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Moment polytopes, semigroup of representations and Kazarnovskii's theorem

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To Stephen Smale, our mathematical hero

Abstract. Two representations of a reductive group G are spectrally equivalent if the same irreducible representations appear in both of them. The semigroup of finite-dimensional representations of G with tensor product and up to spectral equivalence is a rather complicated object. We show that the Grothendieck group of this semigroup is more tractable and we give a description of it in terms of moment polytopes of representations. As a corollary, we give a proof of the Kazarnovskii theorem on the number of solutions in G of a system $f_1 = \cdots = f_m = 0$, where $m = \dim(G)$ and each f_i is a generic function in the space of matrix elements of a representation π_i of G.

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1. Introduction

With a commutative semigroup S one associates its Grothendieck semigroup $\mathrm{Gr}(S)$ which is a semigroup with cancelation. We say that two elements $a, b \in S$ are *analogous* and write $a \sim b$ if there is $c \in S$ with $a + c = b + c$ (where we write the semigroup operation additively). The relation $\sim$ is an equivalence relation and respects the addition. The set of equivalence classes of $\sim$, together with the induced addition, is the *Grothendieck semigroup of* S denoted by $\mathrm{Gr}(S)$. The map which associates with each element its equivalence class, gives a natural homomorphism $\rho : S \to \mathrm{Gr}(S)$.

The semigroup $\mathrm{Gr}(S)$ has the *cancelation property*; i.e., for $a, b, c \in \mathrm{Gr}(S)$, the equality $a + c = b + c$ implies $a = b$. Moreover, for any homomorphism $\varphi : S \to H$, where H is a semigroup with cancelation, there exists a unique homomorphism $\bar{\varphi} : \mathrm{Gr}(S) \to H$ such that $\varphi = \bar{\varphi} \circ \rho$. In particular, analogous elements have the same image under the homomorphism φ.

Any semigroup H with cancelation naturally extends to its *group of formal differences* which consists of pairs of elements from H where we consider two pairs (a, b) and (c, d) equal if $a + d = b + c$. The *Grothendieck group of a semigroup* S is the group of formal differences of $\mathrm{Gr}(S)$. The semigroup $\mathrm{Gr}(S)$ contains significant information about S and is simpler to describe.

In this paper, we discuss a description of analogous elements, the Grothendieck semigroup and the homomorphism $\rho : S \to \mathrm{Gr}(S)$ for the following two examples of semigroups. The first one is the motivating example for the second one which we consider as the main contribution of the present paper.

Example 1. Let $\mathcal{K}$ be the semigroup of nonempty finite subsets in the lattice $\mathbb{Z}^n$ with respect to the addition of subsets. The semigroup $\mathcal{K}$ is rather complicated, but its Grothendieck semigroup is easy to describe. It is isomorphic to the semigroup $\mathcal{P}$ of convex integral polytopes in $\mathbb{R}^n$ with respect to addition (also called the Minkowski sum). The homomorphism ρ sends a finite subset $A \subset \mathbb{Z}^n$ to its convex hull $\Delta(A)$.

From the description of $\mathrm{Gr}(\mathcal{K})$, one obtains a simple proof of the well-known Bernstein–Kushnirenko theorem. We recall its statement here: The support of a Laurent polynomial f in $x_1, \ldots, x_n$ is the finite set of $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ where $x_1^{a_1} \cdots x_n^{a_n}$ appears in f with nonzero coefficient. For a finite nonempty set $A \subset \mathbb{Z}^n$ let L_A denote the subspace of Laurent polynomials with supports in A. Let $A_1, \ldots, A_n \subset \mathbb{Z}^n$ be finite nonempty subsets with convex hulls $\Delta_1, \ldots, \Delta_n$, respectively. The Bernstein–Kushnirenko theorem asserts that the number of solutions in $(\mathbb{C}^*)^n$ of a system $f_1 = \cdots = f_n = 0$, where f_i are generic Laurent polynomials in L_{A_i}, is equal to the mixed volume of the $\Delta_1, \ldots, \Delta_n$ multiplied by $n!$ (see Sections 4 and 6).

Example 2. Let G be a complex connected reductive algebraic group of dimension m. We say that two finite-dimensional representations π_1, π_2 of G are *spectrally equivalent* if they have the same G-spectrum, i.e., the same irreducible representations appear in both of them (but perhaps with different multiplicities). Let $\mathcal{R}_{\mathrm{Spec}}(G)$ be the semigroup of finite-dimensional representations of G with respect to tensor product and up to spectral equivalence. This semigroup is quite complicated. In Section 7 we describe its Grothendieck semigroup: let us identify the weight lattice of G, i.e., the lattice of characters of a maximal torus of G, with $\mathbb{Z}^n$, and let $\mathbb{R}^n$ be its real span. The Weyl group W of G is a finite group generated by reflections acting on $\mathbb{R}^n$ preserving the lattice $\mathbb{Z}^n$. One fixes a fundamental domain C for the action of W and calls it the positive Weyl chamber. Up to the spectral equivalence, a representation π is determined by a finite number of integral points in the positive Weyl chamber; they are the so-called highest weights of the representation π. The convex hull of the union of W-orbits of these highest weights is called the *weight polytope* of the representation. We denote it by $\Delta_W(\pi)$. It is a convex integral W-invariant polytope in $\mathbb{R}^n$. Also we call the intersection of $\Delta_W(\pi)$ with the positive Weyl chamber C, the *moment polytope* of π and denote it by $\Delta_W^+(\pi)$. The main result (Theorem 7.10) asserts that the Grothendieck semigroup of $\mathcal{R}_{\mathrm{Spec}}(G)$ is isomorphic to the semigroup $\mathcal{P}_W$ of convex integral W-invariant polytopes in $\mathbb{R}^n$ together with the Minkowski sum of polytopes. The homomorphism ρ sends a representation π to its weight polytope $\Delta_W(\pi)$. Alternatively, one can describe the Grothendieck semigroup of $\mathcal{R}_{\mathrm{Spec}}(G)$ in terms of the moment polytopes (see Theorem 7.11).

