LOGARITHMIC FUNCTIONAL AND THE WEIL RECIPROCITY LAW

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Dedicated to my old friend Sergei Abramov

This article is devoted to the (one-dimensional) logarithmic functional with argument being one-dimensional cycle in the group $(\mathbb{C}^*)^2$. This functional generalizes usual logarithm, that can be viewed as zero-dimensional logarithmic functional. Logarithmic functional inherits multiplicative property of the logarithm. It generalizes the functional introduced by Beilinson for topological proof of the Weil reciprocity law. Beilinson's proof is discussed in details in this article.
Logarithmic functional can be easily generalized for multidimensional case. It's multidimensional analog (see Ref. 5) proves multidimensional reciprocity laws of Parshin. I plan to return to this topic in upcoming publications.

1. Introduction

We start introduction with a brief discussion of classic results related to this paper: reciprocity law, its topological proof and multidimensional generalization. Further we comment on the logarithmic functional and its relation to the Beilinson functional. We end introduction with layout of material.

1.1. *The Weil reciprocity law*

Given two polynomials of degrees n and m with leading coefficients equal to one, the following equality holds: the product of values of the first polynomial over the roots of the second one is equal to the product of values of the second polynomial over the roots of the first one, multiplied by $(-1)^{mn}$. André Weil has found further generalization of this equality called the *reciprocity law*. The Weil reciprocity law applies to any pair of non-zero rational functions f, g on arbitrary irreducible algebraic curve X over an algebraically closed field K. In this paper we will consider only the case of the field K being the field of complex numbers $\mathbb{C}$. We will give exact statement of the reciprocity law for $K = \mathbb{C}$ in Section 2. Now we just give general comments.

The law is as follows. For each point $a \in X$ some non-zero element $[f, g]_a$ of the field K is defined, that is called the *Weil symbol* of functions f, g at the point a. The Weil symbol depends on functions f, g skew-symmetrically, i.e. $[f, g]_a = [g, f]_a^{-1}$. Besides the Weil symbol is multiplicative in each argument, i.e. for any triplet of non-zero rational functions f, g, φ and any point $a \in X$ the following equalities hold $[f\varphi, g]_a = [f, g]_a[\varphi, g]_a$ and $[f, \varphi g]_a = [f, g]_a[f, \varphi]_a$. Let O be the union of supports of divisors of functions f and g. Weil symbol of functions f, g can differ form 1 only at points a of the finite set O. Therefore the product of Weil symbols over all points a of curve X is well defined: it is equal to the product of symbols $[f, g]_a$ over finite set of points $a \in O$. The Weil reciprocity law states that *the product of Weil symbols $[f, g]_a$ over all point of the curve X is equal to one.*

1.2. *Topological explanation of the reciprocity law over the field* $\mathbb{C}$

In the case of $K = \mathbb{C}$ Beilinson[1] proved the Weil reciprocity law topologically and has found topological generalization of the Weil symbols. Let f, g be non-zero rational functions on connected complex algebraic curve X and let $O \subset X$ be finite set containing support of divisors of functions f and g. Consider piecewise-smooth oriented closed curve $\gamma : [0, 1] \to X \setminus O$, $\gamma(0) = \gamma(1)$ on the manifold X that does not contain points of set O. Beilnson introduced a functional that associates a pair f, g and curve γ with an element $B_\gamma(f, g)$ of the group $\mathbb{C}/\mathbb{Z}$ (i.e. some complex number defined up to an integer additive term). He showed that:

1) For fixed f, g the functional $B_\gamma(f, g)$ considered as a function of cycle γ gives an element of one-dimensional cohomology group of the manifold $X \setminus O$ with coefficients in group $\mathbb{C}/\mathbb{Z}$, i.e. for homologous to zero in $X \setminus O$ integer-valued linear combination $\sum k_i\gamma_i$ of oriented curves γ_i the equality $\sum k_i B_\gamma(f, g) = 0$ holds.

2) The functional skew-symmetrically depends on f and g, i.e. $B_\gamma(f, g) = -B_\gamma(g, f)$.

3) The functional has the following property of multiplicativity: for any three rational functions f, g, φ not having zeros and poles in the set $X \setminus O$ and for any curve $\gamma \subset X \setminus O$, equalities $B_\gamma(f\varphi, g) = B_\gamma(f, g) + B_\gamma(\varphi, g)$ and $B_\gamma(f, \varphi g) = B_\gamma(f, g) + B_\gamma(f, \varphi)$ hold.

4) The functional $B_\gamma(f,g)$ is related to Weil symbols as follows: for any point $a \in O$ and small curve $\gamma \subset X \setminus O$ "running around point a", equality $[f,g]_a = \exp 2\pi i B_\gamma(f,g)$ holds.
In accordance with 4) Weil symbols correspond to the values of Beilinson functional on special curves related to the points of the set O. The sum of those curves over all points in O is homologous to zero in $X \setminus O$, which gives the Weil reciprocity law (see Section 5).
Summarizing, Beilinson functional explains the Weil reciprocity law over field $\mathbb{C}$, and also provides an analog of Weil symbols for arbitrary cycles in $X \setminus O$, not necessary related to the points of the set O.

