CYCLES OF DYNAMICAL SYSTEMS ON THE PLANE AND ROLLE'S THEOREM

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In this paper it is shown that cycles of planar dynamical systems with polynomial vector fields are in many ways similar to ovals of algebraic curves, and more generally to "separating solutions" of such systems, on general algebraic curves. For example, for separating solutions the analog of Bezout's theorem is valid. The proofs are straightforward. They are based on the version of Rolle's theorem and Bezout's theorem for planar algebraic curves offered below.

We note that according to Hilbert's conjecture the total number of limit cycles of a dynamical system with a polynomial vector field is bounded above by the degree of the field. From the validity of the conjecture and the results of the present note it follows that the curve consisting of all the limit cycles of a polynomial dynamical system resembles an algebraic curve. At the present time, even the finiteness of the number of limit cycles has not been proved: Il'yashenko proved that Dyudak's proof [1] contains an unfillable gap (cf. [2]).

1. Rolle's Theorem for Dynamical Systems

We consider a smooth dynamical system on the plane

$$\dot{x}=F(x);\ \ x=x_1,\ x_2;\ \ F=F_1,\ F_2. \tag{1}$$

Definition. An oriented smooth (possibly not connected) curve on the plane is called a *separating solution* for a dynamical system, if a) the curve consists of trajectories of the system (with the natural orientation of the trajectories), b) the curve does not pass through any singular points of the system, c) the curve is the boundary of some domain in the plane with the natural orientation of the boundary.

Example 1. A cycle of a dynamical system is always a separating solution of it: it is oriented either as the boundary of the interior domain in relation to the cycle, or as the boundary of the exterior domain.

Example 2. A noncompact trajectory which goes to infinity as $t \to +\infty$ and as $t \to -\infty$ is a separating solution.

A connected component of any separating solution is either a cycle or a noncompact trajectory which goes to infinity as $t \to \pm\infty$.

Example 3. A nonsingular level line of the function $H(x_1, x_2) = c$, oriented as the boundary of the domain $H(x_1, x_2) > c$, is a separating solution of the Hamiltonian system $\dot{x}_1 = \partial H/\partial x_2$, $\dot{x}_2 = -\partial H/\partial x_1$.

Definition. A *point of contact* of a curve and a dynamical system on the plane is a point of the curve at which the tangent vector to the curve and the vector of the dynamical system are collinear.

One has the following theorem.

THEOREM 1 (Rolle's Theorem for Dynamical Systems on the Plane). Suppose for a continuous curve (i.e., the image of an interval or of a circle under a continuous map into the plane) at each point there exists a tangent vector. Then between two points of intersection of the curve with a separating solution of a dynamical system there is a point of contact.

The ordinary theorems of Rolle, Lagrange, and Cauchy are obtained from Theorem 1 in the case of a dynamical system with constant vector field.

In propositions one more often needs not Theorem 1 itself, but the simpler

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LEMMA 1. Theorem 1 is valid for C^1-smooth curves which intersect a separating solution transversely.

Proof. Let t_1 and t_2 be the parameters of two successive transverse intersections of a curve with a separating solution. This solution divides the plane into two (not necessarily connected) domains: the domain U_1, whose oriented boundary it is, and the complementary domain $U_2 = R^2 \setminus \bar{U}_1$. Suppose for definiteness that at time t_1 the curve enters the domain U_1. Then at time t_2 the curve leaves the domain U_1. Hence the ordered pairs of vectors consisting of the tangent vector to the curve and the vector of the dynamical system, at times t_1 and t_2 determine opposite orientations of the plane. Consequently, at some intermediate time the tangent vector and the vector of the dynamical system are collinear.

Remark. The concept of a point of contact of a curve with a dynamical system was introduced by Poincare in his memoir [3]. Lemma 1 is close to his Theorem 10 (ibid., p. 59) according to which an arc of a curve "supporting" a characteristic (ibid., p. 17) has a point of contact with the dynamical system. Lemma 1 is based on the fact that an arc of a curve, contained between its successive transverse intersections with a separating solution, necessarily supports this solution.