As in the case of Bernstein–Kushnirenko theorem, Example 2 then allows us to obtain a simple proof of the theorem of Kazarnovskii. We recall its statement here: for a representation π of G let L_π denote the space of matrix elements of π, i.e., the subspace spanned by the matrix entries (in some basis) of the representation π regarded as regular functions on G. Let $\pi_1, \ldots, \pi_m$ be finite-dimensional representations of G with moment polytopes $\Delta_1 = \Delta_W^+(\pi_1), \ldots, \Delta_m = \Delta_W^+(\pi_m)$, respectively. The Kazarnovskii theorem computes the number of solutions of a system $f_1 = \cdots = f_m = 0$ on the group G, where f_i are generic functions from the space of matrix elements L_{π_i}, in terms of the polytopes Δ_i. (see Sections 4 and 8).

In Section 9 we rewrite the Kazarnovskii formula for the number of solutions. To each irreducible representation of a classical group there corresponds its *Gelfand–Cetlin polytope*. Given a representation π of a classical group G, one defines the polytope $\tilde{\Delta}(\pi)$ to be the polytope fibred over the moment polytope $\Delta_W^+(\pi)$ and with Gelfand–Cetlin polytopes as fibres. Then, for classical groups, the Kazarnovskii theorem can be formulated exactly as the Bernstein–Kushnirenko theorem: the number of solutions of the system under discussion is equal to the mixed volume of the polytopes $\tilde{\Delta}(\pi_i)$ multiplied by $m!$.

The main theorem (Theorem 7.10) was conjectured by the second author in the early 1990s after the paper [Kazarnovskii87] had appeared. Our main tool in the proof here is the PRV (Parthasarathy–Ranga Rao–Varadarajan) conjecture/theorem which is a deep result about tensor product of irreducible representations (Theorem 7.8).

The weight polytope (or moment polytope) of a representation also plays an important role in questions related to the geometry of the group G, its compactifications and its subvarieties. For some interesting results in this direction see [Kapranov97, Timashev03, Kiritchenko06, Kiritchenko07].

The proof of the Bernstein–Kushnirenko theorem in this paper, up to some improvements, is the same as the one in [Khovanskii92]. The main theorem (Theorem 7.10) allows us to extend this argument and prove the Kazarnovskii theorem. Another ingredient in our proof is the intersection theory of finite-dimensional subspaces of rational functions on varieties (developed in [Kaveh-Khovanskii08]). This is briefly reviewed in Section 2. A crucial step in our proofs is a description of the *completion* of a subspace L_π of matrix elements of a representation π (Theorems 6.3 and 8.7). This is a direct corollary of the description of Grothendieck semigroup of subspaces of matrix elements (Proposition 2.2).

Kushnirenko's theorem is a particular case of Bernstein–Kushnirenko's theorem where all the Newton polytopes of equations are the same. Up to now, its several generalizations are known (see [Brion89] for spherical varieties, and [Kaveh-Khovanskii09] for arbitrary varieties). The Bernstein–Kushnirenko theorem is harder to generalize (see [Kaveh-Khovanskii10]). Its most important generalization so far is the Kazarnovskii theorem which we address in this paper.

2. Intersection theory of finite-dimensional subspaces of regular functions

Let X be a complex n-dimensional irreducible normal affine variety with $\mathbb{C}[X]$ its ring of regular functions. Consider the collection $\mathbf{K}[X]$ of all nonzero finite-dimensional subspaces of $\mathbb{C}[X]$. The *product* of two subspaces $L_1, L_2 \in \mathbf{K}[X]$ is the subspace spanned by all the fg, where $f \in L_1$, $g \in L_2$. With this product, $\mathbf{K}[X]$ is a commutative semigroup.

The *base locus* $Z(L)$ of a subspace $L \in \mathbf{K}[X]$ is the set of all $x \in X$ for which $f(x) = 0$ for any $f \in L$. Let $L_1, \dots, L_n \in \mathbf{K}[X]$ and $Z = \bigcup_i Z(L_i)$. We recall that f is a generic element of a vector space L if f runs over $L \setminus \Sigma$, where Σ is a proper algebraic subset of L.

Definition 2.1. The *intersection index* $[L_1, \dots, L_n]$ is the number of solutions in $X \setminus Z$ of a generic system of equations $f_1 = \cdots = f_n = 0$, where $f_i \in L_i$, $1 \leq i \leq n$.

One shows that the intersection index is well defined (i.e., is independent of the choice of a generic system) [Kaveh-Khovanskii08] and symmetric with respect to permuting the subspaces L_i. Moreover, the intersection index is linear in each argument. The linearity in first argument means that

$$[L_1' L_1'', L_2, \dots, L_n] = [L_1', L_2, \dots, L_n] + [L_1'', L_2, \dots, L_n] \tag{1}$$

for any $L_1', L_1'', L_2, \dots, L_n \in \mathbf{K}[X]$. From (1) one sees that for a fixed $(n-1)$-tuple $L_2, \dots, L_n \in \mathbf{K}[X]$, the map $\pi : \mathbf{K}[X] \to \mathbb{R}$ given by $\pi(L) = [L, L_2, \dots, L_n]$ is a homomorphism from the semigroup $\mathbf{K}[X]$ to the additive group of integers. The existence of such a homomorphism shows that the intersection index induces an intersection index on $\mathrm{Gr}(\mathbf{K}[X])$; i.e., $[L_1, \dots, L_n]$ *remains invariant if we substitute each* L_i *with an analogous subspace* $\tilde{L}_i$.

One can describe the relation of analogous subspaces in a different way as follows (see [Kaveh-Khovanskii08]). A rational function $f \in \mathbb{C}(X)$ is called *integral over the subspace* L if it satisfies an equation

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0$$

with $m > 0$ and $a_i \in L^i$. The collection of all the integral functions over L is a finite-dimensional subspace $\overline{L}$ called the *completion of* L.

Proposition 2.2. (1) *Two subspaces* $L_1, L_2 \in \mathbf{K}[X]$ *are analogous if and only if* $\overline{L}_1 = \overline{L}_2$.

(2) *For any* $L \in \mathbf{K}[X]$, *the completion* $\overline{L}$ *belongs to* $\mathbf{K}[X]$ *and is analogous to* L.

(3) *Moreover, the completion* $\overline{L}$ *contains all the subspaces* $M \in \mathbf{K}[X]$ *analogous to* L.

For $L \in \mathbf{K}[X]$ define the Hilbert function H_L by $H_L(k) = \dim(\overline{L^k})$. The following theorem computes the self-intersection index of a subspace L.