1.3. *Multi-dimensional reciprocity laws*

A.N. Parshin (Refs. 6,7) has found remarkable multi-dimensional generalization of the reciprocity law. It applies to an arbitrary collection $\mathbf{f} = \{f_1, \dots, f_{n+1}\}$ of $n+1$ nonzero rational functions f_j on irreducible n-dimensional manifold M over arbitrary algebraically closed field K. In multi-dimensional generalization instead of point on manifold M one considers flag $F = \{M_0 \subset M_1 \subset \cdots \subset M_n = M\}$ consisting of a chain of embedded germs M_j of algebraic manifolds of increasing dimensions, $\dim M_j = j$, locally irreducible in the neighborhood of the point M_0. Each collection of functions $\mathbf{f}$ and flag F is associated with nonzero element $[\mathbf{f}]_F$ of the field K, which is called *Parshin symbol. Reciprocity laws* state that for some chosen (precisely described) finite sets of flags L the product $\prod_{F \in L} [\mathbf{f}]_F$ is equal to 1.
Let $n = 1$ and manifold M be nonsingular algebraic curve. In this case:
1) flag $F = \{M_0 \subset M\}$ is defined by point $a = M_0$,
2) Parshin symbol $[\mathbf{f}]_F$ of the pair of functions $\mathbf{f} = f_1, f_2$ on flag F coincides with Weil symbol $[f_1, f_2]_a$,
3) the only chosen set of flags L is equal to the union of supports of the divisor of functions f_1, f_2,
4) reciprocity law for this set coincides with the Weil reciprocity law.
In the case of $K = \mathbb{C}$ Brylinski and McLaughlin[2] proved multidimensional multiplicity laws topologically and found topological generalization of Parshin symbols. Their topological construction heavily uses sheaf theory and is not intuitive.
About 10 years ago I have found[4] explicit formula for the product of roots of a system of algebraic equations with general enough set of Newton polyhedrons in the group $(\mathbb{C}^*)^n$. This formula (which is multi-dimensional gener-

alization of Vieta formula) uses Parshin symbols. Its proof however is based on simple geometry and combinatorics and does not use Parshin theory.

1.4. *The logarithmic functional*

The search for formula for the product of the roots of a system of equations convinced me that for $K = \mathbb{C}$ there should be intuitive geometric explanation of Parshin symbols and reciprocity laws. Such explanation based on multidimensional logarithmic functional was finally found in Ref. 5.

This paper is devoted to one-dimensional case. *One-dimensional logarithmic functional* (word "one-dimensional" we will omit further) associates each one-dimensional cycle γ in one-dimensional complex X and piecewise-smooth mapping $(f, g) : X \to (\mathbb{C}^*)^2$ into group $(\mathbb{C}^*)^2$ with a complex number, defined up to an integer additive term (i.e., with an element of the group $\mathbb{C}/\mathbb{Z}$). Logarithmic functional is direct generalization of usual logarithm. *Zero-dimensional logarithmic functional* (see Section 7.1) reduces to the usual logarithmic function. All theorems about logarithmic functional translate almost automatically to multidimensional case. I started presentation from one-dimensional case because of the following reasons:

1) The Weil reciprocity law is formulated much simpler than Parshin reciprocity laws. That's why the introduction of the logarithmic functional is much clearer in one-dimensional case.

2) Properties of logarithmic functional and methods of their prove become apparent enough already in one-dimensional case.

3) In one-dimensional case there already exist simple Beilinson functional giving clear topological proof of the reciprocity law. It has only one drawback: it is not transparent how to generalize it for multi-dimensional case.

One of the goals of this publication is to compare logarithmic functional and Beilinson functional. The later is defined for a pair of rational functions f, g and one-dimensional cycle γ on complex algebraic curve X. Generalizing Beilinson functional we define logarithmic functional in Beilinson form, or in short LB-*functional*, for a pair of smooth functions f, g and one-dimensional cycle on real manifold M.

LB-functional has many properties of Beilinson functional, however there are some differences too: when functions f, g are fixed, LB-functional not always gives a class of one-dimensional cohomology group of the manifold M. For this it is required that the form $df \wedge dg$ is identical to zero on the manifold M. For the rational functions f, g on complex curve X the identity $df \wedge dg \equiv 0$ holds automatically. This fact plays a key role in the proof of the reciprocity law with the help of LB-functional.

We show that logarithmic functional is always representable as LB-functional. This representation relates logarithmic functional with Beilinson functional.

1.5. *Organization of material*

In Section 2 the Weil reciprocity law is given. In section 3 LB-functional is defined for a pair of complex-valued function on a circle. In Section 4 LB-functional is defined for a mapping $(f,g) : M \to (\mathbb{C}^*)^2$ of the manifold M into group $(\mathbb{C}^*)^2$ and closed oriented curve on manifold M. Section 5 discusses Beilinson's proof of the Weil theorem. In Section 6 LB-functional is defined for the mapping $(f,g) : M \to (\mathbb{C}^*)^2$ and one-cycle on the manifold M, being an image of one-cycle γ in one-dimensional complex X of piecewise-smooth mapping $\phi : X \to M$. Section 7 defines logarithmic functional and proves its main properties. Section 8 shows that logarithmic functional can always be represented in the form of LB-functional.

