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ASYMPTOTICS OF INTEGRALS OVER VANISHING CYCLES AND THE NEWTON POLYHEDRON

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In this note we discuss the relation between the vanishing rate of integrals of a holomorphic differential form over cycles which vanish at a critical point of the germ of a holomorphic function and the Newton order of the form; that is, the order computed from the Newton polyhedron of the germ. The vanishing rates of integrals of various forms define (see [1] and [2]) the spectrum of the germ—one of the most useful characteristics of the germ (see [2]–[4]). Saito recently proved [5] a conjecture of Steenbrink [4] on computing the spectrum of a germ from its Newton polyhedron. Saito's proof is highly technical (it uses the language of D-modules, of homological algebra, and of spectral sequences). This note results from reflecting on Saito's theorem. We present a different proof which is an elementary consequence of some known results and which, in our opinion, is completely transparent. This is our first joint work. We gratefully dedicate it to V. I. Arnol'd.

1. The filtrations. Let $f\colon \mathbf{C}^n, 0 \to \mathbf{C}, 0$ be the germ of a holomorphic function at an isolated critical point, and let Ω^p be the space of germs at $0 \in \mathbf{C}^n$ of holomorphic differential p-forms.

The *Hodge filtration* in Ω^p is the filtration by the asymptotics of the corresponding integrals. Namely, define the Hodge order $\alpha(\omega)$ of each form $\omega \in \Omega^n$ to be the largest number with the property that the exponents α in the expansion of the integral

$$\int_{\delta(t)} \omega/df = \sum a_{k,\alpha} t^{\alpha} (\ln t)^k \tag{1}$$

over any continuous family $\delta(t)$ of cycles which vanish at the critical point of the germ f all satisfy $\alpha \geq \alpha(\omega)$. The Hodge order function defines a decreasing filtration in Ω^n: the subspace with index α consists of the forms whose Hodge order is no smaller than α (see [2]).

The Newton filtration. We shall assume that the Newton polyhedron of the Taylor series of the germ in the chosen system of coordinates $x_1, \dots, x_n$ is *convenient*; that is, it intersects every coordinate axis. The (decreasing) *Newton filtration* in Ω^n is then constructed using the Newton polyhedron. Namely, the polyhedron is used in the standard way to define a decreasing filtration on power series in $x_1, \dots, x_n$; the subspace with index α consists of series all of whose monomials have Newton degree at least α. The filtration on power series defines a filtration on differential forms by setting the Newton order of the form $h(x_1, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$ to be equal to the Newton order of the series $h(x_1, \dots, x_n) x_1 \cdot x_2 \cdot \cdots \cdot x_n$ (that is, the Newton order of the form dx_i/x_i is assumed to be zero).

The spaces Ω_f and $\overline{\Omega}_f$. The quotient spaces $\Omega_f = \Omega^n/df \wedge \Omega^{n-1}$ and $\overline{\Omega}_f = \Omega^n/df \wedge d\Omega^{n-2}$ will play an important role in what follows. (The integral (1) is equal to zero if ω

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is representable in the form $df \wedge d\eta$, where $\eta \in \Omega^{n-2}$; in particular, the representatives of any element of $\overline{\Omega}_f$ have the same Hodge order. The motivation for the introduction of the space Ω_f is more complicated (see [6] and [2]). We mention only that this space is finite-dimensional, and its dimension is equal to the Milnor number of the germ f.)

Hodge and Newton filtrations are defined in the quotient spaces Ω_f and $\overline{\Omega}_f$; they are projections of the corresponding filtrations in Ω^n

The spectrum of a decreasing filtration. Every decreasing filtration has a *spectrum*, that is, a collection of numbers considered with multiplicity, the multiplicity of a number α being equal to the dimension of the quotient space with index α modulo the union of the subspaces with indices greater than α (if there is no subspace with index α, then α does not belong to the spectrum). We let $h(\alpha)$, $g(\alpha)$ and $\bar{h}(\alpha), \bar{g}(\alpha)$ be the multiplicity of the number α for the Hodge and Newton filtrations in Ω_f and $\overline{\Omega}_f$, respectively.

2. Formulation of the results. Suppose that a germ f is nondegenerate for its Newton polyhedron (see [2], [8] and [9]) and that the Newton polyhedron is convenient (that is, intersects each of the coordinate axes). Then the following assertions hold.

