $a \in N^{n-1}$ for the restriction of the map F to the separating solution Γ exists and does not exceed N + B.

Proof. Let the point b be a regular value both for the map F and for its restriction to the separating solution Γ. Then at two successive points of intersection of the curve $F^{-1}(b)$ with the separating solution Γ the function $*(\alpha \wedge F^*w^{n-1})$ assumes values of different signs. On the part of the curve lying between two successive intersections, it assumes all intermediate values. To complete the proof it remains to use Lemma 3 and Lemma 2.

Remark. Assertion 2 gives an estimate of the u.n.p. of a point $a \in N^{n-1}$ for the map $F|_\Gamma$, which is independent of the choice of separating solution Γ of the equation $\alpha = 0$.

Definition. Let $\alpha_1,\ldots,\alpha_k$ be an ordered collection of 1-forms on the manifold M. We shall say that the submanifold $\Gamma \subset M$ is a separating solution of the ordered system of Pfaffian equations $\alpha_1 = \ldots = \alpha_k = 0$, if there exists a chain of submanifolds $M = \Gamma_0 \supset \ldots \supset \Gamma_k = \Gamma$, in which each successive manifold Γ_i is a separating solution of the Pfaffian equation $\Gamma_i = 0$ on the preceding manifold Γ_{i-1}, where the first manifold coincides with M and the last with Γ.

Let there be fixed on the manifold M^n a covering function ρ and a volume form v^n together with the corresponding operation $*$. Suppose there is given on the manifold M an ordered system of k Pfaffian equations $\alpha_1 = \ldots = \alpha_k = 0$. We consider a smooth map F of the manifold M^n into a smooth $(n - k)$-dimensional manifold N^{n-k} with volume form w^{n-k}. Below we describe a construction which associates with each map F of M^n into N^{n-k} a collection of 2^k maps of M^n into $N^{n-k}R^k$. The construction depends on the following data: form $\alpha_1,\ldots,\alpha_k$, giving a system of Pfaffian equations; the volume forms on M^n and N^{n-k}; the map F; and the covering function ρ. The following theorem holds here.

THEOREM 1. For any separating solution Γ of a Pfaffian system of equations, and for any point $a \in N^{n-k}$, the u.n.p. of the point a for the restriction of the map F to Γ can be estimated explicitly in terms of the collection of 2^k u.n.p. of the point $(a, 0) \in N^{n-k} \times \mathbf{R}^k$ for the 2^k mappings constructed below from M^n to $N^{n-k} \times R^k$. The estimate is meaningful if all 2^k u.n.p. exist. It does not depend on the choice of the separating solution Γ of the system of Pfaffian equations.

The proof of the theorem consists of producing the collection of mappings and reference to Assertion 2. Both are done in steps. At each step the number of Pfaffian equations is decreased by one. By hypothesis, the manifold Γ is a separating solution of the equation $\alpha_k = 0$ on the manifold Γ_{k-1}. On this manifold the restriction of the function ρ is a covering function, and the form $v^n/\alpha_1 \wedge \ldots \wedge \alpha_{k-1}$ is a volume form. The operation $\tilde{*}$ for this form is defined by the relation $\tilde{*}\,\tau^{n-k+1} = *(\alpha_1 \wedge \ldots \wedge \alpha_{k-n} \wedge \tau^{n-k+1})|_\Gamma$. According to Assertion 2, the u.n.p. of the point a for the map $F|_\Gamma:\Gamma \to N^{n-k}$ can be estimated in terms of the u.n.p. of the point $(a, 0)$ in $N^{n-k} \times \mathbf{R}^1$ of the following two maps: the restriction to Γ_{k-1} of the map (F, ρ) and the restriction to Γ_{k-1} of the map $(F, *(\alpha_1 \wedge \ldots \wedge \alpha_{k-1} \wedge F^*w^{n-k}))$. It is necessary to continue this process. To estimate the u.n.p. of the point $(a, 0) \in N^{n-k} \times \mathbf{R}^1$ for each of the two maps constructed of Γ_{k-1} into $N^{n-k} \times \mathbf{R}^1$, it is necessary to construct two maps for each of Γ_{k-2} into $N^{n-k} \times \mathbf{R}^2$ and use Assertion 2, etc.

