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NEWTON POLYHEDRA AND AN ALGORITHM FOR COMPUTING HODGE-DELIGNE NUMBERS

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ABSTRACT. An algorithm is given for computing the mixed Hodge structure (more precisely, the Hodge-Deligne numbers) for cohomology of complete intersections in toric varieties in terms of Newton polyhedra specifying the complete intersection. In some particular cases the algorithm leads to explicit formulas.

Bibliography: 8 titles.

Homology and cohomology yield one of the most natural topological invariants of an arbitrary variety. It had been known since Hodge's work that the cohomology of projective complex manifolds admits a natural Hodge decomposition. Deligne (see [1] and [2]) generalized Hodge theory to arbitrary complex algebraic varieties by endowing their cohomology with a so-called mixed Hodge structure. In particular, for an arbitrary algebraic variety X and a Hodge structure $H^k(X)$ he defined the Hodge-Deligne numbers $h^{p,q}(H^k(X))$.

The present paper is devoted to a computation of these invariants for varieties X defined in $(\mathbf{C}\setminus 0)^n$ by a system of polynomial equations $f_1 = \cdots = f_k = 0$, where the polynomials are nondegenerate with respect to their Newton polyhedra $\Delta_1, \ldots, \Delta_k$. We recall that the Newton polyhedron of a polynomial $f = \sum a_m x^m$, where $m = (m_1, \ldots, m_n)$ and $x = (x_1, \ldots, x_n)$, is the convex hull of those points $m \in \mathbf{R}^n$ for which $a_m \neq 0$. The Newton polyhedron of a polynomial generalizes the notion of degree and plays a similar role. A system of equations with fixed Newton polyhedra is nondegenerate for almost all values of the coefficients.

Throughout this paper the space $(\mathbf{C}\setminus 0)^n$ is called the n-dimensional torus and is denoted by $\mathbf{T}^n$.

It might seem somewhat artificial that we study complete intersections in the torus $\mathbf{T}^n$ and not in the affine space $\mathbf{C}^n$. We suggest two reasons to justify our choice. First, cutting $\mathbf{C}^n$ by coordinate hyperplanes into tori of various dimensions, we can apply the results on complete intersections in tori to complete intersections in $\mathbf{C}^n$. This principle can be applied not only to the space $\mathbf{C}^n$, but to arbitrary toric varieties, to wit algebraic varieties

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that split naturally into a union of disjoint tori. Thus the case of a torus plays a key role in the computation of Hodge-Deligne numbers for complete intersections in toric varieties. Second, the case of a torus is the simplest one. This is because the torus $\mathbf{T}^n = (\mathbf{C}\setminus 0)^n$ is a commutative algebraic group (with respect to the operation of coordinatewise multiplication). The invariant sheaves on this group decompose with respect to characters. Cohomology with coefficients in such sheaves can be computed separately for each character of the group, which considerably simplifies all computations.

It will be more comvenient to consider cohomology with compact support instead of ordinary cohomology. Like the usual cohomology, cohomology with compact support has a natural (mixed) Hodge structure. The Poincaré duality between cohomology with compact support and the usual cohomology on smooth algebraic varieties is compatible with the Hodge structures. Therefore, in passing to cohomology with compact support we do not lose anything. The convenience achieved thereby consists in the following. For a variety X we consider the numbers

$$e^{p,q} = \sum_k (-1)^k h^{p,q}(H_c^k(X)).$$

Then, if $X = \bigcup_\alpha X_\alpha$ is a decomposition of X into a union of a finite number of locally closed subvarieties, we have $e^{p,q}(X) = \sum_\alpha e^{p,q}(X_\alpha)$. Thus the $e^{p,q}$ behave in a very simple way under decomposition of varieties into pieces, which is a very natural operation when working with toric varieties.

An important role is also played by the numbers $e^p(X)$ defined by $e^p(X) = \sum_q e^{p,q}(X)$.

We list the main results of the paper. First, for a nondegenerate complete intersection X either in a torus or in an arbitrary toric variety we compute the numbers $e^p(X)$ in terms of the Newton polyhedra of the equations defining the complete intersection. Second, for a nondegenerate complete intersection X either in a torus or in an arbitrary toric variety we give a finite algorithm for computing the numbers $e^{p,q}(X)$ in terms of the Newton polyhedra. In the important special case of a hypersurface with prime Newton polyhedron (a polyhedron in $\mathbf{R}^n$ is called *prime* if the number of faces coming together at every vertex is equal to n) we give explicit formulas for the numbers $e^{p,q}(X)$. Third, for a nondegenerate complete intersection X in a compact toric variety we give an algorithm for computing the Hodge numbers $h^{p,q}(X)$ (this result is a consequence of the preceding one, since for compact varieties $h^{p,q}$ is equal to $e^{p,q}$ modulo sign). Fourth, for a noncompact complete intersection X we give an algorithm for computing the Hodge numbers in the following cases: a) X is a complete intersection in a torus, and the Newton polyhedra of the equations defining X have maximal dimension; b) X is a complete intersection in $\mathbf{C}^n$, and the Newton polyhedra of the equations defining X contain the origin and intersect all coordinate axes (this result follows from the computation of the numbers $e^{p,q}(X)$ and an analogue of the Lefschetz theorem).

A few words about the division of labor. The computation of toric cohomology (Proposition 2.10) and generalization of some known results on Hodge structures to the case of cohomology with compact support (the use of forms with "logarithmic poles"; Proposition 1.12) are due to the first author. The introduction of the additive invariant $e^{p,q}$ of algebraic varieties and the applications of this invariant (Corollary 2.5; the algorithm in §5) are due to the second author.

The present paper had been written in 1978, but was not submitted for publication. Its results are used in the first author's paper [5] and in a paper by the second author (in preparation) in which the algorithms from the present paper are carried as far as explicit formulas.

§1. Hodge-Deligne theory

In this section we recall the Hodge-Deligne theory of algebraic varieties over the field of complex numbers $\mathbf{C}$ with special regard to cohomology with compact support. The central role here is played by the characteristic $e^{p,q}(X)$ of an algebraic variety X.

1.1. Let H be a finite-dimensional vector space over the field $\mathbf{Q}$ of rational numbers. A *pure Hodge structure of weight r* on H is a decomposition of the complex vector space

$$H_{\mathbf{C}} = H \otimes \mathbf{C} = \bigoplus_{p+q=r} H^{p,q}$$

such that $H^{p,q} = \overline{H}^{q,p}$ (here the bar denotes the complex conjugation in $H_{\mathbf{C}}$). The introduction of this notion is motivated by the classical Hodge theory according to which the r-dimensional cohomology $H^r(X,\mathbf{Q})$ of a compact Kähler manifold X has a natural Hodge structure of weight r. The dimension of the vector $\mathbf{C}$-space $H^{p,q}$ is called the *Hodge number of type (p,q) of the structure H* and is denoted by $h^{p,q}(H)$.

A Hodge structure of weight r on H gives rise to the so-called *Hodge filtration F* on $H_{\mathbf{C}}$, where

$$F^p = \bigoplus_{s \geqslant p} H^{s,r-s}.$$

This is a descending filtration, and for each integer p

$$H_{\mathbf{C}} = F^p \oplus \overline{F}^{r-p+1}.$$

Conversely, if F is a descending filtration on $H_{\mathbf{C}}$ satisfying the above relation, then putting $H^{p,q} = F^p \cap \overline{F}^q$ we obtain a pure Hodge structure of weight r on $H_{\mathbf{C}}$.

1.2. Let H be a vector space over $\mathbf{Q}$. A (mixed) *Hodge structure on H* consists of a) an ascending weight filtration W on H and b) a descending Hodge filtration F on $H_{\mathbf{C}}$ that are connected by the following relation: the filtration F induces a pure Hodge structure of weight r on the complexification of $\mathrm{Gr}^W H = W_r/W_{r-1}$.

In particular, for each r there is a decomposition

$$(\mathrm{Gr}^W H)_{\mathbf{C}} = \bigoplus_{p+q=r} H^{p,q}.$$

The dimension of the $\mathbf{C}$-space $H^{p,q}$ is called the *Hodge number of type (p,q) of the Hodge structure H* and is denoted by $h^{p,q}(H)$.

1.3. Let H and H' be $\mathbf{Q}$-spaces endowed with Hodge structures. A $\mathbf{Q}$-linear homomorphism $f\colon H \to H'$ is called *compatible with Hodge structures* (or a morphism of Hodge structures) if f is compatible with filtrations W and F, i.e., if $f(W_r) \subset W_r'$ and $f_{\mathbf{C}}(F^p) \subset F'^p$.

The morphisms of Hodge structures are strictly compatible with filtrations W and F (see [1]). From this it follows that the functor $H^{p,q}$ is exact. In other words, if $H' \to H \to H''$ is an exact sequence of Hodge structures, then for every (p,q) the sequence $H'^{p,q} \to H^{p,q} \to H''^{p,q}$ is also exact.

1.4. Deligne has shown [2] that for each complex algebraic variety X the cohomology $H^k(X,\mathbf{Q})$ carries a natural Hodge structure which coincides with the classical (pure) Hodge structure in the case of smooth projective varieties. However we find it more convenient to use cohomology with compact support $H_c^k(X, \mathbf{Q})$. They also carry a natural Hodge structure. We list some basic properties of the Hodge theory of cohomology with compact support of smooth noncompact algebraic varieties:

a) *If $f\colon X \to Y$ is a proper morphism, then the homomorphism $f^*\colon H_c^*(Y) \to H_c^*(X)$ is compatible with the Hodge structures.*

b) *The Künneth isomorphism $H^*(X) \otimes H^*(Y) \to H_c^*(X \times Y)$ is compatible with the Hodge structures, where the left-hand term is the tensor product of Hodge structures.*

c) *If Y is a closed subvariety in X, then the exact sequence*

$$\cdots \to H_c^k(X \setminus Y) \to H_c^k(X) \to H_c^k(Y) \to H_c^{k+1}(X \setminus Y) \to \cdots$$

is a sequence of Hodge structures.

d) *The Hodge numbers $h^{p,q}(H_c^k(X))$ are equal to 0 for $p+q > k$ (and also for $p < 0$ or $q < 0$).*

These four properties hold for all algebraic varieties, but it is easier to prove them for nonsingular varieties and they will be used only in this case. On the other hand, smoothness is essential for the other two properties.

e) *If X is a smooth projective variety, then the Hodge structure on $H_c^*(X) = H^*(X)$ coincides with the classical one.*

f) *Let X be a smooth irreducible n-dimensional variety. Then the Poincaré pairing*

$$H_c^k(X) \otimes H^{2n-k}(X) \to H_c^{2n}(X) = \mathbf{Q}[-n]$$

is compatible with the Hodge structures in ordinary cohomology and cohomology with compact support.