Theorem 2.3 (See [Kaveh-Khovanskii09, Part II]**).** *For any* $L \in \mathbf{K}[X]$, *the limit*

$$a(L) = \lim_{k\to\infty} H_L(k)/k^n$$

exists, and the self-intersection index $[L, \ldots, L]$ *is equal to* $n!a(L)$.

The proof is based on the Hilbert theorem on the dimension and degree of a subvariety of the projective space.

3. Mixed volume and mixed integral

A function $F : \mathcal{L} \to \mathbb{R}$ on a (possibly infinite-dimensional) linear space $\mathcal{L}$ is called a homogeneous polynomial of degree k if its restriction to any finite-dimensional subspace of $\mathcal{L}$ is a homogeneous polynomial of degree k. (For any k, the zero constant function is a homogeneous polynomial of degree k.)

Definition 3.1. A symmetric multilinear function $B(v_1, \ldots, v_k)$, where $v_i \in \mathcal{L}$, defines a homogeneous polynomial P of degree k on $\mathcal{L}$ given by $P(v) = B(v, \ldots, v)$. We say that the symmetric function B is a *polarization of the homogeneous polynomial* P.

If F is a homogeneous polynomial of degree k, then its derivative $D_v F(x)$ in the direction of a vector v is linear in v and homogeneous of degree $k-1$ in x. Let $(v_1, \ldots, v_k)$ be a k-tuple of vectors. For each x, the kth derivative $D^k_{v_1,\ldots,v_k} F(x)$ is a symmetric multilinear function in the v_i and independent of x. One easily verifies the following.

Proposition 3.2. *Any homogeneous polynomial of degree* k *has a unique polarization* B *defined by the formula*

$$B(v_1, \ldots, v_k) = (1/k!) D^k_{v_1,\ldots,v_k} F.$$

A compact convex subset of $\mathbb{R}^n$ is called a *convex body*. The collection of convex bodies with Minkowski sum is a semigroup with cancelation. The multiplication by a nonnegative scalar is associative and distributive with respect to the Minkowski sum. These properties allow us to enlarge the collection of convex bodies to the (infinite-dimensional) linear space $\mathcal{L}$ of *virtual convex bodies* consisting of formal differences of convex bodies (see [Burago-Zalgaller80]). Let $d\mu$ be the standard measure in $\mathbb{R}^n$. For each convex body $\Delta \subset \mathbb{R}^n$ let $\mathrm{Vol}(\Delta) = \int_\Delta d\mu$ be its volume. The following statement is well known.

Proposition 3.3. *The function* Vol *has a unique extension to the linear space $\mathcal{L}$ of virtual convex bodies as a homogeneous polynomial of degree n.*

Definition 3.4. The *mixed volume* $V(\Delta_1, \ldots, \Delta_n)$ of the convex bodies Δ_i is the value of the polarization of the volume polynomial Vol at $(\Delta_1, \ldots, \Delta_n)$.

Fix a homogeneous polynomial F of degree p in $\mathbb{R}^n$. Let $IF(\Delta) = \int_\Delta F d\mu$ denote the integral of F on Δ. One has the following (see, for example, [Khovanskii-Pukhlikov93]).

Proposition 3.5. *The function IF has a unique extension to the linear space $\mathcal{L}$ of virtual convex bodies as a homogeneous polynomial of degree $n + p$.*

Definition 3.6. The mixed integral $IF(\Delta_1, \ldots, \Delta_{n+p})$ of a homogeneous polynomial F over the bodies $\Delta_1, \ldots, \Delta_{n+p}$ is the value of the polarization of the polynomial IF at the bodies $\Delta_1, \ldots, \Delta_{n+p}$.

From definition, the mixed integral of $F \equiv 1$ is the mixed volume.

4. Theorems of Bernstein–Kushnirenko and Kazarnovskii

We start with the Bernstein–Kushnirenko theorem (see [Kushnirenko76] and [Bernstein75]). The characters $x^k = x_1^{k_1} \cdots x_n^{k_n}$ of $(\mathbb{C}^*)^n$ are in one-to-one correspondence with the points $k = (k_1, \ldots, k_n)$ in the lattice $\mathbb{Z}^n$. A *Laurent polynomial* $f = \sum_k c_k x^k$ is a regular function in $(\mathbb{C}^*)^n$. The *support* of a Laurent polynomial f is the set of $k \in \mathbb{Z}^n$ with $c_k \neq 0$. For a nonempty set $A \subset \mathbb{Z}^n$ let L_A be the collection of all Laurent polynomials whose supports are contained in A.

Problem. Given an n-tuple of finite nonempty subsets $A_1, \ldots, A_n \subset \mathbb{Z}^n$, find the intersection index $[L_{A_1}, \ldots, L_{A_n}]$ in $(\mathbb{C}^*)^n$.

The Bernstein–Kushnirenko theorem answers this problem completely.

Definition 4.1. The *Newton polytope* $\Delta(A)$ of a nonempty finite subset $A \in \mathbb{Z}^n$ is the convex hull of A.

Theorem 4.2 (Bernstein–Kushnirenko's theorem). *The intersection index $[L_{A_1}, \ldots, L_{A_n}]$ is equal to*

$$n! V(\Delta(A_1), \ldots, \Delta(A_n)).$$

Next we discuss the Kazarnovskii theorem [Kazarnovskii87]. Let G be a complex connected m-dimensional reductive algebraic group with a maximal torus $T \cong (\mathbb{C}^*)^n$. We identify the lattice of characters of T with $\mathbb{Z}^n$ and its real span with $\mathbb{R}^n$. The Weyl group W of G (which is a finite group of reflections) acts on $\mathbb{R}^n$ and maps $\mathbb{Z}^n$ to itself. One fixes a cone C which is a fundamental domain for the action of W on $\mathbb{R}^n$ and calls it the *positive Weyl chamber*. An integral point in C is called a *dominant weight*. The main result of highest weight theory is that the finite-dimensional irreducible representations of G are in one-to-one correspondence with the dominant weights. For a dominant weight λ we denote its corresponding representation by V_λ. The point λ is called the *highest weight* of the representation V_λ.