2. Formulation of the Weil reciprocity law

Let Γ be a connected compact one-dimensional complex manifold (another words, Γ is irreducible regular complex algebraic curve). *Local parameter* u near point $a \in \Gamma$ is defined as an arbitrary meromorphic function u with zero of multiplicity 1 at the point a. Local parameter is a coordinate function in small neighborhood of the point a.

Let φ be a meromorphic function on the curve Γ and $\sum_{k\leq m} c_m u^m$ be a Laurent series with respect to local parameter u near point a. We will call the *leading monomial* the first nonzero term of the series, i.e. $\chi(u) = c_k u^k$. The leading monomial is defined for any meromorphic function φ not identical to zero.

For every pair of meromorphic functions f, g not identical to zero on a curve Γ, and every point $a \in \Gamma$ the Weil symbol $[f,g]_a$ is defined. It is nonzero complex number given by

$$[f,g]_a = (-1)^{nm} a_m^n b_n^{-m},$$

where $a_m u^m$ and $b_n u^n$ are leading monomials on parameter u of functions f and g at the point a. Weil symbol is defined with the help of parameter u but *it does not depend on the choice of this parameter.*

Let v be another local parameter near point a and let cv be the leading monomial of the function u on parameter v, i.e. $u = cv + \dots$. Then $a_m c^m v^m$ and $b_n c^n v^n$ are leading monomials of functions f and g on parameter v.

The equality $a_m^n b_n^{-m} = (a_m c^m)^n (b_n c^n)^{-m}$ proves the correctness of the definition of Weil symbol.
As it is seen from the definition *Weil symbol multiplicatively depends on functions f and g*. Multiplicativity on f means that if $f = f_1 f_2$ then $[f,g]_a = [f_1 g]_a [f_2 g]_a$. Multiplicativity with respect to g is defined analogously.
For every pair of meromorphic functions f, g not identical to zero on a curve Γ, the Weil symbol $[f,g]_a$ differs from one only in finite set of points a. Indeed *Weil symbol can differ from one only on the union of supports o the divisors of functions f and g.*
The Weil reciprocity law. (see ...) For every pair of meromorphic functions f, g not identical to zero on an irreducible algebraic curve Γ, the equality

$$\prod [f,g]_a = 1$$

holds. Here product is taken over all points a of the curve Γ.
Infinite product above makes sense since only finite number of terms in it are different from one.
Simple algebraic prove of the law can be found in Ref. 3. Simple topological proof based on properties of LB-functional described in Sections 3 and 4, can be found in Section 5 (this proof is reformulation of Beilinson's reasoning from Ref. 1).

3. LB-functional of the pair of complex valued functions of the segment on real variable

Let J be a segment $a \leq x \leq b$ of the real line. For any continuous on segment J function having non-zero complex values $f : J \to \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} \setminus 0$, denote by $\ln f$ any continuous branch of the logarithm of f. Function $\ln f$ is defined up to an additive term $2k\pi i$, where k is an integer. *If $f(a) = f(b)$ then we can define an integer number $\deg_J \frac{f}{|f|}$ – degree of mapping $\frac{f}{|f|} : J/\partial J \to S^1$ of the circle $J/\partial J$, obtained from segment J by identifying its ends a and b, to the unit circle $S^1 \subset \mathbb{C}$.* Obviously, $\deg_J \frac{f}{|f|} = \frac{1}{2\pi i}(\ln f(b) - \ln f(a))$.
Consider a pair of piecewise-smooth complex-valued functions f and g, having no zero values on the segment J, and $f(a) = f(b)$, $g(a) = g(b)$. For such pair of functions we will call LB-functional the complex number

$\mathrm{LB}_J(f,g)$ given by formula

$$\mathrm{LB}_J(f,g) = \frac{1}{(2\pi i)^2}\int\limits_a^b \ln f \frac{dg}{g} - \frac{1}{2\pi i}\deg_J \frac{f}{|f|}\ln g(b).$$

Lemma 3.1. *LB-functional is defined up to an integer additive term and is well defined element of the group $\mathbb{C}/\mathbb{Z}$.*

Proof. Function $\ln f$ is defined up to an additive term $2k\pi i$, $\int\limits_a^b \frac{dg}{g} = 2\pi i \deg_J \frac{dg}{|g|}$, and $\deg_J \frac{dg}{|g|}$ is an integer number. This means that the number $\frac{1}{(2\pi i)^2}\int\limits_a^b \ln f \frac{dg}{g}$ is defined up to an integer additive term. The value $\ln g(b)$ is defined up to an additive term $2m\pi i$ and $\deg_J \frac{df}{|f|}$ is an integer number. This means that the number $-\frac{1}{2\pi i}\deg_J \frac{f}{|f|}\ln g(b)$ is defined up to an integer. □

Now we give slightly more general formula for LB-functional. Let $a = x_0 < x_1 \cdots < x_n = b$ be an increasing sequence of points on the segment $[a,b]$. Given a continuous function $f : J/\partial J \to \mathbb{C}^*$ we construct a discontinuous function ϕ, which is equal to one of continuous branches $\ln_j f$ of the logarithm of f on each interval J_j defined by inequalities $x_j < x < x_{j+1}$. Let *jump* of function ϕ at point x_j be an integer number $m_\phi(x_j) = \frac{1}{2\pi i}(\lim\limits_{t\to x_j^+}\phi(x) - \lim\limits_{t\to x_j^-}\phi(x))$ for $j = 1,\ldots,n-1$, and at point x_n – an integer number $m_\phi(x_n) = \frac{1}{2\pi i}(\lim\limits_{t\to x_n^+}\phi(x) - \lim\limits_{t\to x_0^-}\phi(x))$.