THEOREM 1 [4]. *The spectrum of the Newton filtration in Ω_f is determined from the Newton polyhedron by the formulas in* [4].

New formulas for this spectrum are set forth in Lemma 5. They follow easily from some formulas due to Steenbrink.

THEOREM 2. *Shifting the indices of the Newton filtration in Ω_f by -1 gives the Hodge filtration. The spectra of these filtrations are related by $h(\alpha) = g(\alpha + 1)$.*

THEOREM 3 ([5],[4]). *The set obtained by subtracting* 1 *from all the numbers of the spectrum defined in Theorem* 1 *coincides with the spectrum of the mixed Hodge-Steenbrink structure.*

See [2] for the definition of the spectrum of the mixed Hodge structure of a germ.

PROOF. Theorem 3 is a corollary of the theorem on the coincidence of spectrum of the mixed Hodge-Steenbrink structure with the spectrum of the Hodge filtration in Ω_f ([2], Russian §14.3.Д (English §14.3.E)) together with Theorems 1 and 2.

REMARK. Steenbrink proved Theorem 1 and stated Theorem 3 as a conjecture (see [4]). His formulas are based on Kushnirenko's theory of Newton filtrations [9]. Theorem 3 was proved by Saito [5]. Saito's proof implicitly contains our Theorem 3 (which is cited below).

Results pertaining to the space $\overline{\Omega}_f$ will be cited in section §6.

3. Proof of Theorem 2.

LEMMA 1 (see [2], Russian §13.1.ж (English §13.1.G); [7], §4.5). *The Newton order of any form $\omega \in \Omega^n$ minus* 1 *is no greater than its Hodge order.*

The lemma is proved by a simple local estimate of the integral lifted to the toroidal resolution of the singularity of the germ f (see [2], §8, and [8] for information on toroidal resolutions).

COROLLARY. *The filtration obtained by shifting the indices of the Newton filtration in Ω_f by -1 contains the Hodge filtration.*

LEMMA 2 (on entropy [7]). *If two decreasing filtrations in a finite-dimensional vector space are such that one filtration is imbedded in the other and the sum of the numbers in the spectrum of one is equal to the sum of the numbers in the spectrum of the other, then the filtrations coincide.*

Lemma 2 is a simple assertion of linear algebra.

LEMMA 3. *The sum of the numbers of the spectrum of the Newton filtration in* Ω_f *is equal to the sum of the numbers of the spectrum of the Hodge filtration in* Ω_f *plus the dimension of the space* Ω_f *(that is, plus the Milnor number* μ *of the germ* f*).*

In fact, the sum of the numbers of the Hodge filtration in Ω_f is equal to $\mu(n/2-1)$ by the determinant theorem of [2], §12.1 (and this is one of the central points in the proof of Theorem 2). The sum of the multiplicities of the numbers of the spectrum is equal to the dimension of Ω_f. Thus, to prove Lemma 3, it suffices to establish the following.

LEMMA 4. *Considered as a subset of points (with multiplicities) on the number line, the spectrum of the Newton filtration in* Ω_f *is symmetric about* $n/2$.

Lemma 4 is a corollary of explicit formulas for the Newton spectrum (see Theorem 1 and Lemma 5) and the geometry of integral polyhedra. Lemma 4 will be proved in §5. Theorem 2 is proved.

4. Formulas for the Newton spectrum in Ω_f allow us to compute the spectrum from functions T^I and B^I (defined below) of the variable α. These functions depend on the Newton polyhedron and the nonempty subset $I \subset \{1,\dots,n\}$. To the subset I, we associate the coordinate plane in the space of exponents of monomials in n variables on which the coordinates with indices not contained in I are equal to zero. We label all monomials on this coordinate plane whose Newton degree is equal to α. We define $T^I(\alpha)$ to be the number of such monomials and $B^I(\alpha)$ to be the number of these monomials which do not lie on smaller coordinate planes. Define the functions $p[T]$ and $p[B]$ of the variable t by the formulas:

$$p[T](t) = \sum_I (-1)^{n-|I|}(1-t)^{|I|} \sum_{\alpha\geq 0} T^I(\alpha)t^\alpha + (-1)^n;$$

$$p[B](t) = \sum_I (-t)^{n-|I|}(1-t)^{|I|} \sum_{\alpha> 0} B^I(\alpha)t^\alpha + (-t)^n,$$

where $|I|$ is the number of elements in the set I. Let $p[g]$ be a generating function of the Newton spectrum in Ω_f: $p[g](t) = \sum g(\alpha)t^\alpha$

LEMMA 5. 1) $p[g](t) = p[T](t)$. 2) $p[g](t) = p[B](t)$.