Property e) actually remains true if we assume that the variety x is quasismooth instead of smooth (see [3]). In particular,

g) *For a compact quasismooth algebraic variety X, the Hodge structure on $H^k(X)$ is a pure structure of weight k.*

1.5. Generalizing the notion of Euler characteristic, for each pair of integers (p, q) we introduce the following invariant of algebraic variety X:

$$e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(H_c^k(X)).$$

We observe that $e^{p,q}(X) = e^{q,p}(X)$. For a compact smooth (or quasismooth) variety X we have

$$e^{p,q}(X) = (-1)^{p+q} h^{p,q}(X),$$

where the $h^{p,q}(X)$ are the Hodge numbers of X. So in this case to know the numbers $e^{p,q}(X)$ is the same as to know the Hodge numbers, and in particular, this knowledge allows one to compute the Betti numbers of X.

It is convenient to consider the numbers $e^{p,q}(X)$ as coefficients of a single polynomial $e(X)$ in the variables x and $\bar{x}$:

$$e(X; x, \bar{x}) = \sum_{p,q} e^{p,q}(X) x^p \bar{x}^q.$$

Our main reason for introducing the characteristic e is the following additivity property.

1.6. PROPOSITION. *Suppose that X is a disjoint union of a finite number of locally closed subvarieties X_i, $i \in I$. Then $e(X) = \sum_{i \in I} e(X_i)$.*

We give a proof only for the case when X and all the X_i are smooth varieties. Moreover, we assume that $(X_i)_{i \in I}$ satisfy the following condition: the closure $\overline{X}_i$ of an arbitrary stratum X_i is a union of some X_j. In this case there exists an integer $i \in I$ such that X_i is closed in X. Using 1.4c) and the fact that $H^{p,q}$ is exact (see 1.3), for each pair (p, q) we obtain the equality

$$e^{p,q}(X) = e^{p,q}(X \setminus X_i) + e^{p,q}(X_i).$$

It remains to pass from X to $X \setminus X_i$. The general case can be reduced to this one by a suitable refinement of the partition $(X_i)_{i \in I}$.

1.7. COROLLARY. *Let (X_i) be a finite covering of X by locally closed subvarieties. Then*

$$e(X) = \sum_{i_0 < \cdots < i_k} (-1)^k e(X_{i_0} \cap \cdots \cap X_{i_k}).$$

Another convenient property of our characteristic e is its multiplicativity, which follows from 1.4b).

1.8. PROPOSITION. $e(X \times Y) = e(X) \times e(Y)$.

1.9. COROLLARY. *Let f: $X \to Y$ be a bundle with fiber F which is locally trivial in the Zariski topology. Then $e(X) = e(Y) \times e(F)$.*

In fact, let (Y_i) be an open covering of Y which trivializes f. To prove the corollary it suffices to apply 1.7 to the covering $(f^{-1}(Y_i)$ of the variety X, then apply 1.8 to $f^{-1}(Y_{i_0}) \cap \cdots \cap f^{-1}(Y_{i_k}) = (Y_{i_0} \cap \cdots \cap Y_{i_k}) \times F$, and finally apply 1.7 to the covering (Y_i).

1.10. We consider some elementary examples showing how to use our characteristic e. To begin with,

a) If X consists of a single point, then $e(X) = 1$.

b) Let $X = \mathbf{P}^1$ be the Riemann sphere. Then $e(\mathbf{P}^1) = 1 + x\bar{x}$.

c) Since $\mathbf{P}^1$ is a union of $\mathbf{C}$ and the point ∞, we see that $e(\mathbf{C}) = x\bar{x}$. In general, $e(\mathbf{C}^n) = (x\bar{x})^n$.

d) Considering $\mathbf{P}^n$ as a union of $\mathbf{C}^n$ and the hyperplane at infinity $\mathbf{P}^{n-1}$, we see that $e(\mathbf{P}^n) = 1 + x\bar{x} + \cdots + (x\bar{x})^n$.

e) For the one-dimensional torus $\mathbf{T}^1 = \mathbf{C} \setminus 0$ we have $e(\mathbf{T}^1) = x\bar{x} - 1$. In general, for the n-dimensional torus $\mathbf{T}^n = (\mathbf{T}^1)^n$ we have $e(\mathbf{T}^n) = (x\bar{x} - 1)^n$.

As a matter of fact, it is easy to compute all Hodge-Deligne numbers for a torus; all of them are equal to zero with the exception of $h^{p,p}(H_c^{n+p}(\mathbf{T}^n)) = C_n^p$, where C_n^p is the number of combinations of p elements from a given set of n elements.

f) We give a less elementary example. Let Y be a smooth subvariety of codimension $r + 1$ in a smooth variety X. Let ρ: $\tilde{X} \to X$ be the blowing up of the subvariety Y in X. Then $\rho^{-1}(Y)$ is a bundle over Y with fiber $\mathbf{P}^r$, and so

$$e(\tilde{X}) = e(X) + e(Y)\left[x\bar{x} + \cdots + (x\bar{x})^r\right].$$

In particular, we have the following formula for the Betti numbers b^k:

$$b^k(\tilde{X}) = b^k(X) + b^{k-2}(Y) + \cdots + b^{k-2r}(Y).$$

1.11. The additivity and multiplicativity properties discussed above lead to computation of $e(X)$ only for very simple varieties X; in more complex cases one has to use less trivial techniques. In our case the ultimate success is due to the fact that for a hypersurface in a torus one can compute the sums $\Sigma_q e^{p,q}(X)$ by interpreting them as cohomology of certain coherent sheaves. We discuss this in more detail.

Let X be a smooth algebraic variety, and $\overline{X}$ a smooth compactification of X such that the divisor $D = \overline{X} \setminus X$ has transversal intersections in $\overline{X}$. We denote by $\Omega^p_{(\overline{X},D)}$ the sheaf of germs of regular differential p-forms on $\overline{X}$ vanishing on D. More precisely, if $D_1, \ldots, D_N$ are the irreducible components of D, then $\Omega^p_{(\overline{X},D)}$ is the kernel of the restriction homomorphism $\Omega^p_{\overline{X}} \to \oplus_i \Omega^p_{D_i}$.

We have the following

1.12. PROPOSITION. *There exists a spectral sequence*

$$E_1^{p,q} = H^q\left(\overline{X}, \Omega^p_{(\overline{X},D)}\right) \Rightarrow H_c^{p+q}(X, \mathbf{C}),$$

degenerating at the term E_1 and converging to the Hodge filtration on $H_c^*(X)$.

In the case when X is compact this spectral sequence coincides with the Hodge-de Rham spectral sequence. In the general case we see that the space $H^{k-p}(\overline{X}, \Omega^p_{(\overline{X},D)})$ is isomorphic to the space $F^pH_c^k(X)/F^{p+1}H_c^k(X)$ whose dimension is equal to $\sum_q h^{p,q}(H_c^k(X))$. Denoting by $\chi(\overline{X}, \mathfrak{F})$ the Euler-Poincaré characteristic of a sheaf $\mathfrak{F}$ on X, which by definition is equal to $\sum_k (-1)^k \dim H^k(\overline{X}, \mathfrak{F})$, we obtain the following

1.13. COROLLARY. *In the above notation and under the above assumptions*

$$e^p(X) = \sum_q e^{p,q}(X) = (-1)^p \chi\left(\overline{X}, \Omega^p_{(\overline{X},D)}\right).$$

§2. Toric varieties

Here we briefly recall the structure of toric varieties which are the main objects considered in the present paper. We already mentioned that toric varieties are special partial compactifications of tori. We begin with recalling some properties of tori.

2.1. A *torus* (more precisely, an *n-dimensional torus*) is an algebraic variety $\mathbf{T}^n$ isomorphic to $(\mathbf{C} \setminus 0)^n$. If x_i is the coordinate on the ith factor $\mathbf{C} \setminus 0$, then for each $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$ the monomial $x^m = x_1^{m_1} \cdots x_n^{m_n}$ is a regular function on $\mathbf{T}^n$ (and even a group homomorphism $\mathbf{T}^n \to \mathbf{T}^1$). Moreover, each regular function f on $\mathbf{T}^n$ can be uniquely written in the form of a finite linear combination of the monomials x^m, $m \in \mathbf{Z}^n$. In other words, the ring $\mathbf{C}[\mathbf{T}^n]$ of regular functions on $\mathbf{T}^n$ is isomorphic to the group algebra $\mathbf{C}[\mathbf{Z}^n]$ of the abelian group $\mathbf{Z}^n$, and $\mathbf{T}^n = \operatorname{Spec} \mathbf{C}[\mathbf{Z}^n]$ is the spectrum of this algebra. Since there is no natural basis in the group of characters of the torus $\mathbf{T}^n$, we shall use the more neutral symbol M in place of $\mathbf{Z}^n$.

A function f on the torus $\mathbf{T}^n = \operatorname{Spec} \mathbf{C}[M]$ considered as an element of $\mathbf{C}[M]$ is called a *Laurent polynomial.* If $f = \sum_{m \in M} a_m x^m$, then the set $\operatorname{Supp}(f) = \{m \in M,\ a_m \neq 0\}$ is called the *support* of f. The convex hull of $\operatorname{Supp}(f)$ in the real vector space $M_{\mathbf{R}} = M \otimes \mathbf{R}$ is called the *Newton polyhedron* of f and is denoted by $\Delta(f)$.

Each point t of the torus $\mathbf{T}$ defines a group homomorphism φ_t: $M \to \mathbf{T}^1$ according to the formula $\varphi_t(m) = x^m(t)$. It is easy to see that this allows us to identify the set of all complex-valued points of $\mathbf{T}^n$ with the set $\operatorname{Hom}(M, \mathbf{T}^1)$ of group homomorphisms of M to $\mathbf{T}^1$.