COROLLARY 2. Under the hypotheses of Theorem 1 let the manifolds M^n and N^{n-k} be R^n and R^{n-k}, the forms $\alpha_1,\ldots,\alpha_k$ have polynomial coefficients (i.e., $\alpha_i = \Sigma P_{ij}dx_j$, where P_{ij} are polynomials) and the map F be polynomial. Then the upper number of preimages which figures in Theorem 1 exists and can be estimated explicitly in terms of the degree of the polynomials which are the coefficients of the form α_i and the components of the map F.

For the proof it suffices to use Theorem 1, taking the standard volume forms in $\mathbf{R}^n$ and R^{n-k} and the covering function $\rho = 1/(1 + \Sigma x_i^2)$. Theorem 1 reduces the estimation of the number of solutions of a transcendental system of equations to the estimation of the number of solutions of 2^k polynomial equations, which, in its own right can be obtained from Bezout's theorem.

Applications of Corollary 2 can be found in [3-7]. In the next section we describe another situation in which Theorem 1 proves only the existence of estimates but cannot give their explicit form.

2. Separating Solutions on Nonsingular Semianalytic Sets

With an analytic variety M which is a semianalytic set in a real projective space, we associate the ring $\mathfrak{A}_M$ of functions, analytic on M and meromorphically extendable to $\bar{M}$ (i.e., $f \in \mathfrak{A}_M$, if for each point of $\bar{M}$ one can find a neighborhood in projective space and a meromorphic function in this neighborhood, whose restriction to M coincides with f). With the ring $\mathfrak{A}_M$ we associate the algebra Ω_M of exterior forms, generated by functions from the ring $\mathfrak{A}_M$ and their differentials.

Definition. We call the analytic variety $M \subseteq \mathbf{R}P^N$ simple, if: 1) M is a semianalytic set in $\mathbf{R}P^N$; 2) the ring $\mathfrak{A}_M$ contains a covering function, i.e., a positive function on M, which tends to zero as one tends to the boundary; 3) there exists on M a volume form such that the ratio of any form of highest degree from Ω_M to this volume form is a function of $\mathfrak{A}_M$.

Example. 1) The space $\mathbf{R}^N$, imbedded in the standard way in $\mathbf{R}P^N$, is the simplest example of of a simple manifold. The ring $\mathfrak{A}_{\mathbf{R}^N}$ contains all rational functions on $\mathbf{R}^N$. 2) The product of two simple manifolds is a simple manifold (for any algebraic imbedding of the product of projective spaces in projective space).

THEOREM 2. Suppose given on the simple n-dimensional analytic manifold M a system of Pfaffian equations $\alpha_1 = \ldots = \alpha_k = 0$, where α_i are 1-forms from Ω_M. In addition, let there be fixed $n - k$ functions $f_1, \ldots, f_{n-k}$ from the ring $\mathfrak{A}_M$. Then there exists a number C such that for any separating solution Γ of the system of Pfaffian equations and for any parameters $a_1, \ldots, a_{n-k}$, the system of equations $f_1 = a_1, \ldots, f_{n-k} = a_{n-k}$ has no more than C nondegenerate solutions on Γ.

Proof. If the Pfaffian is not general (i.e., if k = 0), then Theorem 2 follows from Gabriélov's theorem [2]. Theorem 1 of Sec. 1 reduces the general case to the one considered.

Remark. Gabriélov's theorem in its own right can be proved by the method of Sec. 1. Only an insignificant modification of it is needed. In Sec. 1 we eliminated singularities by small perturbations. Instead of this it is necessary to consider singular analytic sets and their stratification. I expect to return to this question in a forthcoming publication.

COROLLARY 1. Under the hypotheses of Theorem 2 suppose the functions f_i depend polynomially on parameters. Then there exists an estimate for the number of solutions of the system, independent of the choice of parameters.

For the proof it suffices to apply Theorem 1 to the product of the manifold by the space of parameters $\mathbf{R}^N$, the system of Pfaffian equations $\pi_1^*\alpha_1 = \ldots = \pi_1^*\alpha_k = 0$, and the collection of functions $\pi_1^* f_1, \ldots, \pi_1^* f_{n-k}, \pi_2^* b_1, \ldots, \pi_2^* b_N$. Here π_1, π_2 are the projections of the Cartesian product onto the first and second factors, b_j are the coordinate functions in $\mathbf{R}^N$.