Consider the left action of the group G on itself. The induced action on the ring of regular functions $\mathbb{C}[G]$ is given by $g \cdot f(h) = f(g^{-1}h)$, where $g \in G$ and $f \in \mathbb{C}[G]$. With a function $f \in \mathbb{C}[G]$ we associate the subspace L_f which is the smallest G-invariant subspace of $\mathbb{C}[G]$ containing f. One can show that L_f is finite dimensional (i.e., $\mathbb{C}[G]$ is a so-called rational G-module).

Definition 4.3. The *spectrum* $\mathrm{Spec}(f)$ of a function $f \in \mathbb{C}[G]$ is the set of all dominant weights λ for which V_λ appears in the decomposition of L_f into a direct sum of irreducible representations.

Definition 4.4. For $A \subset C \cap \mathbb{Z}^n$ let L_A be the G-invariant subspace of $\mathbb{C}[G]$ consisting of all f with $\mathrm{Spec}(f) \subset A$.

Problem. Given an m-tuple of finite nonempty subsets of dominant weights $A_1, \ldots, A_m \subset C \cap \mathbb{Z}^n$, find the intersection index $[L_{A_1}, \ldots, L_{A_m}]$ in G.

The Kazarnovskii theorem gives a complete answer to this problem. To state it we need an extra notation. According to the *Weyl dimension formula* the dimension of a representation V_λ is equal to $F_W(\lambda)$, where F_W is a polynomial on $\mathbb{R}^n$ of degree $(m-n)/2$ defined explicitly in terms of data associated with W. We call F_W the *Weyl polynomial of* W. We denote the homogeneous component of highest degree of F_W by ϕ_W.

Definition 4.5 (Weight polytope). Let $A \subset C \cap \mathbb{Z}^n$ be a finite nonempty set. The *weight polytope* $\Delta_W(A)$ is the convex hull of union of Weyl orbits of elements of A.

Theorem 4.6 (Kazarnovskii's theorem). *Given the finite nonempty subsets $A_1, \ldots, A_m \in C \cap \mathbb{Z}^n$, the intersection index $[L_{A_1}, \ldots, L_{A_m}]$ is equal to the mixed integral*

$$(m!/\#W) I\phi_W^2(\Delta_W(A_1), \ldots, \Delta_W(A_m)),$$

where $\#W$ is the number of elements in the Weyl group W.

Definition 4.7 (Moment polytope). Let $A \subset C \cap \mathbb{Z}^n$ be a finite nonempty set. The *moment polytope* $\Delta_W^+(A)$ is the intersection of $\Delta_W(A)$ with C.

Remark 4.8. (1) Note that contrary to the weight polytope, the moment polytope $\Delta_W^+(A)$ is not necessarily an integral polytope.

(2) The polytope $\Delta^+_W(A)$ can be identified with the moment (or Kirwan) polytope of a certain compactification of the group G as a $K\times K$ Hamiltonian space, where K is a maximal compact subgroup of G. This justifies the term "moment polytope."

Corollary 4.9 (Alternative statement of the Kazarnovskii theorem). *Given the finite nonempty subsets $A_1,\dots,A_m \in C\cap\mathbb{Z}^n$, the intersection index $[L_{A_1},\dots,L_{A_m}]$ is equal to the mixed integral*

$$m!I\phi^2_W(\Delta^+_W(A_1),\dots,\Delta^+_W(A_m)).$$

Proof. The corollary follows from the Kazarnovskii theorem (Theorem 4.6) and the invariance of polynomial ϕ^2_W under the action of W. □

Remark 4.10. (1) The Bernstein–Kushnirenko theorem is a particular case of the Kazarnovskii theorem, i.e., when $G=(\mathbb{C}^*)^n$. For $G=(\mathbb{C}^*)^n$, the Weyl group is trivial and $C=\mathbb{R}^n$. For each $\lambda\in\mathbb{Z}^n$, the representation V_λ is the one-dimensional space on which $(\mathbb{C}^*)^n$ acts by multiplication via the character λ.

(2) Any subspace L_A is G-invariant, so its base locus $Z(L_A)$ is also G-invariant, and hence either it is empty or it is the whole G. But since L_A is nonzero its base locus should be empty. Thus the Kazarnovskii theorem computes the number of solutions of a generic system of equations $f_1=\cdots=f_m=0$ where $f_i\in L_{A_i}$, in the whole group G (rather than $G\setminus Z$).

(3) For a classical group G the formula in the Kazarnovskii theorem can be rewritten in terms of the mixed volume of certain polytopes (see Section 9).

5. Semigroup of finite sets with respect to addition

The sum of nonempty sets $A,B\subset\mathbb{R}^n$ is the set $A+B=\{a+b\mid a\in A,\ b\in B\}$. The sum of nonempty convex bodies (respectively, convex integral polytopes) is again a convex body (respectively, a convex integral polytope). This is the well-known Minkowski sum of convex bodies. Consider the following:

- $\mathcal{K}$, the semigroup of all finite nonempty subsets of $\mathbb{Z}^n$.
- $\mathcal{P}$, the semigroup of all nonempty convex integral polytopes.

Proposition 5.1. *The semigroup $\mathcal{P}$ has cancelation property.*

Proposition 5.1 follows from the more general fact that the semigroup of convex bodies with respect to the Minkowski sum has cancelation property. The next statement is easy to verify.

Proposition 5.2. *The map which associates with a finite nonempty set $A\subset\mathbb{Z}^n$ its convex hull $\Delta(A)$, is a homomorphism of semigroups from $\mathcal{K}$ to $\mathcal{P}$.*

For an integral convex polytope $\Delta\in\mathcal{P}$ let $\Delta_{\mathbb{Z}}\in\mathcal{K}$ be the set of integral points in Δ, i.e., $\Delta_{\mathbb{Z}}=\Delta\cap\mathbb{Z}^n$. It is not hard to verify the following.

Proposition 5.3 (See [Khovanskii92]). *For any subset $A\subset\mathbb{Z}^n$ we have*

$$A+n\Delta(A)_{\mathbb{Z}}=(n+1)\Delta(A)_{\mathbb{Z}}=\Delta(A)_{\mathbb{Z}}+n\Delta(A)_{\mathbb{Z}}.$$

We then have the following description for the semigroup $\mathrm{Gr}(\mathcal{K})$.