2.2. A detailed definition of toric varieties is given in [3] and [8]. Here we only use the fact that a toric variety can be covered by affine toric varieties (charts) of the form $X_\sigma = \operatorname{Spec} \mathbf{C}[\sigma \cap M]$, where σ is a convex polyhedral cone in $M_{\mathbf{R}}$. The points of the variety X_σ are identified with the semigroup homomorphisms $\sigma \cap M \to \mathbf{T}^1$ (where $\mathbf{T}^1$ is

$\mathbf{C} \setminus 0$ and the composition law coincides with multiplication). This description shows how to identify points of $\mathbf{T}^n$ with points of X_σ.

We shall be mainly interested in the toric varieties $\mathbf{P}_\Delta$ corresponding to a polyhedron Δ in $M_\mathbf{R}$. Let Δ be a bounded convex polyhedron in $M_\mathbf{R}$ whose vertices lie in M (in what follows we shall consider only such polyhedra). To each face Γ of the polyhedron Δ (for short, we write $\Gamma \leqslant \Delta$) we associate the cone $\mathrm{Con}(\Delta, \Gamma) = \bigcup_{r \geqslant 0} r \cdot (\Delta - Q)$, where Q is an arbitrary point lying strictly inside Γ. The variety P_Δ is covered by the charts

$$X_{\mathrm{Con}(\Delta,\Gamma)} = \operatorname{Spec} \mathbf{C}[\mathrm{Con}(\Delta, \Gamma) \cap M],$$

where Γ runs through the set of faces of Δ. Actually, it suffices to consider only the vertices P; the chart corresponding to a vertex P will be denoted by U_P. The variety $\mathbf{P}_\Delta$ is the compactification of the torus $\mathbf{T}_\Delta = X_{\mathrm{Con}(\Delta,\Delta)}$; the dimension of this projective variety is equal to $\dim \Delta$. We observe that if $\dim \Delta \neq n$, then the torus $\mathbf{T}_\Delta$ is not isomorphic to $\mathbf{T}^n$.

To each face Γ of the polyhedron Δ there corresponds a closed subvariety in $\mathbf{P}_\Delta$ which is isomorphic to $\mathbf{P}_\Gamma$ and is denoted by the same symbol. The corresponding "big torus" in $\mathbf{P}_\Gamma$ will be denoted by $\mathbf{T}_\Gamma$. If Γ and Γ' are two faces, then $\mathbf{P}_\Gamma \cap \mathbf{P}_{\Gamma'} = \mathbf{P}_{\Gamma \cap \Gamma'}$; this shows that the variety $\mathbf{P}_\Delta$ is very similar to the polyhedron Δ. We will not elaborate on this observation. We only remark that the symbol $\mathbf{P}$ is used in order to emphasize the analogy with projective spaces, which are special cases of toric varieties.

2.3. The variety $\mathbf{P}_\Delta$ may have singularities lying on subvarieties $\mathbf{P}_\Gamma$, $\Gamma < \Delta$. This is due to the fact that Δ is not prime at the corresponding faces. We give the necessary definition, restricting ourselves to the case of vertices.

Let P be a vertex of Δ. The polyhedron Δ is called *prime at P* (*prime at P with respect to M*) if the cone $\mathrm{Con}(\Delta, P)$ is generated by a basis of $M_\mathbf{R}$ (by a basis of the lattice M). A polyhedron is *prime* (*prime with respect to M*) if it is prime at all vertices.

2.4. PROPOSITION. *If Δ is prime* (*prime with respect to M*), *then $\mathbf{P}_\Delta$ is a quasismooth* (*smooth*) *variety.*

In fact, suppose that Δ is prime at P with respect to M. Choosing a suitable isomorphism $M \simeq \mathbf{Z}^n$, we may assume that the semigroup $\mathrm{Con}(\Delta, P) \cap M$ is isomorphic to $\mathbf{N}^n$ and the semigroup ring $\mathbf{C}[\mathrm{Con}(\Delta, P) \cap M]$ is isomorphic to the ring of polynomials $\mathbf{C}[x_1, \ldots, x_n]$. But that means that the chart U_P is isomorphic to the affine space $\mathbf{C}^n$. Since the charts U_P cover $\mathbf{P}_\Delta$, $\mathbf{P}_\Delta$ is a smooth variety. One can use similar arguments in the case when Δ is prime.

Moreover, the same arguments show that if Δ is prime with respect to M, then the divisor $D = \mathbf{P}_\Delta \setminus \mathbf{T}^n$ has transversal intersections on $\mathbf{P}_\Delta$.

2.5. COROLLARY. *If Δ is a prime polyhedron, then the Hodge numbers $h^{p,q}(\mathbf{P}_\Delta)$ are equal to 0 for $p \neq q$ and*

$$h^{p,p}(\mathbf{P}_\Delta) = (-1)^p \sum_{\Gamma \leqslant \Delta} (-1)^{\dim \Gamma} C^p_{\dim \Gamma}.$$

PROOF. Since $\mathbf{P}_\Delta$ is a union of the tori $\mathbf{T}_\Gamma$, $\Gamma \leqslant \Delta$, we see that $e(\mathbf{P}_\Delta) = \sum_{\Gamma \leqslant \Delta} (x\bar{x} - 1)^{\dim \Gamma}$ (see 1.6 and 1.10e)). On the other hand, $\mathbf{P}_\Delta$ is compact, and by the above proposition it is quasismooth; hence $h^{p,q}(\mathbf{P}_\Delta) = (-1)^{p+q} e^{p,q}(\mathbf{P}_\Delta)$ (cf. 1.5).

2.6. Let Δ and Δ' be polyhedra in $M_\mathbf{R}$. We say that Δ' *majorizes* Δ if there exists a map α: $\mathrm{som}(\Delta') \to \mathrm{som}(\Delta)$ of the set of vertices of Δ' to the set of vertices of Δ such that $\mathrm{Con}(\Delta, \alpha(P')) \subset \mathrm{Con}(\Delta', P')$. If such a map α does exist, then it is unique. This map extends to the set of all faces if we define $\alpha(\Gamma')$ to be the face of Δ with vertices $\alpha(P')$, $P' \in \mathrm{som}(\Gamma')$.

If Δ' majorizes Δ, then there is a natural morphism of algebraic varieties $\rho = \rho_{\Delta',\Delta}$: $\mathbf{P}_{\Delta'} \to \mathbf{P}_\Delta$. This morphism maps the chart $U_{P'}$ to the chart $U_{\alpha(P')}$, and for these affine charts it is contragredient to the natural homomorphism of semigroup algebras $\mathbf{C}[\mathrm{Con}(\Delta, \alpha(P')) \cap M] \to \mathbf{C}[\mathrm{Con}(\Delta', P') \cap M]$.

Morphisms of the above type can be used to resolve the singularities of $\mathbf{P}_\Delta$. In fact, for each Δ there exists a prime polyhedron Δ' which majorizes Δ.

2.7. If Δ' majorizes Δ and Δ majorizes Δ' (in this case Δ and Δ' can be called similar), then the spaces $\mathbf{P}_\Delta$ and $\mathbf{P}_{\Delta'}$ are canonically isomorphic. Thus each element of a class of similar polyhedra defines one and the same toric variety. The question then arises: what corresponds to the polyhedron Δ itself? It turns out that this polyhedron defines a polarization of $\mathbf{P} = \mathbf{P}_\Delta$, i.e., an ample invertible sheaf $\mathcal{O}_\mathbf{P}(\Delta)$ on $\mathbf{P}_\Delta$. More precisely, $\mathcal{O}_\mathbf{P}(\Delta)$ is the subsheaf of the sheaf of rational functions on $\mathbf{P}$ whose sections over the affine chart U_P ($P \in \mathrm{som}(\Delta)$) have the form $x^P f(x)$, where $f(x)$ is an arbitrary regular function on U_P, i.e., $f(x)$ is an element of $\mathbf{C}[\mathrm{Con}(\Delta, P) \cap M]$. From [8], Theorem 13, it follows that the invertible sheaf $\mathcal{O}_\mathbf{P}(\Delta)$ is ample, i.e., some multiplicity of this sheaf defines a projective embedding of $\mathbf{P}$.

The preceding construction can be slightly generalized. Namely, suppose that Δ' is a polyhedron majorizing Δ. Consider the invertible sheaf (of fractional ideals) $\mathcal{O}_\mathbf{P}(\Delta) = \rho^*_{\Delta',\Delta}(\mathcal{O}_{\mathbf{P}_\Delta}(\Delta))$ on $\mathbf{P} = \mathbf{P}_\Delta$. Its local description is essentially the same as above: over the affine chart $U_{P'}$ the sections of $\mathcal{O}_\mathbf{P}(\Delta)$ have the form $x^{\alpha(P')}f(x)$, where $\alpha(P')$ is the vertex of Δ corresponding to P' under the majorization map and $f(x)$ is an arbitrary Laurent polynomial from $\mathbf{C}[\mathrm{Con}(\Delta', P') \cap M]$.

Such a description of the sheaves $\mathcal{O}_\mathbf{P}(\Delta)$ yields a simple characterization of its cohomological properties. First of all, for each $m \in \Delta \cap M$ the function x^m is a global section of $\mathcal{O}_\mathbf{P}(\Delta)$. We denote by $L(\Delta)$ the space of all Laurent polynomials with support in Δ.

2.8. PROPOSITION. a) $H^0(\mathbf{P}_{\Delta'}, \mathcal{O}(\Delta)) = L(\Delta)$;
b) $H^i(\mathbf{P}_{\Delta'}, \mathcal{O}(\Delta)) = 0$ *for* $i > 0$.

The proof can be found in [6].