We proceed to the proof of Theorem 1. In comparison with Lemma 1, we encounter two complications: firstly, the curve may be tangent to the separating solution, and secondly, the velocity vector of the curve may be discontinuous.

First we look into the first complication. The intersection of the curve with the separating solution is a closed set. The complement of this set splits into (finite or infinite) intervals. It suffices for us to show that each finite interval of this kind contains a point of contact. Let t_1 and t_2 be the parameters of the ends of such an interval, and t_1 precede t_2 in the sense of the orientation of the curve. Suppose for definiteness for all intermediate values of the parameters the curve lies in the domain U_1. We shall show that there exists a time close to the time t_1, at which the ordered pair of vectors consisting of the tangent vector to the curve and the vector of the dynamical system define the same orientation of the plane as the transversal entry of the curve into the domain U_1. For this we choose a local coordinate system near the point of the curve with parameter t_1, in which the vector field of the dynamical system is constant [4]. In these local coordinates y_1, y_2 suppose the domain U_1 is defined by the inequality $y_2 > 0$, the dynamical system has the form $\dot{y}_1 = 1$, $\dot{y}_2 = 0$, the separating solution is $y_2 = 0$, and the curve is defined by the vector-function $y_1(t)$, $y_2(t)$. By hypothesis, $y_2(t) > 0$ for $t > t_1$ (and $t < t_2$) and $y_2(t_1) = 0$. Hence at some time close to t_1, the derivative of the function $y_2(t)$ will be positive. At this time the ordered pair of vectors also defines the orientation of the plane needed. By the same arguments, there exists a time close to the time t_2, at which the corresponding pair of vectors gives the opposite orientation.

One deals with the second complication analogously. If there does not exist a point of contact, then one can find a point of the curve in any neighborhood of which the corresponding pair of vectors determines both one and the other orientation of the plane; however the vectors are not collinear at any point. In order to see the impossibility of such a situation, it suffices to straighten the vector field near this point and to use Darboux's theorem (that the derivative of a function assumes all intermediate values). Theorem 1 is proved.

Remark. We do not need curves with discontinuous derivative anywhere. They are included in Theorem 1 only to make the interconnection of this theorem and Rolle's theorem obvious. The case of tangency on the contrary is used in what follows.

COROLLARY. Let a curve which is a smooth submanifold of the plane contain no more than N noncompact (and any number of compact) connected components and have no more than k points of contact with a dynamical system. Then it has no more than N + k isolated points of intersection with any separating solution of this system.

In fact, on a compact component there are no more points of intersection than points of contact. On a noncompact one the points of intersection can be only one more than the points of contact.

2. Algebraic Properties of P-Curves

Definition. A curve in the plane is called a *P-curve of degree n*, if there exists an orientation of this curve under which it is a separating solution of a dynamical system, the components of whose vector field are polynomials of degree n.

Smooth planar algebraic curves of degree n + 1 are P-curves of degree n (cf. Example 3 of Sec. 1). Thus, P-curves are a generalization of real algebraic curves. We shall show that P-curves have a number of properties of algebraic curves.

THEOREM 2. The restriction of a polynomial of degree m to a P-curve of degree n has no more than m(n + m) isolated roots on it.

Remark 1. If the polynomial has nonisolated roots on a connected component of a P-curve, then on this component it is identically equal to zero, since a P-curve is analytic.

Remark 2. We take a trajectory of a dynamical system which winds about a limit cycle. Such a trajectory does not bound a domain and is not a P-curve. A straight line intersecting the cycle intersects such a trajectory a countable number of times.

We preface the proof of the theorem with the following

Assertion 1. Let almost all values of a continuous function on a smooth curve have no more than N preimages. Then each value of this function has no more than N isolated preimages.

Proof. Let us assume that among the preimages of the value a there exist N + 1 isolated points. We fix small disjoint intervals around these points, at the ends of which the function is not equal to a. Let us say that a preimage is positive (negative), if at both ends of the surrounding interval the function is greater (less) than a (a preimage can also be nonpositive and nonnegative). Suppose for definiteness the number of positive preimages is not less than the number of negative ones. Then for all small $\varepsilon > 0$ there exist no less than N + 1 preimages of the value $a + \varepsilon$. The contradiction proves Assertion 1.