Theorem 5.4. *The Grothendieck semigroup of $\mathcal{K}$ is isomorphic to $\mathcal{P}$. The homomorphism $\rho : \mathcal{K} \to \mathcal{P}$ is given by $\rho(A) = \Delta(A)$.*

Proof. From Proposition 5.2 it follows that if $A \sim B$, then $\Delta(A) \sim \Delta(B)$. Conversely, from Proposition 5.3 we know that A and $\Delta(A)_{\mathbb{Z}}$ are analogous. By definition, if $\Delta(A) = \Delta(B)$, then $\Delta(A)_{\mathbb{Z}} = \Delta(B)_{\mathbb{Z}}$. So if $\Delta(A) = \Delta(B)$, then A and B are analogous. □

6. Subspaces of matrix elements for $(\mathbb{C}^*)^n$

Among the subspaces of regular functions on $(\mathbb{C}^*)^n$, the subspaces which are invariant under the action of $(\mathbb{C}^*)^n$ are of particular interest. Each (finite-dimensional) invariant subspace is of the form L_A where $A \subset \mathbb{Z}^n$ is a nonempty finite set. The following proposition is obvious.

Proposition 6.1. (1) *For any finite nonempty set $A \subset \mathbb{Z}^n$, the dimension of the subspace L_A is equal to the number of points in A.*

(2) *For finite nonempty sets $A, B \subset \mathbb{Z}^n$ we have $L_A L_B = L_{A+B}$.*

A finite nonempty subset $A \subset \mathbb{Z}^n$ gives a finite-dimensional diagonal representation $\pi_A : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r \subset \mathrm{GL}(r, \mathbb{C})$ where $r = \#A$. The subspace L_A is in fact the space of matrix elements[1] of the representation π_A.

Definition 6.2. According to Proposition 6.1(2), the collection of subspaces L_A is a semigroup with respect to the product of subspaces. We call it the *semigroup of subspaces of matrix elements of* $(\mathbb{C}^*)^n$ and denote it by $K_{\mathrm{mat}}[(\mathbb{C}^*)^n]$.

Theorem 6.3. *The Grothendieck semigroup of $K_{\mathrm{mat}}[(\mathbb{C}^*)^n]$ is isomorphic to $\mathcal{P}$. The homomorphism $\rho : K_{\mathrm{mat}}[(\mathbb{C}^*)^n] \to \mathcal{P}$ is given by $\rho(L_A) = \Delta(A)$. The completion $\overline{L_A}$ of L_A is equal to L_B where $L_B = \Delta_{\mathbb{Z}}(A)$.*

Proof. From Proposition 6.1 we know that the semigroup $K_{\mathrm{mat}}[(\mathbb{C}^*)^n]$ is isomorphic to $\mathcal{K}$. Also by Theorem 5.4 the Grothendieck semigroup of $\mathcal{K}$ is $\mathcal{P}$. The equality $\overline{L_A} = L_B$ now follows from Proposition 2.2. □

Proof of the Bernstein–Kushnirenko theorem. First, let us prove the Kushnirenko theorem, namely *for any nonempty finite subset $A \subset \mathbb{Z}^n$, the self-intersection index $[L_A, \ldots, L_A]$ is equal to $n!\mathrm{Vol}(\Delta(A))$.* According to Theorem 6.3 we have $\overline{L_A^k} = L_{B_k}$, where $B_k = (k\Delta(A))_{\mathbb{Z}}$ is the set of integral points in $k\Delta(A)$. By Proposition 6.1, the dimension of L_{B_k} is equal to $\#B_k$, i.e., the number of integral points in the polytope $k\Delta(A)$. Put $H(k) = \dim(L_{B_k}) = \#B_k$. Note that, as $k \to \infty$, the number of integral points in $k\Delta(A)$ is asymptotically equal to the volume of the polytope $k\Delta(A)$. It follows that the limit $\lim_{k\to\infty} H(k)/k^n$ is equal to $\mathrm{Vol}(\Delta(A))$. (See

[1] Any linear combination of the matrix entries of a representation, regarded as a function on the group, is called a *matrix element* of the given representation.

[Kaveh-Khovanskii09, Part I] for a more detailed study of asymptotics of such kind.) We can now conclude the Kushnirenko theorem as a corollary of Theorem 2.3. The Bernstein–Kushnirenko theorem automatically follows from the Kushnirenko theorem and the multilinearity of intersection index and mixed volume. □

7. Semigroup of representations up to spectral equivalence

In this section we describe the Grothendieck semigroup of the semigroup of finite-dimensional representations with tensor product and up to the spectral equivalence.

We will need a generalization of Theorem 5.4. Let $\mathcal{K}_0 \subset \mathcal{K}$ be a subset in $\mathcal{K}$ equipped with some addition operation $\tilde{+}$ with respect to which $\mathcal{K}_0$ is a semigroup. Assume that $(\mathcal{K}_0, \tilde{+})$ satisfies the following properties:

(1) If $A \in \mathcal{K}_0$, then $\Delta(A)_{\mathbb{Z}} \in \mathcal{K}_0$.
(2) If $A, B \in \mathcal{K}_0$, then $A + B \subset A \tilde{+} B$.
(3) If $A, B \in \mathcal{K}_0$, then $\Delta(A \tilde{+} B) = \Delta(A + B)$.

With the semigroup $(\mathcal{K}_0, \tilde{+})$ let us associate the semigroup $\mathcal{P}_0 \subset \mathcal{P}$ whose elements are the integral polytopes of the form $\Delta(A)$ for $A \in \mathcal{K}_0$ and the addition operation in $\mathcal{P}_0$ is the Minkowski sum (by property (3) the set $\mathcal{P}_0$ is closed under the Minkowski sum).

Repeating word by word the proof of Theorem 5.4, with $(\mathcal{K}_0, \tilde{+})$ instead of $\mathcal{K}$ and $\mathcal{P}_0$ instead of $\mathcal{P}$, we obtain the following theorem.

Theorem 7.1. *The Grothendieck semigroup of* $(\mathcal{K}_0, \tilde{+})$ *is isomorphic to* $\mathcal{P}_0$. *The homomorphism* $\rho : (\mathcal{K}_0, \tilde{+}) \to \mathcal{P}_0$ *is given by* $\rho(A) = \Delta(A)$.

Now let G be a complex connected reductive algebraic group.