2.9. Along with the description of the invertible sheaves $\mathcal{O}_\mathbf{P}(\Delta)$ we shall need a description of the sheaves of differential forms on the space $\mathbf{P} = \mathbf{P}_\Delta$. Here we shall assume that $\mathbf{P}$ is a smooth variety, i.e., that Δ is a prime polyhedron with respect to M, although this assumption is not esssential. As we explained in 1.11, we are especially interested in the structure of the sheaves $\Omega^p_{(\mathbf{P},D)}$, where $D = \mathbf{P} \setminus \mathbf{T}$. We recall that $\Omega^p_{(\mathbf{P},D)}$ is the kernel of the restriction homomorphism

$$\Omega^p_\mathbf{P} \to \bigoplus_\Gamma \Omega^p_{\mathbf{P}_\Gamma},$$

where Γ runs through the set of faces of codimension 1 in Δ. All further results on these sheaves are based on the existence of a canonical isomorphism

$$\Lambda^p(M) \otimes_\mathbf{Z} \mathcal{O}_\mathbf{P}(-D) \xrightarrow{\sim} \Omega^p_{(\mathbf{P},D)}.$$

Here, as usual, $\mathcal{O}_\mathbf{P}(-D)$ denotes the sheaf of ideals of D, i.e., the sheaf of germs of functions vanishing on $D = \mathbf{P} \setminus \mathbf{T}$.

We give an explicit construction of this isomorphism. It transforms the tensor $m_1 \wedge \cdots \wedge m_p \otimes f$, where $m_1, \ldots, m_p \in M$ and f is a local section of $\mathcal{O}(-D)$, to the p-form

$$f\frac{dx^{m_1}}{x^{m_1}} \wedge \cdots \wedge \frac{dx^{m_p}}{x^{m_p}}.$$

It is easy to verify that this yields a homomorphism of sheaves $\Lambda^p(M) \otimes \mathcal{O}(-D) \to \Omega^p_{(\mathbf{P},D)}$. It remains to check that this is an isomorphism. It suffices to do that over each chart U_P. Each such chart is isomorphic to $\mathbf{C}^n$, and under this isomorphism D becomes a union of coordinate hyperplanes; after this the verification presents no more difficulties.

For $p = n$ the above isomorphism shows that the sheaf $\mathcal{O}(-D)$ is isomorphic to the canonical sheaf $\Omega^n_{(\mathbf{P},D)} = \Omega^n_{\mathbf{P}}$.

We denote by $L^*(\Delta)$ the space of Laurent polynomials whose support lies strictly in the interior of the polyhedron Δ. It is easy to see (compare with 2.8a)) that for $\dim \Delta = n$ the space $L^*(\Delta)$ is identified with the space of global sections of the invertible sheaf $\mathcal{O}_{\mathbf{P}}(-D) \otimes \mathcal{O}_{\mathbf{P}}(\Delta)$. This proves the first part of the following assertion.

2.10. PROPOSITION. *Let Δ be an n-dimensional polyhedron in $M_{\mathbf{R}}$, and let Δ' be a polyhedron majorizing Δ and prime with respect to M. Then for $\mathbf{P} = \mathbf{P}_{\Delta'}$*

$$H^i\big(\mathbf{P}, \Omega^p_{(\mathbf{P},D)}(\Delta)\big) = \begin{cases} \Lambda^p(M) \otimes L^*(\Delta), & i = 0, \\ 0, & i > 0. \end{cases}$$

PROOF. We need to show that $\Omega^p_{(\mathbf{P},D)}(\Delta)$ is acyclic, and to do this we may assume that $p = n$. But in this case $\Omega^n_{(\mathbf{P},D)} = \Omega^n_{\mathbf{P}}$ is the canonical sheaf on $\mathbf{P}$, and the acyclicity of $\Omega^n_{\mathbf{P}}(\Delta) = K \otimes \mathcal{O}(\Delta)$ follows from the assertion in §4 of [6].

In particular, under the assumptions of 2.10 we obtain the following formula for the Euler-Poincaré characteristic of the sheaf $\Omega^p_{(\mathbf{P},D)}(\Delta)$:

$$\chi\big(\mathbf{P}, \Omega^p_{(\mathbf{P},D)}(\Delta)\big) = C^p_n l^*(\Delta),$$

where $l^*(\Delta) = \dim L^*(\Delta)$ is the number of integral points in the interior of Δ. It is also clear that $\chi(\mathbf{P}, \Omega^p_{(\mathbf{P},D)}) = (-1)^n C^p_n$.

§3. Lefschetz-type theorems for toric varieties

In this section we present some comparison theorems for properties of the toric variety $\mathbf{P}_\Delta$ and of its hyperplane sections.

3.1. Each regular function f on the torus $\mathbf{T}^n = \operatorname{Spec} \mathbf{C}[M]$ (or, which is the same, each Laurent polynomial $f \in \mathbf{C}[M]$) defines a hypersurface in $\mathbf{T}^n$. This hypersurface is given by the equation $f(x) = 0$; we denote it by Z_f or simply by Z. Each hypersurface in $\mathbf{T}^n$ can be represented in this form.

3.2. Now let Δ be a polyhedron in $M_{\mathbf{R}}$ containing $\operatorname{supp}(f)$. Then $f \in L(\Delta)$ and, in view of 2.7, f can be viewed as a global section of the invertible sheaf $\mathcal{O}_{\mathbf{P}}(\Delta)$ on the toric variety $\mathbf{P}_\Delta$. Being a section of $\mathcal{O}(\Delta)$, f defines a hypersurface in $\mathbf{P}_\Delta$; this hypersurface consists of the points where this section vanishes. The resulting hypersurface will be denoted by $\overline{Z}_{(\Delta,f)}$ or $\overline{Z}$. In the local chart U_P (where P is a vertex of Δ; see 2.2) the subvariety $\overline{Z}$ is given by the equation $x^{-P}f(x) = 0$. Assuming that $\dim \Delta = n$, we see that $\overline{Z} \cap \mathbf{T}^n = Z$, so that in this case $\overline{Z}$ is a compactification of Z (although not necessarily the closure of Z in $\mathbf{P}_\Delta$). We also observe that in view of the ampleness of $\mathcal{O}(\Delta)$ the variety $\mathbf{P}_\Delta \setminus \overline{Z}$ is affine.

In what follows we shall need a slightly more general construction. Suppose again that Δ' majorizes Δ; considering f as a section of the invertible sheaf $\mathcal{O}(\Delta)$ on $\mathbf{P}_{\Delta'}$, we obtain a hypersurface in $\mathbf{P}_{\Delta'}$ which will be denoted by $\overline{Z}_{(\Delta',\Delta,f)}$. It is clear that $\overline{Z}_{(\Delta',\Delta,f)} = \rho_{\Delta',\Delta}^{-1}(\overline{Z}_{(\Delta,f)})$. This is again a compactification of Z, and in the affine chart $U_{P'}$ ($P' \in \operatorname{som}(\Delta')$) this hypersurface is given by the equation $x^{-\alpha(P')}f(x) = 0$, where α is the majorization map (see 2.6).

From this it is easy to understand the structure of the intersection of $\overline{Z}_{(\Delta',\Delta,f)}$ with the torus $\mathbf{T}_{\Gamma'}$, where Γ' is a face of Δ'. Let $\alpha(\Gamma')$ be the corresponding face of Δ. Shifting Δ and Δ', we may assume that Γ' and $\alpha(\Gamma')$ contain 0; let $M_{\Gamma'}$ be the sublattice in M cut out by the subspace in $M_{\mathbf{R}}$ spanned by Γ'. Finally, let $f_{\alpha(\Gamma')}$ be the trace of the Laurent polynomial f on the face $\alpha(\Gamma')$. Then $\overline{Z}_{(\Delta',\Delta,f)} \cap \mathbf{T}_{\Gamma'}$ is the hypersurface in the torus $\mathbf{T}_{\Gamma'} = \operatorname{Spec} \mathbf{C}[M_{\Gamma'}]$ given by the equation $f_{\alpha(\Gamma')} = 0$.

3.3. In the following two assertions we compare the fundamental group π_1 and the Picard group Pic of the hypersurface $\overline{Z}$ with the corresponding groups of $\mathbf{P}_\Delta$ (the structure of these last groups is described in [3]). Since we will not use these results, we give only very general statements. The proofs of the propositions below easily follow from the Lefschetz-Grothendieck theory, but the details would lead us astray.

So, let $f \in L(\Delta)$, where Δ is an n-dimensional polyhedron, and let $\overline{Z} = \overline{Z}_{(\Delta,f)}$ be the corresponding hypersurface in $\mathbf{P} = \mathbf{P}_\Delta$.

3.4. PROPOSITION. a) *If $n \geqslant 2$, then $\overline{Z}$ is connected.*

b) *If $n \geqslant 3$, then $\pi_1(\overline{Z})$ is a finite cyclic group.*

c) *If $n \geqslant 3$ and Δ is prime with respect to M, then $\pi_1(\overline{Z}) = 0$.*

Examples show that if Δ is not prime with respect to M, then $\overline{Z}$ is not necessarily simply connected even if Δ is a simplex of dimension $\geqslant 3$. Perhaps this is the most interesting aspect in which hypersurfaces in our situation differ from hypersurfaces in projective spaces.

3.5. PROPOSITION. a) *If $n \geqslant 3$, then* $\operatorname{Pic}\overline{Z}$ *is a group of finite type and* $\operatorname{Pic}\mathbf{P} \to \operatorname{Pic}\overline{Z}$ *is a monomorphism.*

b) *If $n \geqslant 4$ and Δ is a prime polyhedron, then* $\operatorname{Pic}\mathbf{P}$ *is a subgroup of finite index in* $\operatorname{Pic}\overline{Z}$.

c) *If $n \geqslant 4$ and Δ is prime with respect to M, then* $\operatorname{Pic}\mathbf{P} = \operatorname{Pic}\overline{Z}$.

3.6. The above two facts concern an arbitrary Laurent polynomial f; from now on we shall assume that f is nondegenerate. We say that *a Laurent polynomial $f \in L(\Delta)$ is nondegenerate with respect to* Δ if the hypersurface $\overline{Z} = \overline{Z}_{(\Delta,f)}$ transversally intersects all strata of $\mathbf{P}$. In other words, for each face $\Gamma \leqslant \Delta$ the variety $\overline{Z} \cap \mathbf{T}_\Gamma$ must be smooth and have codimension 1 in $\mathbf{T}_\Gamma$. In particular, $\overline{Z}$ must not pass through the "vertices" of $\mathbf{P}$ (i.e., points of the form $\mathbf{P}_P$, where $P \in \operatorname{som}(\Delta)$), and so Δ must coincide with the Newton polyhedron of f. This allows us to speak simply about nondegeneracy of f.