Lemma 3.2. *Let for functions f, g and segment J LB-functional be defined. Then the equality*

$$LB_J(f,g) = \frac{1}{(2\pi i)^2}\int\limits_J \phi\frac{dg}{g} - \frac{1}{2\pi i}\sum_{i=1}^{n} m_\phi(x_i)\ln g(x_i)$$

holds.

Proof. First show that $\mathrm{LB}_J(f,g) =$

$$\frac{1}{(2\pi i)^2}\left(\sum_{i=0}^{n}\int\limits_{J_i}\ln_i f\frac{dg}{g} + \sum_{i=0}^{n-1}\ln_i(x_i)\ln g(x_i) - \sum_{i=1}^{n}\ln_i(x_{i+1})\ln g(x_i)\right).$$

Let's change function ϕ on interval J_j by adding to the branch of $\ln_j f$ number $2k\pi i$ leaving ϕ without changes on other intervals. As the result the value of LB-functional above is incremented by the number

$\frac{2k\pi i}{(2\pi i)^2}\left(\int\limits_{J_i}\frac{dg}{g}+\ln g(x_i)-\ln g(x_{i+1})\right)$, which is equal to zero. Since the change of the branch of the logarithm on any of the intervals J_j does not affect the result, we can assume that the function ϕ is taken as a continuous branch of the logarithm of function f on the whole segment $J=[a,b]$. In this case the formula for the LB-functional holds (it coincides with the definition of the LB-functional). The claim of lemma follows, since $2\pi i m_\phi(x_j)=\ln_j f(x_j)-\ln_{j+1} f(x_j)$ for $j=1,\ldots,n-1$, and $2\pi i m_\phi(x_n)(\phi)=\ln_n f(x_n)-\ln_0 f(x_0)$ □

If one of functions f or g is constant then it is easy to compute LB-functional. The following is obvious

Lemma 3.3. *Let for functions f, g and segment J LB-functional be defined. If $f\equiv C$ then $LB_J(f,g)=\frac{1}{2\pi i}\ln C\deg_J\frac{g}{|g|}$. If $g\equiv C$ then $LB_J(f,g)=-\frac{1}{2\pi i}\ln C\deg_J\frac{f}{|f|}$.*

The following obvious lemma shows that under change of variable LB-functional behaves as an integral of a differential form.

Lemma 3.4. *Let for functions f, g and segment J functional $LB_J(f,g)$ be defined, and let $\phi: J_1\to J$ be a piecewise-smooth homeomorphism of segment J_1 into segment J. Then functional $LB_{J_1}(f\circ\phi,g\circ\phi)$ is defined. Additionally, if ϕ preserves orientation then $LB_{J_1}(f\circ\phi,g\circ\phi)=LB_J(f,g)$, and if ϕ changes orientation then $LB_{J_1}(f\circ\phi,g\circ\phi)=-LB_J(f,g)$.*

Now we discuss how the LB-functional changes under homotopy of the pair of functions f,g. Let I be the unit segment $0\le t\le 1$ and $F: I\times J\to C^*$, $G: I\times J\to C^*$ be piecewise-smooth functions such, that $F(t,a)=F(t,b)$, $G(t,a)=G(t,b)$ for every fixed t. Let $f_0(x)=F(0,x)$, $f_1(x)=F(1,x)$, $g_0(x)=G(0,x)$, $g_1(x)=G(1,x)$.

Theorem 3.1. *The equality holds:*

$$LB_J(f_1,g_1)-LB_J(f_0,g_0)=\frac{1}{(2\pi i)^2}\int\limits_{I\times J}\frac{dF}{F}\wedge\frac{dG}{G}.$$

Proof. Differential of the form $\ln F\frac{dG}{G}$ is equal to $\frac{dF}{F}\wedge\frac{dG}{G}$. Using Stokes formula we get

$$\int\limits_{I\times J} \frac{dF}{F} \wedge \frac{dG}{G} = \int\limits_a^b \ln f_1 \frac{dg_1}{g_1} - \int\limits_a^b \ln f_0 \frac{dg_0}{g_0} - \int\limits_0^1 (\ln F(t,b) - \ln F(t,a)) \frac{dG}{G}. \tag{1}$$

Since the degree of the mapping is homotopy invariant, the difference $\ln F(t,b) - \ln F(t,a)$ does not depend on parameter t and is equal to both numbers $2\pi i \deg_J \frac{f_1}{|f_1|}$ and $2\pi i \deg_J \frac{f_0}{|f_0|}$. Therefore

$$\int\limits_0^1 (\ln F(t,b) - \ln F(t,a)) \frac{dG}{G} = 2\pi i \deg_J \frac{f_1}{|f_1|} \ln g_1(a) - 2\pi i \deg_J \frac{f_0}{|f_0|} \ln g_0(a) \tag{2}$$

and the statement of the theorem follows. □

Corollary 3.1. *Let f and g be piecewise-smooth functions on real line periodic with period $A = b - a$ with values in $\mathbb{C}^*$ and let J_c be a segment $a + c \leq x \leq b + c$ of the length A. Then $LB_{J_c}(f,g)$ does not depend on the choice of the point c.*