Definition 7.2. The *spectrum* $\mathrm{Spec}(\pi)$ of a finite-dimensional representation π of G is the set of all dominant weights λ where V_λ appears in the decomposition of π as a direct sum of irreducible representations.

We say that two finite-dimensional representations π_1, π_2 are *spectrally equivalent* if their spectrums coincide (multiplicities of the irreducible representations appearing in π_1 and π_2 can be different). Clearly the tensor product of representations respects the spectral equivalence.

Definition 7.3. We denote the semigroup of all the finite-dimensional representations of G up to the spectral equivalence, and with respect to the tensor product of representations, by $\mathcal{R}_{\mathrm{Spec}}(G)$.

Let us also introduce the following semigroups:

- $\mathcal{K}_W$, the subsemigroup of $\mathcal{K}$ consisting of finite subsets which are invariant under W;
- $\mathcal{P}_W$, the subsemigroup of $\mathcal{P}$ consisting of convex integral polytopes which are invariant under W.

Definition 7.4. For a representation π let $\mathrm{Spec}_W(\pi) \in K_W$ be the union of all Weyl orbits of elements of $\mathrm{Spec}(\pi)$, i.e., $\mathrm{Spec}_W(\pi) = \{w(\lambda) \mid \lambda \in \mathrm{Spec}(\pi),\ w \in W\}$. We call the convex hull of the set $\mathrm{Spec}_W(\pi)$ the *weight polytope* of π and denote it by $\Delta_W(\pi)$. Also the intersection of the weight polytope with the positive Weyl chamber will be called the *moment polytope* of π and will be denoted by $\Delta_W^+(\pi)$.

Consider the map $\mathrm{Spec}_W : \mathcal{R}_{\mathrm{Spec}}(G) \to \mathcal{K}_W$ which associates with a representation π the set $\mathrm{Spec}_W(\pi)$.

Proposition 7.5. (1) *The map* Spec_W *is onto; i.e., for any subset* $A \in \mathcal{K}_W$ *there is a representation* π *with* $\mathrm{Spec}_W(\pi) = A$.

(2) *For any polytope* $\Delta \in \mathcal{P}_W$ *there is a representation* π *with* $\Delta_W(\pi) = \Delta$.

Proof. (1) Let A_0 be the intersection of A with the positive Weyl chamber. Let π be the direct sum of the irreducible representations V_λ for $\lambda \in A_0$. Then $\mathrm{Spec}_W(\pi) = A$.

Part (2) follows from (1). □

The tensor product of representations induces a binary operation on the set $\mathcal{K}_W$ which we denote by $\tilde{+}$. From definition and Proposition 7.5(1) the map $\mathrm{Spec}_W : (\mathcal{R}_{\mathrm{Spec}}(G), \otimes) \to (\mathcal{K}_W, \tilde{+})$ is an isomorphism of semigroups.

For a representation π of G, let $\chi(\pi) \subset \mathbb{Z}^n$ be the set of characters of the restriction of π to the torus T. It is well known that the set $\chi(\pi)$ is invariant under the Weyl group W. From the representation theory of torus T we have the following statement.

Proposition 7.6. *For any two representations* π_1, π_2 *one has* $\chi(\pi_1 \otimes \pi_2) = \chi(\pi_1) + \chi(\pi_2)$.

And from the highest weight theory for the group G one obtains the following.

Proposition 7.7. *The convex hull of* $\chi(\pi)$ *coincides with* $\Delta_W(\pi)$.

We will need the following key fact regarding tensor product of finite-dimensional representations. It is commonly known as the PRV conjecture. It was first conjectured by K. Parthasarathy, R. Ranga Rao and V. Varadarajan in [PRV67]. Later it was proved by S. Kumar in [Kumar88].

Theorem 7.8. *For any two finite-dimensional representations* π_1, π_2 *of* G*, we have* $\mathrm{Spec}_W(\pi_1) + \mathrm{Spec}_W(\pi_2) \subseteq \mathrm{Spec}_W(\pi_1 \otimes \pi_2)$.

From Propositions 7.6 and 7.7 one readily obtains the following property of the weight polytope.

Proposition 7.9. *For any two finite-dimensional representations* π_1, π_2 *of* G*, we have* $\Delta_W(\pi_1) + \Delta_W(\pi_2) = \Delta_W(\pi_1 \otimes \pi_2)$.

The following theorem is the main result of the paper which describes the Grothendieck semigroup of $(\mathcal{R}_{\mathrm{Spec}}(G), \otimes)$.

Theorem 7.10 (Main theorem). *The Grothendieck semigroup for $\mathcal{R}_{\mathrm{Spec}}(G)$ is isomorphic to $\mathcal{P}_W$. The homomorphism $\rho : \mathcal{R}_{\mathrm{Spec}}(G) \to \mathcal{P}_W$ maps a representation π to its weight polytope $\Delta_W(\pi)$.*

Proof. It is enough to show that the semigroup $(\mathcal{K}_W, \tilde{+})$ is isomorphic to the semigroup $\mathcal{P}^W$. According to Propositions 7.5 and 7.9 and Theorem 7.8 the semigroup $(\mathcal{K}_W, \tilde{+})$ satisfies properties (1)–(3) stated before Theorem 7.1. The theorem now follows from Theorem 7.1. □

Let us give another formulation of Theorem 7.10. Denote by $\mathcal{P}_W^+$ the semigroup of all polytopes which can be represented as $\Delta \cap C$ for $\Delta \in \mathcal{P}_W$ together with the Minkowski sum. It is easy to see that the map $\pi : \mathcal{P}_W \to \mathcal{P}_W^+$ defined by $\pi(\Delta) = \Delta \cap C$ is an isomorphism of semigroups.

Theorem 7.11. *The Grothendieck semigroup of $\mathcal{R}_{\mathrm{Spec}}(G)$ is isomorphic to $\mathcal{P}_W^+$. The homomorphism $\rho : \mathcal{R}_{\mathrm{Spec}}(G) \to \mathcal{P}_W^+$ maps a representation π to the moment polytope $\Delta_W^+(\pi)$.*

8. Subspaces of matrix elements for reductive groups

The subspaces L_A appearing in the Kazarnovskii theorem (Theorem 4.6) can be realized in an alternative way: they are in fact the subspaces of matrix elements of representations of G. Let π be a finite-dimensional representation. Denote by $L_\pi \subset \mathbb{C}[G]$ the linear subspace spanned by the matrix elements of the representation π. It is invariant under the left action (as well as the right action) of G on $\mathbb{C}[G]$. We will regard it as a G-submodule of $\mathbb{C}[G]$ (for the left action). One has the following.