Theorem 2 is sufficient for most applications. For example, the central technical assertion of [8] (Lemma 1 of Sec. 3) follows from it. Our further goal is to construct the category of manifolds in which this theorem works.

We dwell on certain properties of simple manifolds.

We call a finite set of N functions on a manifold separating, if these functions give an imbedding (without self-intersections or singularities) of the manifold into $\mathbf{R}^N$.

Assertion 1. The ring $\mathfrak{A}_M$ of any simple manifold contains a separating set.

Proof. Real projective space can be imbedded in $\mathbf{R}^N$.

By an affine domain in a simple manifold is meant a domain defined by a condition $f \neq 0$, where $f \in \mathfrak{A}_M$.

Assertion 2. 1) The union and intersection of a finite number of affine domains are an affine domain. 2) The complement of the zeros of a form from Ω_M is an affine domain. 3) Each affine domain $U \subseteq M \subseteq \mathbf{R}P^N$ is a simple submanifold of $\mathbf{R}P^N$.

Proof. The sum of the squares of functions vanishes precisely at those points where all the functions are equal to zero. This proves that the union of affine domains is an affine domain. The rest of the assertions are proved just as simply.

3. Pfaffian Manifolds

Definition. An analytic submanifold X of codimension k in a simple analytic manifold M is called a simple Pfaffian manifold if there exist a finite collection of affine domains U_i and a finite collection of 1-forms $\alpha_{i1}, \ldots, \alpha_{ik} \in \Omega_M$ such that:

1) in the domain U_i there exists a separating solution X_i of the system of Pfaffian equations $\alpha_{i1} = \ldots = \alpha_{ik} = 0$ such that the intersection of the submanifold X with the domain U_i consists of connected components of the manifold X_i;

2) the domains U_i cover X.

Definition. A system of equations defines a submanifold *indivisibly*, if: 1) it defines the submanifold; 2) any covector which vanishes on the tangent space of the submanifold is a linear combination of the differentials of the equations.

Assertion 1. A submanifold Y of codimension m, indivisibly defined by a finite system of equations $f_1 = \ldots = f_N = 0$, $f_i \in \mathfrak{A}_M$ in a Pfaffian submanifold $X \subseteq M$, is a Pfaffian submanifold of M.

Proof. Let U_i be the covering which figures in the definition of the Pfaffian submanifold X. With an index i and an increasing collection of indices $J = \{j_1, \ldots, j_m\}$, $1 \leqslant j \leqslant \ldots \leqslant j_m \leqslant N$, we associate the domain $U_{i,J}$ in which the form $\alpha_{i1} \wedge \ldots \wedge \alpha_{ik} \wedge df_{j_1} \wedge \ldots \wedge df_{j_m}$ does not vanish. In this domain, the manifold defined in X by the equations $f_{j_1} = \ldots = f_{j_m} = 0$ is a separating solution of the system $\alpha_{i1} = \ldots = \alpha_{ik} = df_{j_1} = \ldots = df_{j_m} = 0$ (under the condition that $m > 0$), and the manifold $Y \cap U_{i,J}$ consists of connected components of it.

In the same way one verifies

Assertion 2. The product of simple Pfaffian submanifolds of simple manifolds M_1 and M_2 is a simple Pfaffian submanifold of $M_1 \times M_2$.

Definition. 1) We call an analytic manifold X with ring of analytic functions $\mathfrak{A}_X$ a simple Pfaffian manifold, and the functions of the ring $\mathfrak{A}_X$ simple functions, if there exists an imbedding π of the manifold X in a simple analytic manifold M, such that $\pi(X)$ is a simple Pfaffian submanifold of M, and the ring $\mathfrak{A}_X$ is induced from the ring $\mathfrak{A}_M$ by the imbedding π. 2) We call a map of one simple Pfaffian manifold into another a simple map, if it induces a homomorphism of the rings of simple functions.

A composition of simple mappings is a simple mapping. However, a map which is inverse to a simple one is not simple in general. Our next goal is the extension of rings of functions on simple Pfaffian manifolds with the help of the addition of inverse maps.