Proof. Consider homotopy $F(t,x) = f(t+x)$, $G(t,x) = g(t+x)$. This homotopy preserves LB-functional since $dF \wedge dG \equiv 0$. From previous theorem the proof follows. □

Lemma 3.5. *The equality $LB_J(f,g) = -LB_J(g,f)$ holds.*

Proof. Using equalities $\ln(b) = \ln(a) + 2\pi i \deg_J \frac{f}{|f|}$, $\ln g(b) = \ln g(a) + 2\pi i \deg_J \frac{g}{|g|}$ and Newton-Leibnitz formula we get:

$$\int\limits_a^b d[\ln f \ln g] = 2\pi i \deg_J \frac{f}{|f|} \ln g(a) + 2\pi i \deg_J \frac{g}{|g|} \ln f(a) + \tag{3}$$

$$(2\pi i)^2 \deg_J \frac{f}{|f|} \deg_J \frac{g}{|g|}.$$

On the other hand

$$\int\limits_a^b d[\ln f \ln g] = \int\limits_a^b \ln g \frac{df}{f} + \int\limits_a^b \ln f \frac{dg}{g}.$$

Two expressions for the integral $\int\limits_a^b d[\ln f \ln g]$ must coincide. □

Lemma 3.6. *For any piecewise-smooth function $f : J \to \mathbb{C}^*$, having equal values at the end-points a and b of the segment J, $LB_J(f, f) = \frac{1}{2} \deg_J \frac{f}{|f|}$.*

Proof. Substituting $g = f$ in (3) we obtain

$$2 \int_a^b \frac{1}{2\pi i} \ln f \frac{df}{f} = 2 \deg_J \frac{f}{|f|} \ln f(c) + 2\pi i \deg_J^2 \frac{f}{|f|}.$$

The claim follows, since $\deg_J^2 \frac{f}{|f|} \equiv \deg_J \frac{f}{|f|} \mod 2$. □

Lemma 3.7. *For any three piecewise-smooth functions f, φ, g with values in the group $\mathbb{C}^*$ on segment J with end-points a and b, such that $f(a) = f(b)$, $\varphi(a) = \varphi(b)$, $g(a) = g(b)$, the equality*

$$LB_J(f\varphi, g) = LB_J(f, g) + LB_J(\varphi, g)$$

holds.

Proof. In order to prove this it is enough to use the following facts:
1) in group $\mathbb{C}/\mathbb{Z}$ the equality $\frac{1}{2\pi i} \ln(f\varphi) = \frac{1}{2\pi i} \ln f + \frac{1}{2\pi i} \ln \varphi$ holds.
2) for any pair of continuous functions f and φ that do not have zero values and $f(a) = f(b)$, $\varphi(a) = \varphi(b)$, the equality $\deg_J \frac{f\varphi}{|f\varphi|} = \deg_J \frac{f}{|f|} + \deg_J \frac{\varphi}{|\varphi|}$ holds. □

The following lemma is obvious and we only give a formulation of it.

Lemma 3.8. *Let segment J with end-points a and b is split by a point c ($a < c < b$) into two segments: J_1 with end-points a, c and J_2 with end-points c, b. Let f, g be a pair of piecewise-smooth functions on J with values in group $\mathbb{C}^*$ such, that $f(a) = f(c) = f(b)$, $g(a) = g(c) = g(b)$. Then the functionals $LB_J(f, g)$, $LB_{J_1}(f, g)$, $LB_{J_2}(f, g)$ are defined and*

$$LB_J(f, g) = LB_{J_1}(f, g) + LB_{J_2}(f, g).$$

4. *LB*-functional of the pair of complex valued functions and one-dimensional cycle on real manifold

Let M be a smooth real manifold and $K(M)$ – multiplicative group with elements being smooth complex-valued functions on M not having values of 0. Let $\gamma : J \to M$ be piecewise-smooth closed curve on M. Element $I_J(\gamma^* f, \gamma^* g)$ of the group $\mathbb{C}/\mathbb{Z}$ will be called *LB-functional of the pair of functions $f, g \in K(M)$ and oriented closed curve γ* and will be denoted as $\mathrm{LB}_\gamma(f, g)$.

Lemma 4.1. *LB-functional of the pair of functions $f, g \in K(M)$ and oriented closed curve γ does not change under orientation preserving re-parametrization of the curve γ.*

Proof. For re-parametrization of the curve γ preserving the end-point $\gamma(a) = \gamma(b)$ the statement is proved in Lemma 3.4. Independence of the LB-functional from the choice of point $\gamma(a) = \gamma(b)$ is proved in Corollary 3.1. □

An element $\mathrm{LB}_\gamma(f, g)$ of the group $\mathbb{C}/\mathbb{Z}$, defined by formula $\mathrm{LB}_\gamma(f, g) = \sum k_i \mathrm{LB}_{\gamma_i}(f, g)$, is called *LB-functional of pair of functions $f, g \in K(M)$ and cycle $\gamma = \sum k_i \gamma_i$*, where $k_i \in \mathbb{Z}$ and γ_i parameterized closed curves on manifold M.
For any cycle γ in manifold M and any function $f \in K(M)$ denote by $\deg_\gamma \frac{f}{|f|}$ the degree of mapping $\frac{f}{|f|} : \gamma \to S^1$ of the cycle γ into unit circle S^1.