Proposition 8.1. $L_\pi = L_A$ *where* $A = \mathrm{Spec}(\pi)$.

Corollary 8.2. *If two representations π_1 and π_2 are spectrally equivalent then $L_{\pi_1} = L_{\pi_2}$.*

It is well known that every irreducible representation V_λ appears in the (left) regular representation of G on $\mathbb{C}[G]$ with multiplicity equal to $\dim(V_\lambda)$. From this we get the following.

Proposition 8.3. *Let $L_\pi = \sum_\lambda m_\lambda V_\lambda$ be a decomposition of L_π into irreducible representations. Put $A = \mathrm{Spec}(\pi)$. Then*

(1) *if $\lambda \in A$, we have $m_\lambda = \dim(V_\lambda) = F_W(\lambda)$;*
(2) *if $\lambda \notin A$, then $m_\lambda = 0$.*

From Proposition 8.3 and the Weyl dimension formula we obtain the following.

Corollary 8.4. *For any finite nonempty subset $A \subset C \cap \mathbb{Z}^n$, $\dim(L_A) = \sum_{\lambda \in A} F_W^2(\lambda)$.*

The following proposition is straightforward to verify.

Proposition 8.5. *For any two representations π_1, π_2, one has $L_{\pi_1 \otimes \pi_2} = L_{\pi_1} L_{\pi_2}$.*

Definition 8.6. By Proposition 8.5 the subspaces L_A form a semigroup (with respect to the product of subspaces). We call this semigroup the *semigroup of matrix elements of* G and denote it by $K_{\mathrm{mat}}[G]$.

Theorem 8.7. *The semigroup $K_{\mathrm{mat}}[G]$ is isomorphic to $\mathcal{R}_{\mathrm{Spec}}(G)$ and its Grothendieck semigroup is isomorphic to $\mathcal{P}_W^+$. The map $\rho : K_{\mathrm{mat}}[G] \to \mathcal{P}_W^+$ is given by $\rho(L_A) = \Delta_W^+(A)$. The completion of L_A is L_B where $B = \Delta_W^+(A)_{\mathbb{Z}}$.*

Proof. According to Corollary 8.2 and Propositions 8.3 and 8.5, the map $\pi \mapsto L_\pi$ is an isomorphism of semigroups $\mathcal{R}_{\mathrm{Spec}}(G)$ and $K_{\mathrm{mat}}[G]$. Then by Theorem 7.11 the Grothendieck semigroup of $K_{\mathrm{mat}}[G]$ is isomorphic to $\mathcal{P}_W^+$. The equality $\overline{L_A} = L_B$ follows from Proposition 2.2. □

We can now prove the Kazarnovskii theorem. As in the Bernstein–Kushnirenko theorem, first we prove it for the self-intersection index.

Lemma 8.8 (Analogue of the Kushnirenko theorem for a group G). *For any finite nonempty set $A \subset C \cap \mathbb{Z}^n$, the self-intersection index $[L_A, \dots, L_A]$ is equal to $m! I\phi_W^2(\Delta_W^+(A))$.*

Proof. According to Theorem 8.7 we have $\overline{L_A^k} = L_{B_k}$ where $B_k = (k\Delta_W^+(A))_{\mathbb{Z}}$. By Corollary 8.4 the dimension of L_{B_k} is equal to $\sum_{\lambda \in B_k} F_W^2(\lambda)$. Put $H(k) = \dim(L_{B_k})$. One sees that, as $k \to \infty$, the sum $\sum_{\lambda \in B_k} F_W^2(\lambda)$ asymptotically is equal to $k^m \int_{\Delta_W^+(A)} \phi_W^2 d\mu$, because the polynomial F_W^2 has degree $m - n$ and its homogeneous component of highest degree is ϕ_W^2. It follows that $\lim_{k\to\infty} H(k)/k^m$ is equal to $\int_{\Delta_W^+(A)} \phi_W^2 d\mu$. (See [Kaveh-Khovanskii09, Part I] for a more detailed study of asymptotics of such kind.) We can now conclude the lemma from Theorem 2.3. □

Proof of the Kazarnovskii theorem. The Kazarnovskii theorem follows from Lemma 8.8 and the multilinearity of the mixed integral of a homogeneous polynomial as well as the multilinearity of the intersection index. □

9. Intersection index as mixed volume

In this section we see how to rewrite the formula in the Kazarnovskii theorem as a mixed volume of certain polytopes (instead of mixed integral). To this end, we use the so-called *Gelfand-Cetlin polytopes.*

In their classical paper [Gelfand-Cetlin50], Gelfand and Cetlin constructed a natural basis for any irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ and showed how to parametrize the elements of this basis with integral points in a certain convex polytope. These polytopes are called the *Gelfand–Cetlin polytopes.* Since then similar constructions have been done for other classical groups and analogous polytopes were defined (see [Berenstein-Zelevinsky88]). We will also call them *Gelfand–Cetlin polytopes* or for short *G-C polytopes.*

Consider the list of groups $\mathbb{C}^*$, $\mathrm{SL}(n_1, \mathbb{C})$, $\mathrm{SO}(n_2, \mathbb{C})$ and $\mathrm{SP}(2n_3, \mathbb{C})$, for any $n_1, n_2, n_3 \in \mathbb{N}$. We will say that G is a *classical group* if G is in this list, or if G can be constructed from the groups in the list using the operations of taking direct product and/or taking quotient by a finite central subgroup. In this sense, the general linear group and the orthogonal group are classical groups.

Let G be a classical group. As usual let $m = \dim(G)$, and we identify the weight lattice of G with $\mathbb{Z}^n$, its real span by $\mathbb{R}^n$, and denote the positive Weil chamber by C. In summary we have the following result.