Definition. 1) Let M_1 and M_2 be two simple Pfaffian manifolds. We call the map $f: M_1 \to M_2$ admissible, if there exist a simple Pfaffian manifold M and simple maps of it $\pi_1: M \to M_1$ and $\pi_2: M \to M_2$ such that the projection π_1 is an analytic diffeomorphism between M and M_1, and $f = \pi_2 \pi_1^{-1}$.

2) We call a function $f: M_1 \in \mathbf{R}^1$ on a simple Pfaffian manifold admissible, if there exist a simple Pfaffian manifold M, a simple map of it $\pi_1: M \to M_1$, and a simple function $g: M \to \mathbf{R}^1$ such that π_1 is an analytic diffeomorphism and $f = g\pi_1^{-1}$.

3) The manifold M with projection $\pi_1: M \to M_1$, which figures in 1), 2), is called a resolution of the corresponding mapping or function.

Assertion 3. A finite number of admissible functions and maps have a common resolution (i.e., there exists one manifold M with projection $\pi_1: M \to M_1$, on which the fixed finite number of admissible functions and maps of the manifold become simple).

COROLLARY 1. On a simple Pfaffian manifold the admissible functions form a ring.

Proof of the Corollary. Any two admissible functions can be considered on a common resolution. On this resolution one can carry out arithmetic operations on the functions.

The proof of Assertion 3, and also all the other assertions of the present section, is based on Assertion 4 which is formulated below and is easily verified.

Let M_1, M_2, M_3 be smooth manifolds and $\pi_1: M_1 \to M_3$, $\pi_2: M_2 \to M_3$ be smooth maps, whose images intersect transversely in M_3 [i.e., if $\pi_1(a) = \pi_1(b) = c$, then the images of the tangent spaces at the points a and b to M_1 and M_2 generate the tangent space to M_3 at the point c].

Assertion 4. The set of points $(a, b) \in M_1 \times M_2$ for which $\pi_1(a) = \pi_2(b)$ is a smooth submanifold. Further, for any finite separating set of functions $\{f_i\}$ on M_3 the system of equations $\pi_1^* f_i(a) = \pi_2^* f_i(b)$ in $M_1 \times M_2$ defines this submanifold somehow.

To prove Assertion 3 it is necessary to consider with two resolutions $\pi_1 : M_1 \to M$ and $\pi_2 : M_2 \to M$ the resolution $\pi : \Gamma \to M$, where Γ is the submanifold in $M_1 \times M_2$, defined by the condition $\pi_1(a) = \pi_2(b)$, and π is the projection defined by the condition $\pi = \pi_1 \rho_1 = \pi_2 \rho_2$, where ρ_i are the projections of the product $M_1 \times M_2$ on the factors. Assertions 1 and 4 guarantee that π is a resolution.

Assertion 5. Let $\{f_i\}$ be a finite separating set of admissible functions on the simple Pfaffian manifold M. The map g of the simple Pfaffian manifold K into M is admissible if and only if all the functions $\{g^* f_i\}$ are admissible.

Proof. Let $\pi : \tilde{M} \to M$ be a resolution for the functions $\{f_i\}$ and $\rho : \tilde{K} \to K$ be a resolution for the functions $\{g^* f_i\}$. The submanifold of $\tilde{K} \times \tilde{M}$, defined by the condition $g\rho(a) = \pi(b)$, together with its projection onto K gives a resolution for the map $g : K \to M$.

Assertion 6. A map of one simple Pfaffian manifold into another is admissible if and only if it induces a homomorphism of the rings of admissible functions.

Proof. If one adds another function to a separating set of functions, then the set remains separating. Hence Assertion 5 follows from Assertion 4.

COROLLARY 2. The composition of admissible maps is an admissible map.

Definition. A domain on a simple Pfaffian manifold is called admissible, if it is the diffeomorphic image under an admissible map of some simple Pfaffian manifold. This manifold together with the diffeomorphism is called a resolution of the admissible domain. A map or a function on an admissible domain is called admissible, if it becomes admissible on a resolution of the domain.

Assertion 7. The intersection of a finite number of admissible domains in an admissible domain.