We proceed to the proof of Theorem 2. Let Q be a polynomial of degree m. It suffices for us to show that for almost all values a the polynomial Q has no more than m(n + m) preimages on a P-curve. By Sard's theorem, for almost all a, the equation $Q = a$ defines a smooth curve on the plane, which intersects a fixed separating solution transversely. The points of contact of the curve $Q = a$ and the dynamical system (1) satisfy the polynomial relation

$$Q'_{x_2}F_1 + Q'_{x_1}F_2 = 0 \qquad (2)$$

of degree (n + m − 1). Relation (2) either defines an algebraic curve on the plane or holds identically. In the first case, according to Bezout's theorem, the curve $Q = a$ has no more than m(n + m − 1) points of intersection with the curve (2) for almost all a. The curve $Q = a$, like any algebraic curve of degree m, has no more than m noncompact components. Hence, according to the corollary of Sec. 1, it has no more than m(n + m) intersections with a separating solution. If (2) holds identically, then the polynomial Q is constant on trajectories of the dynamical system. In this case, the equation $Q = a$ in general does not have isolated roots on a separating solution of the system. Theorem 2 is proved.

Assertion 2. The estimate in Theorem 2 cannot be improved more than three times (for $n > 0$).

In the proof we shall use only the following facts: a) among the P-curves of degree n, there are contained all smooth algebraic curves of degree n + 1; b) among the P-curves of degree $n > 0$, there are also contained nonalgebraic curves.

The Bezout estimate of the number of points of intersection of real algebraic curves is sharp. Hence in Theorem 2 it is impossible to make an estimate smaller than A = m(n + 1). On the other hand, through any B = (m + 1)(m + 2)/2 − 1 points of the plane one can pass an algebraic curve of degree m. Taking these points on a nonalgebraic component of a P-curve, we see that it is impossible to make an estimate in Theorem 2 smaller than B (the P-curves of degree 0 constitute an exception here — all of them are straight lines and do not contain nonalgebraic components). Further, for $m \leqslant 2n$ the estimate of Theorem 2 is less than 3A, and for $m \geqslant 2n$, less than 3B.

COROLLARY 1. A P-curve of degree n has no more than n + 1 noncompact components.

Proof. It is straightforward to prove (cf. [7]) the following assertion: if for a curve which is a submanifold on the plane the number of transverse intersections with any line does not exceed N, then the curve has no more than N noncompact components. It remains to use Theorem 2 for a linear function.

The estimate of Corollary 1 is sharp: there exist algebraic curves of degree $n + 1$ with $n + 1$ noncompact components (the simplest example is a curve consisting of parallel lines).

COROLLARY 2. All cycles of a dynamical system with a polynomial field of degree 2 are convex.*

Proof. For any nonconvex oval there exists a line which intersects it in no less than four points. By Theorem 2, a line may intersect a P-curve of degree 2 only in three points.

Remark. Noncompact trajectories of a field of degree 2 may be nonconvex. For example, a noncompact component of an algebraic curve of degree 3 may turn out to be nonconvex. The tangent to such a curve at a point of inflection intersects it exactly once. It is straightforward to show that for any trajectory of a field (not only a P-curve) of degree 2 the tangent to a trajectory at a point of inflection intersects the trajectory precisely once.

COROLLARY 3. The restriction of a polynomial of degree m to a P-curve of degree n has no more than $(n + m - 1)(2n + m - 1)$ isolated critical points on this curve.

Proof. The Lie derivative of a polynomial of degree m along a polynomial field of degree n is a polynomial of degree $n + m - 1$. Now one gets the estimate needed from Theorem 2.

COROLLARY 4. A P-curve of degree n has no more than n^2 compact components.

Proof. According to Corollary 3, a linear function has no more than $2n^2$ critical points on a P-curve of degree n. On the other hand, on each oval it has a maximum and a minimum.