PROOF. Steenbrink [4] found another formula for the Newton spectrum. The first equality of the lemma follows from Steenbrink's formula upon using the inclusion-exclusion principle. The second equality follows from the first using the inclusion-exclusion principle and the binomial theorem.

5. Symmetry of the Newton spectrum in Ω_f. Two formulas for the Newton spectrum are cited in Lemma 5. We shall prove below that the spectrum determined by the first formula is symmetric, relative to the point $n/2$, to the spectrum determined by the second. In terms of generating functions, the symmetry of the spectrum is contained in the equality $t^n p[g](t^{-1}) = p[g](t)$. The latter is equivalent to the equality $t^n p[T](t^{-1}) = p[B](t)$ established below.

The geometry of integral polyhedra. The functions B^I and T^I behave rather wildly on the unit interval. But if the argument changes only by integers, these functions vary polynomially. More precisely, the following assertions are true.

LEMMA 6. *For every* $\alpha \geq 0$ *there exists a polynomial* T^I_α *and for every* $\alpha > 0$ *there exists a polynomial* B^I_α *such that*

$$T^I_\alpha(m) = T^I(\alpha+m) \quad \textit{and} \quad B^I_\alpha(m) = B^I(\alpha+m)$$

for every nonnegative integer m.

LEMMA 7. *If $1 > \alpha \geq 0$, the polynomials T^I_α and $B^I_{1-\alpha}$ are related by the expression* $T^I_\alpha(-u) = (-1)^{|I|-1} B^I_{1-\alpha}(u-1)$.

Lemmas 6 and 7 follow easily from the results of [10]. Lemma 7 is a manifestation of the duality between the interior integral points and the integral points on the polyhedron (compare [11] and [12]).

Generating functions of polynomials. To establish the symmetry of the Newton spectrum in Ω_f, we use the following lemma about generating functions on polynomials. It can be verified by a direct summation.

LEMMA 8. *If P is a polynomial in one variable, then the series $\sum_{m\geq 0} P(m)t^{-m}$ and $\sum_{m>0} -P(-m)t^m$, which converge for $|t| > 1$ or $|t| < 1$ respectively, determine the same rational function.*

LEMMA 9. *The series $\sum_{\alpha\geq 0} T^I(\alpha)t^{-\alpha}$ and $(-1)^I \sum_{\alpha>0} B^I(\alpha)t^\alpha$ determine the same function.*

PROOF. We have

$$\sum_{\alpha\geq 0} T^I(\alpha)t^{-\alpha} = \sum_{1>\alpha\geq 0} t^{-\alpha} \sum_{m\geq 0} T^I(\alpha+m)t^{-m}$$

By Lemmas 6, 8 and 7,

$$\sum_{m\geq 0} T^I(\alpha+m)t^{-m} = \sum_{m>0} -T^I_\alpha(-m)t^m$$

$$= -(-1)^{|I|-1} \sum_{m>0} B^I_{1-\alpha}(m-1)t^m = (-1)^{|I|} t \sum_{m\geq 0} B^I_{1-\alpha}(m)t^m$$

Furthermore,

$$(-1)^{|I|} \sum_{\alpha>0} B^I(\alpha)t^\alpha = (-1)^{|I|} \sum_{1>\alpha\geq 0} t^{1-\alpha} \sum_{m\geq 0} B^I_{1-\alpha}(m)t^m$$

The equality $t^n p[T](t^{-1}) = p[B](t)$ follows immediately from Lemma 9. This establishes the symmetry of the Newton spectrum about the point $n/2$. Lemma 4 is proved.

6. Formulation of the assertions about the space $\overline{\Omega}_f$.

THEOREM 4. *The generating function $p[\bar{h}](t) = \sum \bar{h}(\alpha)t^\alpha$ of the spectrum of the Hodge filtration in $\overline{\Omega}_f$ is given by the formulas*

1. $p[\bar{h}](t) = (1-t)^{-1}t^{-1}p[T](t)$,
2. $p[\bar{h}](t) = (1-t)^{-1}t^{-1}p[B](t)$,

where the functions $p[T]$ and $p[B]$ are as in Lemma 5.