The proof of Assertion 7 is a repetition of the proof of Assertion 3.

Definition. By a Pfaffian atlas on a smooth manifold M is meant a finite open covering of it $M = \bigcup U_i$ together with diffeomorphisms φ_i of the domains U_i into simple Pfaffian manifolds, such that for all i, j the domain of definition of the diffeomorphism $\varphi_i \varphi_j^{-1}$ is an admissible domain, and the diffeomorphism is an admissible map. Two atlases are called equivalent if their union is an atlas. A Pfaffian manifold structure is a collection of equivalent atlases.

A Pfaffian domain is a domain on a Pfaffian manifold, which in each chart of some atlas is admissible. One defines Pfaffian maps and functions analogously. A Pfaffian form is a form which in each chart of some atlas lies in the exterior algebra generated by the Pfaffian functions and their differentials. A Pfaffian vector field is a differentiation on the ring of Pfaffian functions (which does not leave this ring).

We list some properties of Pfaffian manifolds.

1. A real algebraic manifold has a unique structure as a Pfaffian manifold, compatible with the algebraic structure (i.e., such that all semialgebraic domains are Pfaffian domains and all algebraic functions are Pfaffian functions).
2. The manifold of nonsingular points of a semianalytic set has a unique structure as a Pfaffian manifold, compatible with the analytic structure (i.e., such that all semianalytic domains are Pfaffian domains and all analytic functions which can be meromorphically extended to the boundary of the domain are Pfaffian functions).
3. The tangent and cotangent bundles over a Pfaffian manifold and their tensor products, the Cartesian products of several Pfaffian manifolds, and spaces of jets of mappings of one Pfaffian manifold into another have natural Pfaffian manifold structures.
4. The complement of the set of zeros of some Pfaffian function or Pfaffian form and the complement of the preimage of a point under a Pfaffian map are Pfaffian domains. The union and intersection of a finite number of Pfaffian domains and one or several connected components of a Pfaffian domain are Pfaffian domains.

5. The preimage of a regular value under a Pfaffian map, a submanifold indivisibly defined as the intersection of zero level surfaces of a finite number of Pfaffian functions, are Pfaffian submanifolds.

6. A submanifold which is a separating solution of $\alpha_1 = \ldots = \alpha_k = 0$ in which α_i are Pfaffian forms, one or several components of such a submanifold, and also a submanifold which admits a representation of the sort described in each chart of some Pfaffian atlas are Pfaffian submanifolds.

7. Compositions of Pfaffian maps and their jet extensions are Pfaffian maps. If the graph of some map of one Pfaffian manifold into another is a Pfaffian submanifold of their Cartesian product, then the map is a Pfaffian map. If there exists an analytic inverse of a Pfaffian map, then it is a Pfaffian map.

Remark. The construction of Pfaffian manifolds from simple Pfaffian manifolds is done in two steps. At the first step one extends the supply of admissible functions on simple manifolds with the help of inverse maps which are diffeomorphisms. At the second step one performs ordinary gluings (one glues simple manifolds with the help of the extended supply of functions). Both steps of the construction can be applied to another initial supply of manifolds. For example, if one starts with real affine algebraic manifolds, then among the glued objects one will find, for example, all real algebraic manifolds, and among the admissible maps, single-valued branches over real manifolds of multivalued algebraic maps. Here is another example. Instead of simple analytic manifolds one can take the manifold $\mathbf{R}^n$ with the ring of polynomials; instead of simple Pfaffian manifolds, separating solutions of Pfaffian equations with polynomial 1-forms in $\mathbf{R}^n$. As a result of applying the two steps of the construction one gets another category of Pfaffian manifolds. In this category one can prove not only the finiteness of the connected components of a manifold, etc. (cf. the following section), but give explicit upper estimates of the number of connected components, etc. More details about this category can be found in [3, 7].

4. Finiteness Theorems

By a realization of a distribution of linear subspaces in the tangent bundle of a Pfaffian manifold is meant a choice of a finite covering of the manifold by Pfaffian domains together with a distribution in each domain given by a system of equations $\alpha_1 = \ldots = \alpha_k = 0$, where α_i are Pfaffian 1-forms (one for each domain). We call an integral manifold of the distribution a leaf compatible with a realization if in each domain of the covering it consists of connected components of some separating solution of the corresponding system of equations $\alpha_1 = \ldots = \alpha_k = 0$.