If f is nondegenerate with respect to Δ and Δ' majorizes Δ, then $\overline{Z}' = \overline{Z}_{(\Delta',\Delta,f)}$ also transversally intersects the strata of $\mathbf{P}_{\Delta'}$. In particular, if in addition Δ' is prime with respect to M, then $\overline{Z}'$ is a smooth variety and $D_Z = D \cap \overline{Z}'$ is a divisor with transversal crossings (see 2.4). Finally we observe that from Bertini's theorem it follows that a "generic" element from $L(\Delta)$ is nondegenerate with respect to Δ (see [6]).

The following Lefschetz type theorem compares the cohomology of a toric variety $\mathbf{P} = \mathbf{P}_\Delta$ and its nondegenerate hyperplane section $\overline{Z} = \overline{Z}_{(\Delta,f)}$. Here and up to the end of this section Δ denotes a polyhedron of dimension $n = \operatorname{rk} M$.

3.7. THEOREM. *The Gysin homomorphism $H^i(\overline{Z}, \mathbf{C}) \to H^{i+2}(\mathbf{P}, \mathbf{C})$ is an isomorphism for $i > n - 1 = \dim \overline{Z}$ and is a surjection for $i = n - 1$.*

PROOF. The Gysin homomorphism fits into the exact sequence

$$H^{i+1}(\mathbf{P} \setminus \overline{Z}) \to H^i(\overline{Z}) \to H^{i+2}(\mathbf{P}) \to H^{i+2}(\mathbf{P} \setminus \overline{Z})$$

of hypercohomology of the exact sequence of complexes

$$0 \to \Omega^\cdot_{\mathbf{P}} \to \Omega^\cdot_{\mathbf{P}}(\log D) \to \Omega^{\cdot -1}_{\overline{Z}} \to 0.$$

Here the complexes and the sheaves Ω and $\Omega(\log)$ on the toroidal varieties $\mathbf{P}$ and $\overline{Z}$ are understood in the sense of [3], §§13 and 15. The left-hand homomorphism corresponds to the Poincaré residue. Since $\mathbf{P} \setminus \overline{Z}$ is an affine toroidal variety, its cohomology groups $H^i(\mathbf{P} \setminus \overline{Z}, \mathbf{C})$ vanish for $i > n$ (see [3], §§13 and 6). This completes the proof.

3.8. COROLLARY. *For each open toric subvariety $U \subset \mathbf{P}$ the homomorphism $H^i_c(\overline{Z} \cap U) \to H^{i+2}_c(U)$ is bijective for $i > n - 1$ and surjective for $i = n - 1$.*

PROOF. Since U is obtained from $\mathbf{P}$ by throwing out several strata, it suffices to verify that the assertion remains true if we throw out from U a closed irreducible $\mathbf{T}$-invariant subvariety F. We denote $U \setminus F$ by V and consider the following commutative diagram with exact rows (see 1.4c)):

$$\begin{array}{ccccccccc}
\to & H^{i-1}_c(\overline{Z} \cap F) & \to & H^i_c(\overline{Z} \cap V) & \to & H^i_c(\overline{Z} \cap U) & \to & H^i_c(\overline{Z} \cap F) & \to \\
& \downarrow \gamma_{i-1} & & \downarrow \alpha_i & & \downarrow \beta_i & & \downarrow \gamma_i & \\
\to & H^{i+1}_c(F) & \to & H^{i+2}_c(V) & \to & H^{i+2}_c(U) & \to & H^{i+2}_c(F) & \to
\end{array}$$

Since $\overline{F}$ (the closure of F in $\mathbf{P}$) is a toric variety of smaller dimension and $\overline{Z} \cap \overline{F}$ is an ample nondegenerate hypersurface in $\overline{F}$, the assertion of the corollary holds for F in view of the inductive assumption. The proof is now completed by a routine diagram search.

In particular, for $U = \mathbf{T}^n$ we obtain the following

3.9. PROPOSITION. *The Gysin homomorphism $H^i_c(Z, \mathbf{C}) \to H^{i+2}_c(\mathbf{T}^n, \mathbf{C})$ is an isomorphism for $i > n - 1 = \dim Z$.*

3.10. REMARK. The dual statement to the effect that "the natural homomorphism $H^i(\mathbf{T}) \to H^i(Z)$ is bijective for $i < n - 1$" was first proved by D. N. Bernshteĭn. His very transparent proof based on Morse theory is not yet published. The first author has given a purely algebraic proof of this fact. The above proof is the third one known to the authors. We remark that the above results also carry over to complete intersections (see §6).

3.11. Since the cohomologies of the torus $\mathbf{T}$ and the Hodge structure on them are well known (see 1.10e)), Proposition 3.9 gives complete information about the Hodge structure on $H^i_c(Z)$ for $i > \dim Z$. In fact, for such i there is an isomorphism of Hodge structures $H^i_c(Z) = H^{i+2}_c(\mathbf{T})$ [1]. On the other hand, if $i < \dim Z$, then $H^i_c(Z) = 0$; this is a general fact about smooth affine varieties. Therefore the "middle-dimension" cohomology groups $H^{n-1}_c(Z)$ are of main interest. It is also seen that the Hodge-Deligne numbers $h^{p,q}(H^{n-1}_c(Z))$ can be easily recovered from $e^{p,q}(Z)$.

These remarks and the fact that $h^{p,q}(H_c^{n-1}(Z)) = 0$ for $p + q > n - 1$ (see 1.4d)) allow us to derive from 3.9 the following formula: for $p + q > n - 1$

$$e^{p,q}(Z) = e^{p+1,q+1}(\mathbf{T}) = \begin{cases} 0, & p \neq q, \\ (-1)^{n+p+1} C_n^{p+1}, & p = q. \end{cases}$$

§4. Computation of the χ_y-characteristic

In this section we compute the χ_y-characteristic of a hypersurface Z in the torus $\mathbf{T}$, i.e., the sum $\sum_q e^{p,q}(Z)$. Along with the Lefschetz type theorems, this is the second important ingredient of our computation of $e^{p,q}(Z)$. In what follows it will be assumed that the Newton polyhedron Δ has dimension $n = \dim M_{\mathbf{R}}$.

4.1. Let $f \in L(\Delta)$ be a nondegenerate Laurent polynomial. We denote by $\overline{Z}$ the hypersurface $\overline{Z}_{(\Delta',\Delta,f)}$ in $\mathbf{P} = \mathbf{P}_{\Delta'}$; $\overline{Z}$ is a smooth compactification of $Z = Z_f$ whose divisor at infinity $D_Z = D \cap \overline{Z}$, $D = \mathbf{P} \setminus \mathbf{T}$, is a divisor with transversal crossings. In view of 1.13, $\sum_q e^{p,q}$ modulo sign coincides with the Euler-Poincaré characteristic of the sheaf $\Omega^p_{(\overline{Z},D_Z)}$ on $\overline{Z}$. The computation of this last characteristic (see [4]) can be carried over to $\mathbf{P}$ in a fairly standard way. We briefly recall how to do that, since Hirzebruch considers the sheaves $\Omega^p_{\overline{Z}}$ and proceeds somewhat differently.

To begin with, we observe that there is the following exact sequence of coherent sheaves on $\mathbf{P}$:

$$0 \to \Omega^p_{(\overline{Z},D_Z)} \otimes_{\mathcal{O}_\mathbf{P}} \mathcal{O}_\mathbf{P}(-\overline{Z}) \to \Omega^{p+1}_{(\mathbf{P},D)} \otimes_{\mathcal{O}_\mathbf{P}} \mathcal{O}_{\overline{Z}} \to \Omega^{p+1}_{(\overline{Z},D_Z)} \to 0.$$

This sequence is analogous to the corresponding sequence of usual sheaves of differentials (see [4], §16.3) and is easily obtained from it by passing to the kernels of the corresponding restriction homomorphisms. Recalling that $\overline{Z}$ is defined by a section of the invertible sheaf $\mathcal{O}_\mathbf{P}(\Delta)$, we see that $\mathcal{O}_\mathbf{P}(\overline{Z}) \cong \mathcal{O}_\mathbf{P}(\Delta)$. Taking the tensor product of the preceding exact sequence with the invertible sheaves $\mathcal{O}_\mathbf{P}((k+1)\Delta)$ and passing to the Euler-Poincaré characteristics, we obtain

$$\chi\left(\overline{Z}, \Omega^p_{(\overline{Z},D_Z)}\right) = \sum_{k \geqslant 0} (-1)^k \chi\left(\mathbf{P}, \Omega^{p+k+1}_{(\mathbf{P},D)}((k+1)\Delta) \otimes_{\mathcal{O}_\mathbf{P}} \mathcal{O}_{\overline{Z}}\right).$$

Using the exact sequence

$$0 \to \mathcal{O}_\mathbf{P}(-\Delta) \to \mathcal{O}_\mathbf{P} \to \mathcal{O}_{\overline{Z}} \to 0,$$

we see that

$$\chi\left(\overline{Z}, \Omega^p_{(\overline{Z},D_Z)}\right) = \sum_{k \geqslant 0} (-1)^k \left[\chi\left(\mathbf{P}, \Omega^{p+k+1}_{(\mathbf{P},D)}((k+1)\Delta)\right) - \chi\left(\mathbf{P}, \Omega^{p+k+1}_{(\mathbf{P},D)}(k\Delta)\right)\right].$$

All this is commonplace; special features of the toric situation manifest themselves in that by Proposition 2.10 the summands of the above sum can be expressed in terms of the number of points in the polyhedra $k\Delta$. Performing a series of elementary transformations, we obtain the final result:

$$\chi\left(\overline{Z}, \Omega^p_{(\overline{Z},D_Z)}\right) = (-1)^{n+1} C_n^{p+1} - \sum_{k \geqslant 1} (-1)^k C_{n+1}^{p+k+1} l^*(k\Delta).$$

We recall that $l^*(k\Delta)$ is the number of integral points lying in the interior of the polyhedron $k\Delta$.