We note that when $\alpha \geq n-2$ the number $\bar{h}(\alpha)$ is equal to the multiplicity of the number $\exp(2\pi i\alpha)$ in the spectrum of the monodromy operator of f; in particular, $\bar{h}(\alpha) = \bar{h}(\alpha+1)$.

THEOREM 5. *In $\overline{\Omega}_f$, the filtration obtained by shifting the indices of the Newton filtration by -1 coincides with the Hodge filtration.*

Theorem 5 means the following: if the Newton order of a form $\omega \in \Omega^n$ is equal to α and if this order cannot be increased by adding to ω forms of the type $df \wedge d\eta$, where $\eta \in \Omega^{n-2}$, then the Hodge order of ω is equal to $\alpha - 1$.

We formulate a corollary of Theorems 4 and 5. Let H_α denote the quotient space of holomorphic forms of Newton order at least $\alpha+1$ with respect to the following equivalence relation: two forms are equivalent if their integrals over any continuous family $\delta(t)$ of vanishing cycles differ by $o(t^{\alpha+\varepsilon})$, where ε is a suitable sufficiently small positive number.

COROLLARY. *The dimension of the space H_α is equal to the multiplicity of the number α in the Hodge spectrum of $\overline{\Omega}_f$ and can be computed using the formulas in Theorem 4. In particular, if $\alpha \geq n-2$, then* $\dim H_\alpha = \dim H_{\alpha+1}$.

7. Proof of Theorem 4. The integration operator ∂^{-1}. The operator ∂^{-1}, which associates the class $[df \wedge \eta], d\eta = \omega$, to the class $[\omega] \in \overline{\Omega}_f$, is well-defined on $\overline{\Omega}_f$. By the formula for differentiating integrals, we have

$$\frac{d}{dt}\int_{\delta(t)} \eta = \int_{\delta(t)} \omega/df$$

for any family $\delta(t)$ of vanishing cycles. Thus, ∂^{-1} increases the Hodge order by 1. According to a theorem of Brieskorn [6], we have $\bigcap \partial^{-k}\overline{\Omega}_f = 0$. The space Ω_f coincides with the coimage of the operator ∂^{-1} From the enumerated properties of ∂^{-1} and simple considerations from linear algebra, it follows that the Hodge spectra in $\overline{\Omega}_f$ and Ω_f are related by the expression $\bar{h}(\alpha) = \sum_{k\geq 0} h(\alpha - k)$. For generating functions, this means that $\sum \bar{h}(\alpha)t^\alpha = (1-t)^{-1}\sum h(\alpha)t^\alpha$ Theorem 4 now follows from Theorems 1 and 2.

8. Proof of Theorem 5.

LEMMA 10. *The operator ∂^{-1} increases the Newton order by at least* 1.

In fact, if $\omega = hdx$, then gdx is a representative of the class $\partial^{-1}[\omega]$, where $g = f'_{x_1}\int_0^{x_1} h(s, x_2, \ldots, x_n)\,ds$.

Linear algebra. Let $L\colon V \to V$ be an operator on an (infinite-dimensional) linear space with a trivial kernel and a finite-dimensional coimage. Let $d_1, d_2\colon V \to R$ be functions with the following properties: $d_i(x+y) \geq \min(d_i(x), d_i(y))$; $d_i(x+y) = d_i(x)$ when $d_i(x) < d_i(y)$; and $d_i(ax) = d_i(x)$ when $x, y \in V$ and a is a nonzero number. The function d_i determines a decreasing filtration $\{V_\alpha^i = \{v \in V | d_i(v) \geq \alpha\}\}$. Assume that: 1) the spaces V_α^i have finite codimension in V, 2) the projections of the filtrations $\{V_\alpha^1\}$ and $\{V_\alpha^2\}$ to $V/L(V)$ coincide, 3) $d_1 \geq d_2$, and 4) $d_1 \circ L = d_1 + \delta$ and $d_2 \circ L \geq d_2 + \delta$, where δ is some (fixed) number.

LEMMA 11. *Under the above hypotheses,* $d_1 = d_2$.

Lemma 11 is an assertion belonging to linear algebra, and we omit its proof. Theorem 5 follows from Lemma 11 and Theorem 2 upon letting V, L, d_1, d_2 and δ be $\overline{\Omega}_f, \partial^{-1}$, the Hodge order, the Newton order minus 1, and the number 1, respectively.

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