Example. Let g be a Pfaffian map of a manifold into a k-dimensional manifold. With the map there is associated a distribution which assigns to each point the kernel of the differential of the map. Below we describe a realization of this distribution with which all leaves which are preimages of regular values of the map are compatible. We cover the manifold by a finite number of simple Pfaffian manifolds and in each of them we choose a finite separating set of functions $\{f_i\}$. We cover each simple manifold in its own right by domains U_I, where $I = \{i_1,\ldots,i_k\}$ is a set of k indices, U_I is the complement of the zeros of the form $\omega_I = df_{i_1} \wedge \ldots \wedge df_{i_k}$. One covers the manifold-preimage by the domains $g^{-1}U_I$, in each of which one fixes the system $df_{i_1} = \ldots = df_{i_k} = 0$ and gives the necessary realization of the distribution of kernels of the differential of the map g.

THEOREM 3. Suppose given on a Pfaffian manifold which has some proper Pfaffian imbedding in $\mathbf{R}^N$ a distribution with a fixed realization. Then there exists a number C such that each leaf of the distribution which is compatible with the realization has the homotopy type of a cell complex, in which the number of cells is $\leqslant C$.

COROLLARY 1. Let a Pfaffian manifold admit a proper Pfaffian imbedding in $\mathbf{R}^N$. Then the sum of the Betti numbers is finite. Further, for any Pfaffian map of this manifold, the sum of the Betti numbers of the preimage of any regular value of the map is uniformly bounded (outside the dependence on the choice of regular value).

Proof of Theorem 3. We consider the function $\varphi_a = \sum (f_i - a_i)^2$, where f_i are coordinate functions defining a proper imbedding of the manifold in $\mathbf{R}^N$, and $a = (a_1,\ldots,a_N)$ is a point in $\mathbf{R}^N$. According to Morse theory [9], it suffices for us to find a uniform estimate of the number of nondegenerate critical points of the restriction of the function φ_a to a leaf of the distribution, independent of the choice of the parameter a and the leaf.

For this, it suffices to estimate the number of critical points on the part of the leaf which lies in a Pfaffian domain in which: 1) there exists a volume form equal to $dg_1 \wedge \ldots \wedge dg_m$, where m is the dimension of the manifold and g_i are certain Pfaffian functions in the domain; 2) there exists a covering Pfaffian function; 3) the distribution is defined by a system of equations $\alpha_1 = \ldots = \alpha_k = 0$. At a critical point the restriction of φ_a to a leaf of the form $\omega_a = d\varphi_a \wedge \alpha_1 \wedge \ldots \wedge \alpha_k$ vanishes. Let $I = \{i_1, \ldots, i_p\}$ be a set of p positive increasing indices, where p is the dimension of the leaf (i.e., $p + k = m$) and $i_p \leqslant m$. With the set I we associate the system of equations $h_1 = \ldots = h_p = 0$ on a separating solution of the system $\alpha_1 = \ldots = \alpha_k = 0$, where $h_j = *(\omega_a \wedge df_{i_1} \wedge \ldots \wedge \widehat{df_{i_j}} \wedge \ldots \wedge df_{i_p})$. Here the map * is constructed with respect to the volume form in the domain and the symbol ^ over the differential of a function df_{i_j} indicates that this differential does not appear in the product. A Morsean critical point of the restriction of a function to a fiber is a nondegenerate solution of one of the equations of the system constructed. Now Theorem 3 follows from Corollary 1 of Sec. 2.

THEOREM 4. The number of connected components of the preimage of any point under a Pfaffian map is finite and admits a uniform estimate independent of the choice of points in the preimage.