4.2. REMARK. Here and in what follows we are interested only in the dimension of the space of sections of the sheaves $\Omega^p_{(\mathbf{P},D)}(k\Delta)$. However Proposition 2.10 yields a finer structure of a grading of type M on these spaces. Sometimes it is very important to consider this finer structure (see [5]).

4.3. In order to represent the result obtained in 4.1 in a more compact form, we introduce some functions associated to polyhedra. Let Δ be a polyhedron in $M_{\mathbf{R}}$ (replacing M by a sublattice if necessary, we may assume that Δ spans $M_{\mathbf{R}}$); we consider the Poincaré series of the interior of Δ:

$$Q_\Delta(t) = \sum_{k>0} l^*(k\Delta)t^k,$$

where l^* denotes the number of interior lattice points of a polyedron. We shall soon see that the function

$$\Phi_\Delta(t) = Q_\Delta(t)(1-t)^{\dim\Delta+1}$$

is a polynomial of t of degree $\dim\Delta + 1$. We denote the coefficients of $\Phi_\Delta(t)$ by $\varphi_i(\Delta)$; thus

$$\Phi_\Delta(t) = \sum_{i\geqslant 0} \varphi_i(\Delta)t^i.$$

It is clear that $\varphi_0(\Delta) = 0$, $\varphi_1(\Delta) = l^*(\Delta)$, and, in general,

$$\varphi_i(\Delta) = (-1)^i \sum_{j\geqslant 1} (-1)^j C^{i-j}_{\dim\Delta+1} l^*(j\Delta).$$

Comparing this expression (for $i = n - p$) with the final formula of 4.1, we obtain the following formula (for an n-dimensional Δ):

4.4.

$$(-1)^{n-1}\sum_q e^{p,q}(Z) = (-1)^p C_n^{p+1} + \varphi_{n-p}(\Delta).$$

In particular, it follows that $\varphi_i(\Delta) = 0$ for $i > n+1$ and $\varphi_{n+1}(\Delta) = 1$.

4.5. REMARK. Summing up the formulas 4.4 for all p, we obtain the following formula for the Euler characteristic:

$$E(Z) = \sum_k (-1)^k \dim H^k_c(Z) = \sum_k (-1)^k \dim H^k(Z),$$

which, modulo sign, coincides with $\sum_i \varphi_i(\Delta) = \Phi_\Delta(1)$. Since $\Phi_\Delta(1) = n!V(\Delta)$, where $V(\Delta)$ is the n-dimensional volume of Δ, we again get (see [7])

$$E(Z) = (-1)^{n-1} n! V(\Delta).$$

4.6. REMARK. Occasionally instead of the functions $\varphi_i(\Delta)$ associated with the interior of the polyhedron it is more convenient to use similar functions associated with the polyhedron itself. To do this we consider the Poincaré series for Δ

$$P_\Delta(t) = \sum_{k\geqslant 0} l(k\Delta)t^k$$

and the polynomial (of degree at most $\dim\Delta$)

$$\psi_\Delta(t) = P_\Delta(t)(1-t)^{\dim\Delta+1} = \sum_i \psi_i(\Delta)t^i.$$

From the Serre duality for the sheaves $\mathcal{O}(k\Delta)$ on the variety $\mathbf{P}_\Delta$ it can be deduced that $\varphi_i(\Delta) = \psi_{\dim\Delta+1-i}(\Delta)$. Therefore 4.4 can be rewritten in the following form:

$$(-1)^{n-1}\sum_q e^{p,q}(Z) = (-1)^p C_n^{p+1} + \psi_{p+1}(\Delta).$$

This formula is convenient for small p, while 4.4 is more convenient when p is close to n.

§5. The Hodge-Deligne numbers of a hypersurface in a torus

This section is the central one in our paper. In it we show how the knowledge of $\sum_q e^{p,q}(Z)$ allows us to recover all the numbers $e^{p,q}(Z)$ for a nondegenerate hypersurface Z in a torus and hence all Hodge-Deligne numbers for Z. In many cases we are able to give explicit formulas for these numbers.

5.1. Let Δ be a polyhedron in $M_{\mathbf{R}}$, and let $f \in \mathbf{C}[M]$ be a Laurent polynomial which is nondegenerate with respect to Δ. We shall give an algorithm for computing the numbers $e^{p,q}(Z)$ for the hypersurface $Z = Z_f$ in the torus $\mathbf{T}^n = \operatorname{Spec}\mathbf{C}[M]$.

To begin with, we observe that it suffices to consider the case when $\dim\Delta = n = \operatorname{rk} M$. In fact, in general Z is a product of the hypersurface Z' with the same equation in a torus of dimension $\dim\Delta$ and the torus $\mathbf{T}'$ of the complementary dimension: $Z = Z' \times \mathbf{T}'$. From the multiplicativity of e (Propositon 1.8) it follows that

$$e(Z) = e(Z')\cdot e(\mathbf{T}') = e(Z')(x\bar{x} - 1)^{\dim\mathbf{T}'}.$$

5.2. So, in addition to the assumptions of 5.1, we shall assume that $\dim\Delta = n$. Let Δ' be a prime (or prime with respect to M) polyhedron majorizing Δ, let $\mathbf{P} = \mathbf{P}_{\Delta'}$, and let $\overline{Z} = \overline{Z}_{(\Delta',\Delta,f)}$. For a face $\Gamma' \leqslant \Delta'$ we denote by $Z_{\Gamma'}$ the hypersurface $\overline{Z} \cap \mathbf{T}_{\Gamma'}$ in the torus $\mathbf{T}_{\Gamma'}$ (see 2.2 and 3.2; the Newton polyhedron of $Z_{\Gamma'}$ coincides with $\alpha(\Gamma')$); then $\overline{Z} = \bigcup_{\Gamma' \leqslant \Delta'} Z_{\Gamma'}$. Hence

$$e^{p,q}(\overline{Z}) = e^{p,q}(Z_{\Delta'}) + \sum_{\Gamma' < \Delta'} e^{p,q}(Z_{\Gamma'}).$$

We observe that $Z_{\Delta'} = Z$ and for $\Gamma' < \Delta'$ the hypersurface $Z_{\Gamma'}$ has smaller dimension, so that arguing by induction we may assume that the numbers $e^{p,q}(Z_{\Gamma'})$ are known.

By 3.11, the numbers $e^{p,q}(Z)$ for $p + q > n - 1$ are also known; hence the same is true for the numbers $e^{p,q}(\overline{Z})$ for $p + q > n - 1$. By Poincaré duality for the smooth (or quasismooth) variety $\overline{Z}$, the numbers $e^{p,q}(\overline{Z}) = e^{n-1-p,n-1-q}(\overline{Z})$ are also known for $p + q < n - 1$. Using the formula establishing the relationship between $e(\overline{Z})$ and $e(Z)$, we obtain all numbers $e^{p,q}(Z)$ for $p + q < n - 1$.

Since the sums $\sum_q e^{p,q}(Z)$ are also known (see 4.1 or 4.4), we obtain the last missing number $e^{p,n-1-p}(Z)$.

5.3. Remark. The above method for computing the numbers $e^{p,q}(Z)$ is quite transparent and constructive, and without doubt it is an algorithm. It is clear that the numbers $e^{p,q}(Z)$ are determined by the polyhedron Δ; more precisely, they depend on combinatorial properties of Δ (i.e., on the contiguity scheme of the faces) and on the numbers $l^*(k\Delta)$ or $l(k\Delta)$ for all faces Γ and all integral k (incidentally, it suffices to consider only k in the interval $0 < k < \dim\Gamma$).

In some special cases which will be discussed below the above arguments are sufficient for deriving explicit formulas for the numbers $e^{p,q}(Z)$. Explicit formulas in the general case have recently been obtained by the second author.

5.4. REMARK. As we explained in 3.11, starting from $e^{p,q}(Z)$ one can compute all Hodge-Deligne numbers $h^{p,q}(H_c^k(Z))$. Thus, starting from the one-parameter family of numbers $\sum_q e^{p,q}(\Delta)$, we obtain a three-parameter family of numbers.

5.5. The most important special case when the numbers $e^{p,q}(Z)$ can be explicitly computed is the case when the polyhedron Δ is prime. However we begin with the compact case, i.e., with the computation of the Hodge numbers for $\overline{Z} = \overline{Z}_{(\Delta,f)}$, which is the canonical compactification of Z.

Decomposing $\overline{Z}$ into a union of Z_Γ (where Γ runs through the set of faces of Δ) as in 5.2 and using the additivity of $e^{p,q}$, we obtain

$$\begin{aligned}\sum_q e^{p,q}(\overline{Z}) &= \sum_{\Gamma\leqslant\Delta}\sum_q e^{p,q}(Z_\Gamma)\\ &= (-1)^{p+1}\sum_{\Gamma\leqslant\Delta}(-1)^{\dim\Gamma}C_{\dim\Gamma}^{p+1} - \sum_{\Gamma\leqslant\Delta}(-1)^{\dim\Gamma}\varphi_{\dim\Gamma-p}(\Gamma).\end{aligned}$$

We observe that the first summand of the last expression is nothing else but $h^{p+1,p+1}(\mathbf{P})$ (see 2.5).

Now we apply the Lefschetz Theorem 3.7, which shows that $e^{p,q}(\overline{Z}) = e^{p+1,q+1}(\mathbf{P})$ for $p + q > n - 1$. In view of the Poincaré duality for $\overline{Z}$ (here is the place where we use the assumption that Δ is prime), $e^{p,q}(\overline{Z})$ is not equal to zero only if $p = q$ or $p + q = n - 1$. Thus if $p + q = n - 1$ and $p \neq q$, then

$$e^{p,q}(\overline{Z}) = -\sum_{\Gamma\leqslant\Delta}(-1)^{\dim\Gamma}\varphi_{\dim\Gamma-p}(\Gamma).$$

If $p + q = n - 1$ and $p = q$, then

$$e^{p,p}(\overline{Z}) = \sum_{\Gamma\leqslant\Delta}(-1)^{\dim\Gamma}\left[(-1)^{p+1}C_{\dim\Gamma}^{p+1} - \varphi_{\dim\Gamma-p}(\Gamma)\right].$$

The remaining $e^{p,q}(Z)$ either vanish or can be computed via symmetry. In particular, for $n > 1$

$$h^{n-1,0}(\overline{Z}) = h^{0,n-1}(\overline{Z}) = \varphi_1(\Delta) = l^*(\Delta).$$

Continuing to assume that Δ is prime, we give formulas for $e^{p,q}(Z)$. Understandably, we shall restrict ourselves to the case when $p > q$.