Theorem 4.1. *For any cycle γ in the manifold M the following equalities hold:*
1) $LB_\gamma(f, g) = -LB_\gamma(g, f)$ for any pair of functions $f, g \in K(M)$.
2) $LB_\gamma(f\varphi, g) = LB_\gamma(f, g) + LB_\gamma(\varphi, g)$ for any $f, \varphi, g \in K(M)$.
3) $LB_\gamma(f, f) = \frac{1}{2} \deg_\gamma \frac{f}{|f|}$ for any $f \in K(M)$.
4) $LB_\gamma(C, f) = -LB_\gamma(f, C) = \ln C \deg_\gamma \frac{f}{|f|}$ for any function $f \in K(M)$ and any non-zero constant C.

Proof. Statement 1) follows from Lemma 3.5, statement 2) follows from Lemma 3.7, statement 3) – from Lemma 3.6 and statement 4) – from Lemma 3.3. □

Theorem 4.2. *If for a pair of functions $f, g \in K(M)$ on manifold M the equality $df \wedge dg \equiv 0$ holds, then LB-functional $LB_\gamma(f, g)$ depends only on homology class of the cycle $\gamma = \sum k_i \gamma_i$.*

Proof. According to Theorem 3.1 under conditions of Theorem 4.2 LB-functional along closed curve does not change under homotopy of the curve. By homotopying if necessary every component of the cycle γ it is possible to assume that cycle consists from closed curves, passing through the fixed point c. Consider the fundamental group $\pi_1(M, c)$ of the manifold M with basis point c. For functions f and g satisfying condition of the theorem, the mapping $I : \pi_1(M, c) \to \mathbb{C}/\mathbb{Z}$ of the closed curve $\gamma \in \pi_1(M, c)$ into element

$I_\gamma(f,g)$ of group $\mathbb{C}/\mathbb{Z}$ is defined correctly and is group homomorphism. Any homomorphism of the fundamental group into Abel group is passed trough the homomorphism of the fundamental group into group of one-dimensional homologies. $\square$

5. Topological proof of the Weil reciprocity law

Beilinson's functional is a particular case of LB-functional, in which f, g is a pair of analytic functions on one-dimensional complex manifold, and γ is one-dimensional real cycle on this manifold. Under this restrictions we will call LB-functional Beilinson's functional and write $B_\gamma(f,g)$ instead of $\mathrm{LB}_\gamma(f,g)$.

Example 5.1. (Beilinson, see Ref. 1) Let M be one-dimensional complex manifold and $K_a(M) \subset K(M)$ be a subgroup of group $K(M)$ consisting of analytic functions not equal to 0 anywhere. For any two functions $f, g \in K_a(M)$ the equality $df \wedge dg \equiv 0$ holds. In accordance with Theorem 4.2, for any pair of functions $f, g \in K_a(M)$ one-dimensional co-chain on manifold M, that associates with a cycle γ element $B_\gamma(f,g) \in \mathbb{C}/\mathbb{Z}$, is an element of $H^1(M_a, \mathbb{C}/\mathbb{Z})$. According to Theorem 4.1, this class has the following properties:
1) $B_\gamma(f,g) = -B_\gamma(g,f)$ for any pair of functions $f, g \in K_a(M_a)$.
2) $B_\gamma(f\varphi, g) = B_\gamma(f,g) + B_\gamma(\varphi, g)$ for any $f, \varphi, g \in K_a(M_a)$.
3) $B_\gamma(f,f) = \frac{1}{2}\deg_\gamma \frac{f}{|f|}$ for any $f \in K_a(M_a)$.
4)$B_\gamma(C,f) = -B_\gamma(f,C) = \ln C \deg_\gamma \frac{f}{|f|}$ for any function $f \in K_a(M_a)$ and arbitrary nonzero constant C.

Let X be a connected compact one-dimensional complex manifold with a boundary, and f, g be nonzero meromorphic functions on the manifold X, that are regular on the boundary $\gamma = \partial X$ of manifold X and not equal to zero in the points of boundary ∂X. Under these assumptions the following lemma holds.

Lemma 5.1. *Let a single valued branch of the function* $\ln f$ *exists on the manifold* X*. Then* B*-functional* $B_\gamma(f,g)$ *of functions* f, g *and of boundary of the manifold* X *is equal to* $\sum_{p\in X} ord_a g \frac{1}{2\pi i}\ln f(a)$*, where* $ord_a g$ *is order of meromorphic function* g *at the point* a*. (The infinite sum appearing above is well defined since only finitely many terms in this sum are not equal to zero.)*

Proof. The statement follows from the equality

$$\frac{1}{(2\pi i)^2}\int\limits_{\partial X}\ln f\frac{dg}{g}=\frac{1}{2\pi i}\sum_{p\in X}(\mathrm{ord}_p g)\ln f(p). \qquad \square$$

Let $U\subset\mathbb{C}$ be a simply-connected domain with smooth boundary γ, containing point $0\in U$. Let f,g be meromorphic functions in the domain U, such that their restrictions on punctured domain $U\setminus 0$ are analytic functions, not taking values of 0. Let a_1z^k, a_2z^m be leading terms of Laurent series of functions f and g at the point 0: $f=a_1z^k+\ldots$, and $g=a_2z^m+\ldots$.