Examples are known of algebraic curves of degree $n + 1$ which have $n(n - 1)/2$ compact components.

COROLLARY 5. A P-curve of degree n has no more than $(3n - 1)(4n - 1)$ points of inflection (there can be straight lines among the components of a P-curve).

In fact, at points of inflection of the trajectories of the vector field F, the vectors $\dot{x} = F(x)$ and $\ddot{x} = (\partial F/\partial x)(x)F(x)$ are collinear. At these points the determinant of the matrix whose rows are the vector F and the vector $(\partial F/\partial x)F$ vanishes. For a field F of degree n, this determinant is a polynomial of degree $3n - 1$. By Theorem 2 this determinant has no more than $(3n - 1)(4n - 1)$ isolated zeros on any separating solution of the system $\dot{x} = F(x)$. The components of a separating solution containing nonisolated zeros of the determinant are straight lines.

THEOREM 3 (Bezout's Theorem for P-Curves). Two P-curves of degrees n and m have no more than $(n + m)(2n + m) + n + 1$ isolated points of intersection.

Proof. At points of collinearity of polynomial fields F and G of degrees n and m, the polynomial Q of degree $n + m$ which is the determinant of the matrix with rows F and G vanishes. Let Γ be a separating solution of the system $\dot{x} = F(x)$. Let us assume in addition that all zeros of the polynomial Q on the curve Γ are isolated. Then according to Theorem 2, they do not exceed $(n + m)(2n + m)$ in number. Thus, the curve Γ has no more than $(n + m) \times (2n + m)$ points of contact with the system $\dot{x} = G(x)$. The number of noncompact components of this curve does not exceed $n + 1$ (Corollary 1 of Sec. 2). Hence, the number of points of intersection of the curve Γ with any separating solution of the system $\dot{x} = G(x)$ does not exceed $(n + m)(2n + m) + n + 1$ (the corollary of Sec. 1).

Now we turn to the general situation, when the set Q can have nonisolated zeros and consequently can vanish identically on certain components of the curve Γ. Such components of the curve Γ are trajectories of the system $\dot{x} = G(x)$ and either do not intersect or else coincide with components of a separating solution of this system. In both cases such components make no contribution to the number of isolated points of intersection of two separating solutions and in no way interfere with the calculation made. The theorem is proved.

We give another assertion which is related to P-curves.

Assertion 3. Let a vector field of degree n be tangent to the boundary of a compact domain and not vanish on the boundary. Then the Euler characteristic of the domain does not exceed in modulus the number $(1/2)(n^2 + n)$.

*As the referee of the present note kindly pointed out, this result is not new: it is due to Tun Tszin'-Chzhu (cf. Matematika, the periodic collection of translations of foreign papers, 6, No. 2 (1962), Lemma 3 on p. 153).

Proof. One can assume that all singular points of the field are of finite multiplicity (otherwise the components of the field can be divided by a common factor). We denote by ind^+ the total index of all singular points of the vector field on the plane having positive index, and by ind^- the negative ones. One has

$$ind^+ - ind^- \leqslant n^2,$$
$$|ind^+ + ind^-| \leqslant n.$$

The first of these inequalities follows from Bezout's theorem, and the proof of the second can be found in [5]. Under the hypotheses of Assertion 3, the Euler characteristic of a domain is equal to the sum of the indices of the singular points of the field lying inside the domain. One can now derive the estimate needed from the inequalities given.

3. Generalizations

The class of P-curves can be generalized as follows: one can permit that among the trajectories of which the P-curve consists, there occur singular points. Generalized cycles are examples of such generalized P-curves. Singular algebraic curves are another example. All assertions of the present note extend without difficulty to generalized P-curves.

There is also a multidimensional generalization. This generalization consists of constructing an extensive class of transcendental varieties, similar to algebraic varieties. Similarly to the way P-curves are obtained by solving polynomial differential equations, these varieties are obtained by successive solution of Pfaffian polynomial equations. A sketch of the construction can be found in [6, 7]. A detailed account will be in print soon.

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