Theorem 9.1 (G-C polytopes). *For any classical group G and for any $\lambda \in C$, one can explicitly construct a polytope $\Delta_{GC}(\lambda) \subset \mathbb{R}^{(m-n)/2}$, called the Gelfand–Cetlin polytope of λ, with the following properties.*

(1) *If λ is integral, then the dimension of V_λ is equal to the number of integral points in $\Delta_{GC}(\lambda)$.*
(2) *The map $\lambda \mapsto \Delta_{GC}(\lambda)$ is linear; i.e., for any two $\lambda, \gamma \in C$ and $c_1, c_2 \geq 0$ we have $= \Delta_{GC}(c_1\lambda + c_2\gamma) = c_1\Delta_{GC}(\lambda) + c_2\Delta_{GC}(\gamma)$.*

Part (2) in the above theorem is an immediate corollary of the defining inequalities of the G-C polytopes for the classical groups.

Definition 9.2. Let A be a finite nonempty set of dominant weights of G. Define the polytope $\tilde{\Delta}(A) \subset C \times \mathbb{R}^{(m-n)}$ by

$$\tilde{\Delta}(A) = \bigcup_{\lambda \in \Delta_W^+(A)} \{(\lambda, x, y) \mid x, y \in \Delta_{GC}(\lambda)\}.$$

In other words, the projection on the first factor maps $\tilde{\Delta}(A)$ to the weight polytope $\Delta_W^+(A)$ and the fibre over each λ is the double G-C polytope $\Delta_{GC}(\lambda) \times \Delta_{GC}(\lambda)$.

Theorem 9.3 (Reformulation of the Kazarnovskii theorem). *Given finite nonempty subsets $A_1, \ldots, A_m \subset C \cap \mathbb{Z}^n$, the intersection index $[L_{A_1}, \ldots, L_{A_m}]$ is equal to the mixed volume $V(\tilde{\Delta}_W^+(A_1), \ldots, \tilde{\Delta}_W^+(A_m))$ multiplied by $m!$.*

Proof. Because the polytope $\Delta_{GC}(\lambda)$ depends linearly on λ, for a nonempty finite subset A of dominant weights the map $A \mapsto \tilde{\Delta}(\pi_A)$ is a linear map with respect to the addition of subsets. Hence to prove the theorem it is enough to verify that the self-intersection index of the subspace L_A is equal to the volume $\mathrm{Vol}(\tilde{\Delta}_W^+(A))$ multiplied by $m!$. The number of integral points in $k\Delta_{GC}(\lambda) = \Delta_{GC}(k\lambda)$, for large k, is asymptotically equal to the volume of $k\Delta_{GC}(\lambda)$. Now using Theorem 9.1(1) and the Weil dimension formula we have $\mathrm{Vol}_{(m-n)/2}(\Delta_{GC}(\lambda)) = \phi_W(\lambda)$. So the $(m-n)$-dimensional volume of $\Delta_{GC}(\lambda) \times \Delta_{GC}(\lambda)$ is equal to $\phi_W^2(\lambda)$. The theorem then follows from Lemma 8.8 and the Fubini theorem. □

Remark 9.4. (1) The construction of $\tilde{\Delta}(\pi)$ goes back to A. Okounkov who introduced such polytopes for spherical varieties in order to answer a question posed by the second author (see [Okounkov97]).

(2) The Gelfand–Cetlin approach has been generalized to any reductive group by the works of Littelmann [Littelmann98] and Bernstein and Zelevinsky [Berenstein-Zelevinsky01]. These are called the *string polytopes*. Unlike the case of Gelfand–Cetlin polytopes, in general, the dependence of a string polytope $\Delta(\lambda)$ on the dominant weight λ is not linear.

For $G = \mathrm{GL}(n, \mathbb{C})$ the construction of the polytope $\tilde{\Delta}(A)$ is especially simple and explicit.

Let $G = \mathrm{GL}(n, \mathbb{C})$. Then $m = n^2$, $T = (\mathbb{C}^*)^n$, the weight lattice is $\mathbb{Z}^n$ and the Weil group $W = S_n$ acts on $\mathbb{R}^n$ by permuting the coordinates. The (standard) positive Weil chamber C is the cone $\{\lambda = (\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \dots \geq \lambda_n\}$. We say that an $n \times n$ matrix $M = \{x_{i,j}\}$ with real entries is *row-column decreasing* if its entries satisfy the inequalities

(i) $x_{i,j} \geq x_{i,j+1}$ for $j < n$,
(ii) $x_{i,j} \geq x_{i+1,j}$ for $i < n$.

Let $\mathcal{M}(n)$ be the set all $n \times n$ real row-column decreasing matrix. Let $\delta : \mathbb{R}^{n^2} \to \mathbb{R}^n$ be the projection which sends a real $n \times n$ matrix $M = \{x_{i,j}\}$ to its diagonal $\delta(M) = (x_{1,1}, \dots, x_{n,n})$.

For the group $\mathrm{GL}(n, \mathbb{C})$ the polytope $\Delta_{GC}(\lambda) \times \Delta_{GC}(\lambda)$ is the polytope in the space of $n \times n$ real matrices, consisting of all $M \in \mathcal{M}(n)$ such that $\delta(M) = \lambda$. This follows directly from the original work of Gelfand and Cetlin [Gelfand-Cetlin50].

Definition 9.5. For a finite nonempty set A of highest weights for $\mathrm{GL}(n, \mathbb{C})$, define the *Newton polytope* $\Delta_{\mathrm{Newt}}(A)$ to be the set of all matrices $M \in \mathcal{M}(n)$ such that $\delta(M) \in \Delta^+_W(A)$.

From the defining inequalities of G-C polytopes for $\mathrm{GL}(n, \mathbb{C})$ one easily sees that polytope $\Delta_{\mathrm{Newt}}(A)$ coincides with the polytope $\tilde{\Delta}(A)$. Now the Kazarnovskii theorem for $G = \mathrm{GL}(n\mathbb{C})$ can be reformulated in terms of the mixed volumes of the above Newton polytopes.

Theorem 9.6 (Kazarnovskii theorem for $\mathrm{GL}(n, \mathbb{C})$). *For finite nonempty subsets $A_1, \dots, A_{n^2} \subset C \cap \mathbb{Z}^n$, the intersection index $[L_{A_1}, \dots, L_{A_{n^2}}]$ is equal to $(n^2)! V(\Delta_{\mathrm{Newt}}(A_1), \dots, \Delta_{\mathrm{Newt}}(A_{n^2}))$.*

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