Lemma 5.2. *Under the listed above assumptions*

$$B_\gamma(f,g)=\frac{km}{2}+m\ln a_1-k\ln a_2.$$

Proof. Represent f,g as $f=f_1z^k$, $g=g_1z^m$, where f_1,g_1 are analytic in the domain U. Observe, that $f_1(0)=a_1\neq 0$, $g_1(0)=a_2\neq 0$. Using multiplicativity property of B-functional we get $B_\gamma(f,g)=kmB_\gamma(z,z)+mB_\gamma(f_1,z)+kB_\gamma(z,g_1)+B_\gamma(f_1,g_1)$. Using Lemma 5.1 and skew-symmetric property of B-functional we get $B_\gamma(f_1,z)=\ln a_1$, $B_\gamma(z,g_1)=-\ln a_2$ and $B_\gamma(f_1,g_1)=0$. Further $B_\gamma(z,z)=\frac{1}{2}\deg_\gamma\frac{z}{|z|}=\frac{1}{2}$. $\square$

Topological proof of the Weil theorem is actually completed. We just need to reformulate obtained results. Group $\mathbb{C}/\mathbb{Z}$ is isomorphic to the multiplicative group $\mathbb{C}^*$ of the field of the complex numbers. Required isomorphism is given by mapping $\tau:\mathbb{C}/\mathbb{Z}\to\mathbb{C}^*$ defined by formula $\tau(a)=\exp(2\pi ia)$. For any one-dimensional cycle γ in the manifold M and for any pair of functions $f,g\in K(M)$ we call *exponential B-functional* $\tilde{B}_\gamma(f,g)$ an element of the group $\mathbb{C}^*$, defined by formula $\tilde{B}_\gamma(f,g)=\exp(2\pi iB_\gamma(f,g))$.
Using the notion of exponential B-functional the last Lemma can be reformulated as follows

Lemma 5.3. *Under assumptions of Lemma 5.2, exponential B-functional $\tilde{B}_\gamma(f,g)$ coincides with the Weil symbol $[f,g]_0$ of functions f and g at point 0.*

Let Γ be a compact complex curve, f,g – meromorphic functions on Γ such that they are not identical to zero on any connected component of Γ, D – a union of supports of divisors of functions f and g. Let U_p be a small open disk containing point $p\in D$, and $\gamma_p=\partial U_p$ be the border of disk U_p, $U=\bigcup\limits_{p\in D}U_p$, $\gamma=\sum\limits_{p\in D}\gamma_p$. Consider manifold $W=M\setminus U$. Cycle γ homologically equals to zero on manifold W, since $\partial W = -\gamma$. Therefore, the equality $B_\gamma(f,g) = 0$ holds and $\exp(2\pi i B_\gamma(f,g)) = \prod_{p\in P} \exp(2\pi i B_{\gamma_p}(f,g)) = 1$. By Lemma 5.3 we have $\exp(2\pi i B_{\gamma_p}(f,g)) = [f,g]_p$ which proves the Weil theorem.

6. Generalized LB-functional

In this section we will give another (more general, but in fact equivalent) definition of LB-functional. It will be easier to generalize to multidimensional case. Any one-dimensional cycle in $(\mathbb{C}^*)^2$ (as in any other manifold) can be viewed as integer combination of oriented closed curves, i.e. as integer combination of images of oriented circle. Therefore, it is sufficient to define LB-functional for a pair of functions on a standard circle, as it was done above. When $n > 1$ n-dimensional cycle in manifold M can be viewed as an image of the mapping $F : X \to M$ of some n-dimensional cycle γ in some n-dimensional simplicial complex X. Here we give a definition of LB-functional for an image in the group $(\mathbb{C}^*)^2$ of one-dimensional cycle, laying in one-dimensional simplicial complex X under mapping $F : X \to (\mathbb{C}^*)^2$.

Let X be one-dimensional simplicial complex. Let us fix orientation on each edge of complex X. Denote by $S(\Delta_j, Q_p)$ the incidence coefficient between edge Δ_j and vertex Q_p. It is equal to zero if and only if vertex Q_p does not belong to the edge Δ_j, and it is equal to +1 or -1 depending on the sign of Q_p as a boundary of oriented edge Δ_j.

For any continuous function $f : X \to \mathbb{C}^*$ on edges of complex X it is possible to choose single-valued branches of function $\ln f$. We will need a definition of jump function for such a collection of branches of function $\ln f$ on one-dimensional cycle $\gamma = \sum k_j \Delta_j$ of complex X. Let a single-valued branch $\ln_j f : \Delta_j \to \mathbb{C}^*$ of the multi-valued function $\ln f$ be fixed on the edge Δ_j of complex X. Denote by ϕ a collection of branches $\ln f_j$. We will view collection ϕ as discontinuous function on complex X (the collection ϕ defines a function on the complement of complex X to the set of its vertices, but at vertices of complex X this function is multi-valued). Define on vertices of complex X jump function for function ϕ and cycle γ. If vertex Q_p is adjacent to the edge Δ_j then on the vertex Q_p a restriction of function $\ln_j f$ given on the edge Δ_j is defined. We define the value of the jump function $m_{\phi,\gamma}$ for function ϕ and cycle $\gamma = \sum k_j \Delta_j$ at the vertex Q_p of complex X by formula

$$m_{\phi,\gamma}(Q_p) = \frac{1}{2\pi i} \sum S(\Delta_j, Q_p) k_j \ln_j f(Q_p),$$

where summation is done over all edges Δ_j of complex X.