Proof. It suffices to estimate uniformly in a simple Pfaffian manifold M the number of connected components of the set of solutions of the system of equations $f_1 = a_1, \ldots, f_k = a_k$, where f_i are Pfaffian functions, a_i are parameters. For this we consider in the manifold $M \times \mathbf{R}^1$ the set defined by the system $u(\rho - \varepsilon_1) = r$, $\Sigma(f_i - a_i)^2 = \varepsilon_2$, where u is a coordinate function in $\mathbf{R}^1$, ε_1, ε_2, r are nonnegative numbers, ρ is a covering function on M. For almost all r, ε_2 the system defines a smooth manifold. According to Theorem 3, the sum of the Betti numbers of this manifold is bounded independently of the choice of parameters a_i, ε_j, r by some number C. Using the compactness of the subset of M defined by the inequality $\rho \geqslant \varepsilon_1 > 0$, and simple arguments of general topology, it is easy to show that the number of connected components being estimated does not exceed C.

5. Abelian Integrals

Let X and Λ be nonsingular complex quasiprojective algebraic varieties, $\pi: X \to \Lambda$ be a rational map which is regular on X and is a topological locally trivial bundle. We fix a regular rational r-form on X, which is closed on any fiber of the bundle. The integral of such a form over an r-dimensional cycle lying in a fiber and varying continuously under passage from one fiber to another is a multivalued analytic function on the base Λ, parametrizing the fibers. Such complex analytic functions on Λ are called multivalued Abelian functions. An algebraic function is an example of an Abelian function (X and Λ can have the same dimension, the cycle can be a point, and the form a function). A single-valued Abelian function means a branch of a multivalued Abelian function over some fixed domain $U \subset \Lambda$, which is a real-semialgebraic set (under consideration of the complex manifold Λ as a real manifold of double the dimension). We stress that in the definition of a single-valued Abelian function one fixes the domain on which it is considered. By an Abelian map into $\mathbf{C}^N$ of a real semialgebraic domain U of a complex manifold is meant a map, all of whose components are single-valued Abelian functions. Let the image of the domain U under the Abelian map f lie in an algebraic variety (this variety can coincide with $\mathbf{C}^N$), in some domain V of which there is given an Abelian map g. Then in the domain $f^{-1}(V) \cap U$ there is defined the composition of the Abelian maps. Continuing this process one can construct compositions of Abelian maps together with their domains of definition.

THEOREM 5. The preimage of any point under a composition of Abelian maps has a finite number of connected components (in the domain of definition of the composition). This number is bounded by a constant, independent of the choice of point in the image manifold.

Proof. It follows from the local description of Abelian functions given in [8] that the realification of an Abelian map is a Pfaffian map. Hence Theorem 5 follows from Theorem 4. We note that the fact that an Abelian map is Pfaffian is derived from its local description. We consider the plane of a complex variable z, slit, for example, along the ray of negative real numbers. The realification of a branch of the function ln z in the slit plane is a Pfaffian map. In fact, $\mathrm{Re} \ln z = \ln\sqrt{x^2 + y^2}$ and $\mathrm{Im} \ln z = \arctan y/x + 2k\pi$, where $z = x + iy$.

The logarithm, root, and arctangent are the simplest Pfaffian functions. The realifization of a branch of the function z^{α} is a Pfaffian map if and only if α is a real number (for real α the real and imaginary parts of the function z^{α} can be expressed in terms of a real power function and the arctangent, for complex α one adds functions of the type of $\sin \ln x$, which oscillate near zero).

According to [8], a locally Abelian function on some resolution can be expressed in terms of analytic functions, logarithms and functions z^{α} for rational α. Hence its realification is a Pfaffian map.

Remark 1. We consider a one-dimensional complex disk which intersects a manifold which is a branch of a multivalued Abelian function in one point. A monodromy operator is associated with circuit about this point in the disk. That the exponent α is real is equivalent with the fact that all eigenvalues of the monodromy operator lie on the unit circle. Precisely this property of Abelian functions [10, 11] corresponds to their finiteness property. In Theorem 5 instead of Abelian functions one could consider more general functions, satisfying equations of Fuchs type, for which the eigenvalues of the local monodromy operators lie on the unit circle.

Remark 2. A single-valued real-analytic branch of the real or imaginary part of a multivalued Abelian function can be considered over a Pfaffian submanifold, and in particular, over semialgebraic domains of real algebraic submanifolds. The assertion of Theorem 5 is valid for such single-valued branches of Abelian functions and their compositions (its proof remains as before). In this extended form Theorem 5 contains the result of [8].

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