5.6. THEOREM. *Suppose that Δ is a prime n-dimensional Newton polyhedron. Then for $p > q$*

$$e^{p,q}(Z) = (-1)^{n+p+q}\sum_{\dim\Gamma=p+q+1}\left(\sum_{\Gamma'\leqslant\Gamma}(-1)^{\dim\Gamma'}\varphi_{\dim\Gamma'-p}(\Gamma)\right).$$

PROOF. We shall use the following formula (which is a consequence of 1.7):

$$e^{p,q}(Z) = \sum_{\Gamma\leqslant\Delta}(-1)^{n-\dim\Gamma}e^{p,q}(\overline{Z}_\Gamma),$$

where $\overline{Z}_\Gamma = \overline{Z}\cap\mathbf{P}_\Gamma$. By the results of the preceding section, for $p > q$ all the $e^{p,q}(\overline{Z}_\Gamma)$ are zero with the exception of the case $p + q = \dim\Gamma - 1$, when

$$e^{p,q}(\overline{Z}_\Gamma) = -\sum_{\Gamma'\leqslant\Gamma}(-1)^{\dim\Gamma'}\varphi_{\dim\Gamma'-p}(\Gamma').$$

This proves the theorem.

In particular, for $q = 0$ and $p > 0$ we have

$$h^{p,0}(H_c^{n-1}(Z)) = (-1)^{n-1} e^{p,0}(Z) = \sum_{\dim \Gamma = p+1} l^*(\Gamma).$$

As we shall see below, this formula holds for an arbitrary Δ.

5.7. REMARK. The formula from 5.6 can be rewritten in the form

$$e^{p,q}(Z) = (-1)^{n+p+q} \sum_{\Gamma \leqslant \Delta} (-1)^{\dim \Gamma} C_{n-\dim \Gamma}^{n-p-q-1} \varphi_{\dim \Gamma - p}(\Gamma).$$

5.8. PROPOSITION. *For an arbitrary polyhedron Δ in $M_{\mathbf{R}}$ and arbitrary $p > 0$*

$$e^{p,0}(Z) = (-1)^{n-1} \sum_{\dim \Gamma = p+1} l^*(\Gamma),$$

where Γ runs through the set of $(p+1)$-dimensional faces of Δ.

PROOF. First of all, we may assume that $\dim \Delta = n$. In fact, in the general case Z is multiplied by the torus of dimension $n - \dim \Delta$ and $e^{p,0}$ by $(-1)^{n-\dim \Delta}$. Assuming that Δ is n-dimensional, we now use the argument from 5.2. Let Δ' be a prime polyhedron majorizing Δ, let $\mathbf{P} = \mathbf{P}_{\Delta'}$, and let $\overline{Z} = \overline{Z}_{(\Delta', \Delta, f)}$. Let Γ' be a face of Δ' which does not coincide with Δ'; since $\dim Z_{\Gamma'} < n - 1$, it is clear that $e^{n-1-p,n-1}(Z_{\Gamma'}) = 0$. Moreover, the numbers $e^{n-1-p,n-1}(Z)$ are also equal to 0 for $0 < p < n - 1$ (see 3.11). In what follows we shall assume that $0 < p < n - 1$; for $p = n - 1$ the proposition immediately follows from 4.3.

In view of the Poincaré duality for $\overline{Z}$, $e^{p,0}(\overline{Z}) = 0$. On the other hand,

$$e^{p,0}(\overline{Z}) = \sum_{\Gamma' \leqslant \Delta'} e^{p,0}(Z_{\Gamma'}).$$

If $\Gamma' \neq \Delta'$, then the hypersurface $Z_{\Gamma'} = \overline{Z} \cap \mathbf{T}_{\Gamma'}$ has smaller dimension and we can use the inductive assumption. Hence to prove the proposition it remains to verify that

$$\sum_{\Gamma' \leqslant \Delta'} (-1)^{\dim \Gamma' - 1} \sum_{\substack{\Gamma \leqslant \alpha(\Gamma') \\ \dim \Gamma = p+1}} l^*(\Gamma) = 0,$$

where $\alpha(\Gamma')$ is the face of Δ corresponding to Γ' under the majorization map 2.6. The expression on the left can be rewritten as follows:

$$-\sum_{\substack{\Gamma \leqslant \Delta \\ \dim \Gamma = p+1}} l^*(\Gamma) \left\{ \sum_{\substack{\Gamma' \leqslant \Delta' \\ \Gamma \leqslant \alpha(\Gamma')}} (-1)^{\dim \Gamma'} \right\}.$$

Hence it suffices to verify that for each face Γ of the polyhedron Δ which is different from Δ we have

$$\sum_{\substack{\Gamma' \leqslant \Delta' \\ \Gamma \leqslant \alpha(\Gamma')}} (-1)^{\dim \Gamma'} = 0.$$

To do this, it is convenient to pass to the fans Σ_Δ and $\Sigma_{\Delta'}$ dual to the polyhedra Δ and Δ'. More precisely, to each face Γ of the polyhedron Δ we associate the cone σ_Γ dual to $\mathrm{Con}(\Delta, \Gamma)$; similarly, to $\Gamma' \leqslant \Delta'$ we associate the cone $\sigma_{\Gamma'}$. The condition $\Gamma \leqslant \alpha(\Gamma')$ becomes then the condition $\sigma_{\Gamma'} \subset \sigma_\Gamma$. The collection of cones $\sigma_{\Gamma'}$, where $\Gamma \leqslant \alpha(\Gamma')$, forms a conic polyhedral decomposition of the cone σ_Γ. Since $\dim \Gamma' = n - \dim \sigma_{\Gamma'}$, the above equality turns into

$$\sum_{\sigma_{\Gamma'} \subset \sigma_\Gamma} (-1)^{\dim \sigma_{\Gamma'}} = 0,$$

which is almost obvious. This completes the proof of Proposition 5.8.

5.9. COROLLARY. *For* $\dim \Delta = n \geqslant 4$

$$e^{n-2,1}(Z) = (-1)^{n-1}\left[\varphi_2(\Delta) - \sum_{\dim \Gamma = n-1} \varphi_1(\Gamma)\right]$$

(*we recall that* $\varphi_2(\Delta) = l^*(2\Delta) - (n+1)l^*(\Delta)$ *and* $\varphi_1(\Gamma) = l^*(\Gamma)$).

5.10. COROLLARY. $e^{0,0}(Z) = (-1)^{n-1}(\Pi - 1)$, *where* Π *is the number of integral points lying in the* 1-*skeleton of* Δ.

In fact, arguing as in the beginning of the proof of 5.8 we may reduce the problem to the case when $\dim \Delta = n$. Since $e^{0,p}(Z) = e^{p,0}(Z)$, we have

$$(-1)^{n-1}\left(e^{0,1}(Z) + \cdots + e^{0,n-1}(Z)\right) = \sum_{\dim \Gamma \geqslant 2} l^*(\Gamma).$$

On the other hand, according to 4.6

$$(-1)^{n-1}\sum_q e^{0,q}(Z) = n + \psi_1(\Delta).$$

Since $\psi_1(\Delta) + n + 1 = l(\Delta) = \sum_{\Gamma \leqslant \Delta} l^*(\Gamma)$,

$$(-1)^{n-1}e^{0,0}(Z) = \sum_{\dim \Gamma \leqslant 1} l^*(\Gamma) - 1 = \Pi - 1.$$

5.11. The formulas obtained in 5.8–5.10 allow us to give explicit computations of Hodge numbers for arbitrary Newton polyhedra of dimension at most 4. Here we restrict ourselves to the Hodge-Deligne numbers for the most interesting "middle-dimension" cohomology groups $H_c^{n-1}(Z)$.

a) $n = 1$. In this case Δ is an interval of length l, and Z consists of $l = l^*(\Delta) + 1 = l(\Delta) - 1$ points.

b) $n = 2$. In this case Z is a smooth curve of genus $l^*(\Delta)$, Π of whose points are known out. From now on Π denotes the number of integral points in the 1-skeleton of Δ. The table of Hodge numbers for $H_c^1(Z)$ has the form

$l^*(\Delta)$	0
$\Pi - 1$	$l^*(\Delta)$

c) $n = 3$. Here Z is a noncomplete surface; the table of Hodge numbers for $H_c^2(Z)$ has the form

$l^*(\Delta)$	0	0
$\sum_\Gamma l^*(\Gamma)$	$h^{1,1}$	0
$\Pi - 1$	$\sum_\Gamma l^*(\Gamma)$	$l^*(\Delta)$

where $h^{1,1} = \varphi_2(\Delta) - \Sigma_\Gamma l^*(\Gamma)$. Here the sum is taken over the two-dimensional faces of Δ. We also recall that $\varphi_2(\Delta) = l^*(2\Delta) - 4l^*(\Delta)$.

Incidentally, the canonical compactification $\overline{Z} = \overline{Z}_{(\Delta, f)}$ is a quasismooth surface whose Hodge numbers are as follows:

$$h^{0,0}(\overline{Z}) = h^{2,2}(\overline{Z}) = 1,$$
$$h^{2,0}(\overline{Z}) = h^{0,2}(\overline{Z}) = l^*(\Delta),$$
$$h^{1,1}(\overline{Z}) = \varphi_2(\Delta) - 3 - \sum_\Gamma (l^*(\Gamma) - 1),$$
$$h^{0,1} = h^{1,0} = h^{2,1} = h^{1,2} = 0.$$

d) $n = 4$. In what follows Γ runs through the set of 3-dimensional faces of Δ and F runs through the set of 2-dimensional faces of Δ. The Hodge table for $H_c^3(F)$ has the form

$*(\Delta)$	0	0	0
$\sum_\Gamma l^*(\Gamma)$	$h^{1,2}$	0	0
$\sum_F l^*(F)$	$h^{1,1}$	$h^{2,1}$	0
$\Pi - 1$	$\sum_F l^*(F)$	$\sum_\Gamma l^*(\Gamma)$	$l^*(\Delta)$

where

$$h^{2,1} = h^{1,2} = \varphi_2(\Delta) - \sum_\Gamma l^*(\Gamma),$$
$$h^{1,1} = \varphi_3(\Delta) - \varphi_2(\Delta) + \sum_\Gamma l^*(\Gamma) - \sum_F l^*(F).$$

Incidentally, $\varphi_2(\Delta) = l^*(2\Delta) - 5l^*(\Delta)$ and $\varphi_3(\Delta) = \psi_2(\Delta) = l(2\Delta) - 5l(\Delta) + 10$.

5.12. Knowing the numbers $e^{p,q}$ for the hypersurfaces in tori, we thereby know them for hypersurfaces in arbitrary toric varieties. We consider in more detail the case of hypersurfaces in the n-dimensional affine space $\mathbf{A} = \mathbf{C}^n$. In this case the lattice M has a natural basis and can justifiably be denoted by $\mathbf{Z}^n$. The Newton polyhedron of a polynomial $f(x_1, \dots, x_n)$ lies in the positive orthant $\mathbf{R}^n_+$. As above, we assume that the polynomial f is nondegenerate; the corresponding hypersurface $f(x) = 0$ in $\mathbf{A}$ will be denoted by Z^a.

Our approach to the study of Z^a is the same: we represent $\mathbf{A}^n$ as a union of tori and sum up the corresponding $e^{p,q}$. More precisely, for each $I \subset \{1, \dots, n\}$ we denote by Z_I the hypersurface defined by the equation $f_I = 0$ in the torus $(\mathbf{C} \setminus 0)^{|I|}$. The Newton polyhedron of f_I coincides with $\Delta_I = \Delta \cap \mathbf{R}^I$; clearly, if Δ does not intersect $\mathbf{R}^I$, then $Z_I = (\mathbf{C} \setminus 0)^{|I|}$. Summing over I, we see that

$$e^{p,q}(Z^a) = \sum_I e^{p,q}(Z_I).$$

This again yields the Euler characteristic of Z^a (see [7], §4):

$$E(Z^a) = \sum_I (-1)^{|I|-1} |I|! V_{|I|}(\Delta_I),$$

but still gives little information about the cohomology of Z^a.

A more complete answer can be given if, in addition, we assume that the Newton polyhedron Δ is "convenient", i.e., coincides with $\mathbf{R}^n_+$ in some neighborhood of 0. First, in this case $H_c^k(Z^a) = 0$ for $k < n-1$ (since Z^a is affine and toroidal), and second, $H_c^k(Z^a) = H_c^{k+2}(\mathbf{A})[1]$ for $k > n-1$ (Corollary 3.8). In other words, all $H_c^k(Z^a)$ vanish with the exception of $H_c^{n-1}(Z^a)$ and $H_c^{2n-2}(Z^a) = \mathbf{C}[1-n]$. Thus the only nontrivial Hodge structure is that on $H_c^{n-1}(Z^a)$ for which

$$(-1)^{n-1}h^{p,q}(H_c^{n-1}(Z^a)) = \begin{cases} e^{p,q}(Z^a), & (p,q) \neq (n-1,n-1), \\ e^{p,q}(Z^a) - 1, & (p,q) = (n-1,n-1). \end{cases}$$

§6. Complete intersections

In this section we show how the above results can be applied to the computation of Hodge structures for complete intersections in tori.

6.1. Let $\mathbf{T}^n$ again be the n-dimensional torus with lattices of characters M, and let $f_1, \dots, f_r \in \mathbf{C}[M]$ be r Laurent polynomials defining r hypersurfaces $Z_i = Z_{f_i}$ ($i = 1, \dots, r$) in $\mathbf{T}^n$. Let Δ_i be the Newton polyhedron of f_i. In what follows we assume that the system $f_1, \dots, f_r$ is nondegenerate (see [7] for a definition of this notion).

For a nondegenerate system $f_1, \dots, f_r$, the hypersurfaces $Z_1, \dots, Z_r$ intersect transversally and define a complete intersection $Y = Z_1 \cap \cdots \cap Z_r$. Moreover, if Δ is an arbitrary n-dimensional polyhedron majorizing $\Delta_1, \dots, \Delta_r$, then the hypersurfaces $\overline{Z}_1, \dots, \overline{Z}_r$ in $\mathbf{P}_\Delta$ intersect transversally (and are transversal to the divisor $D = \mathbf{P}_\Delta \setminus \mathbf{T}^n$); this constitutes a geometric definition of nondegeneracy of the system $f_1, \dots, f_r$.

6.2. To compute $e^{p,q}(Y)$ we add auxiliary variables $\lambda_1, \dots, \lambda_r$ (Lagrange multipliers), and in the toric variety $\mathbf{C}^r \times \mathbf{T}$ we consider the hypersurface Z_F with the equation

$$F(\lambda, x) = \lambda_1 f_1(x) + \cdots + \lambda_r f_r(x) - 1 = 0.$$

The Newton polyhedron Λ of the polynomial F is the convex hull (in the space $\mathbf{R}^r \times M_{\mathbf{R}}$) of the following $(r+1)$ polyhedra: $\{0\}$, $\{e_1\} \times \Delta_1, \dots, \{e_r\} \times \Delta_r$, where $e_1, \dots, e_r$ is the natural basis of $\mathbf{Z}^r$. It is easy to verify that F is nondegenerate with respect to Λ.

Restricting the projection $\mathbf{C}^r \times \mathbf{T}^n \to \mathbf{T}^n$ to Z_F, we obtain a map $\pi\colon Z_F \to \mathbf{T}^n$. It is easy to see that for a point x from Y the fiber $\pi^{-1}(x)$ is empty. If $x \notin Y$, then the fiber $\pi^{-1}(x)$ is a linear affine subspace in $\mathbf{C}^r$. Moreover, Z_F is a locally trivial (in the Zariski topology) bundle over $\mathbf{T}^n \setminus Y$ with fiber $\mathbf{C}^{r-1}$. Therefore by 1.9

$$e(Z_F) = (x\bar{x})^{r-1} e(\mathbf{T}^n \setminus Y)$$

or

$$e^{p,q}(Y) = e^{p,q}(\mathbf{T}^n) - e^{p+r-1,q+r-1}(Z_F).$$

This formula reduces the study of the complete intersection Y to the study of the hypersurface Z_F. We observe, however, that Z_F is a hypersurface not in a torus, but in $\mathbf{C}^r \times \mathbf{T}^n$.

6.3. The complete intersection $Y \subset \mathbf{T}^n$ is a smooth affine variety, $\dim Y = n - r$. Therefore $H_c^i(Y) = 0$ for $i < n - r$. Moreover, as in the case of hypersurfaces, we have the following Lefschetz-type theorem for Y (D. N. Bernshteĭn).

6.4. THEOREM. *Suppose that all Newton polyhedra $\Delta_1, \dots, \Delta_r$ have dimension $n = \operatorname{rk} M$. Then the Gysin homomorphism*

$$H_c^i(Y) \to H_c^{i+2r}(\mathbf{T}^n)$$

is an isomorphism for $i > n - r$.

Thus the Hodge structure on $H_c^k(Y)$ is also well known for $k > n - r$. Together with the formula for the numbers $e^{p,q}(Y)$ from 6.2, this also allows us to recover the Hodge numbers $h^{p,q}(H_c^{n-1}(Y))$.

6.5. Remark. The assumption that the polyhedra $\Delta_1, \ldots, \Delta_r$ in the statement of Theorem 6.4 are n-dimensional may be somewhat weakened (but not totally discarded). Namely, it suffices to require that *for each nonempty $I \subset \{1, \ldots, r\}$ the dimension of the polyhedron $\sum_{i \in I} \Delta_i$ be not less than $n + |I| - r$*. In that case we can also compute all Hodge-Deligne numbers of Y.

6.6. Remark. Without any conditions on Δ_i, summing the formulas from 6.2 for all p and q, we see that $E(Y) = -E(Z_F)$. Applying the formula from 4.5 for the Euler characteristic of a hypersurface in a torus, we see that

$$E(Z_F) = \sum_I (-1)^{n+|I|-1}(n + |I|)! V_{n+|I|}(\Lambda_I).$$

Here I runs through the subsets of $\{1, \ldots, r\}$, $\Lambda_I = \Lambda \cap (\mathbf{R}^I \times M_{\mathbf{R}})$ and $V_{n+|I|}(\Lambda_I)$ denotes the $(n + |I|)$-dimensional volume of Λ_I.

We denote by $\Delta_1 * \cdots * \Delta_r$ the convex hull of the polyhedra $\{e_1\} \times \Delta_1, \ldots, \{e_r\} \times \Delta_r$ in the space $\mathbf{R}^r \times M_{\mathbf{R}}$. Then Λ is a pyramid with vertex $\{0\}$ and base $\Delta_1 * \cdots * \Delta_r$. In general, for $I \subset \{1, \ldots, r\}$ we denote by Δ^{*I} the polyhedron $*_{i \in I} \Delta_i$; then Λ_I is a pyramid over Δ^{*I}. Since

$$(n + |I|) \cdot V_{n+|I|}(\Lambda_I) = V_{n+|I|-1}(\Delta^{*I}),$$

the formula for $E(Y)$ can be rewritten as follows:

$$E(Y) = \sum_I (-1)^{n+|I|}(n + |I| - 1)! V_{n+|I|-1}(\Delta^{*I}).$$

Comparing this with the formula from [7], Theorem 2, we obtain

$$(n + r - 1)! Y_{n+r-1}(\Delta_1 * \cdots * \Delta_r) = n! \sum_{\bar{k}, |\bar{k}| = n} V_n(\Delta^{\bar{k}}).$$

Here, for a multi-index $\bar{k} = (k_1, \ldots, k_r)$, $V_n(\Delta^{\bar{k}})$ denotes the mixed volume

$$V_n\Big(\underbrace{\Delta_1, \ldots, \Delta_1}_{k_1 \text{ times}}, \ldots, \underbrace{\Delta_r, \ldots, \Delta_r}_{k_r \text{ times}}\Big).$$

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