Lemma 6.1. *The jump function is an integer valued function on the vertices of complex X.*

Proof. On every edge Δ_j adjacent to the vertex Q_p we can choose branches $\ln_j f$ so they are equal at Q_p. With such choice of the function ϕ the number $m_{\phi,\gamma}(Q_p)$ is equal to zero, because the chain $\gamma = \sum k_j \Delta^j$ is a cycle. With the change of branches $\ln_j f$ the change in the value of jump function is integer number. □

Consider piecewise-smooth mapping $(f, g) : X \to (\mathbb{C}^*)^2$ of one-dimensional complex X and one-dimensional cycle $\gamma = \sum k_j \Delta_j$ in X. Let $\phi = \{\ln_j f\}$ be a collection of single-valued branches of function $\ln f$ on edges Δ_j of complex X. For pair of functions f, g and cycle γ we will call LB-functional the complex number $\mathrm{LB}_\gamma(f, g)$ defined by formula

$$\mathrm{LB}_\gamma(f,g) = \frac{1}{(2\pi i)^2} \int\limits_\gamma \phi \frac{dg}{g} - \frac{1}{2\pi i} \sum_{Q \in V} m_{\phi,\gamma}(Q) \ln g(Q),$$

where summation is over set V of all vertices of complex X and $m_{\phi,\gamma}$ is jump function for function ϕ and cycle γ.
The following lemmas a proved similarly to Lemma 3.1 and 3.2.

Lemma 6.2. *LB-functional $LB_\gamma(f, g)$ is defined up to an integer additive term and is correctly defined element of the group $\mathbb{C}/\mathbb{Z}$.*

Lemma 6.3. *LB-functional $LB_\gamma(f, g)$ for the mapping $(f, g) : X \to (\mathbb{C}^*)^2$ does not change under a simplicial subdivision of the complex X.*

Let discuss the change of LB-functional under homotopy of the mapping (f, g). Let I be a unit segment $0 \le t \le 1$ and $(F, G) : I \times X \to (\mathbb{C}^*)^2$ be a piecewise-smooth mapping, which coincides with the mapping $(f_0, g_0) : X \to (\mathbb{C}^*)^2$ when $t = 0$ and with the mapping $(f_1, g_1) : X \to (\mathbb{C}^*)^2$ when $t = 1$. Similarly to the Theorem 3.1 we can prove

Theorem 6.1. *The equality*

$$LB_\gamma(f_1, g_1) - LB_\gamma(f_0, g_0) = \frac{1}{(2\pi i)^2} \int\limits_{I\times\gamma} \frac{dF}{F} \wedge \frac{dG}{G}$$

holds.

7. Logarithmic function and logarithmic functional

In this section we define logarithmic functional, formulate and prove its major properties. Definitions and results of this section can be easily extended to the multi-dimensional case and we plan to return to this topic in future publications.

7.1. *Zero-dimensional logarithmic functional and logarithm*

In order to highlight an analogy between logarithm and logarithmic functional we will define zero-dimensional functional here.

Consider group $\mathbb{C}^*$ with coordinate z. Group $\mathbb{C}^*$ is homotopy-equivalent to the circle T^1, defined by the equation $|z| = 1$. Denote by $e = 1$ the unit element of the group $\mathbb{C}^*$.

Let X be finite set of points and $K(X)$ – multiplicative group with elements being complex-valued functions on X having no 0 values. Thus element f of the set $K(X)$ is a mapping $f : X \to \mathbb{C}^*$.

Let D be zero-dimensional cycle from the group $H_0(X, \mathbb{Z})$, i.e. D is a linear combination of points from the set X with integer coefficients: $D = \sum k_j x_j$, where $k_j \in \mathbb{Z}$ i $x_j \in X$.

Any mapping $f : X \to (\mathbb{C}^*)^2$ of the finite set X into group $\mathbb{C}^*$ is homotopic to the mapping of X into point e since group $\mathbb{C}^*$ is connected.

We will call *zero-dimensional logarithmic functional* a functor that associates with the pair f, D, consisting of function $f \in K(M)$ and zero-dimensional cycle $D = \sum k_j x_j$, the complex number

$$\ln(f, D) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{dz}{z},$$

where γ is arbitrary piecewise-smooth curve in group $\mathbb{C}^*$ with the boundary $\partial\gamma$ equal to $\sum k_j f(x_j) - (\sum k_j)e$.

Obviously, the number $\ln(f, D)$ is defined up to an additive integer term and is correctly defined element of the group $\mathbb{C}/\mathbb{Z}$. The following formula holds

$$\ln(f, D) = \frac{1}{2\pi i} \sum k_j \ln f(x_j),$$

where $\ln f(x_j)$ is any of values of multi-valued function $\ln f$ at point x_j.

Thus zero-dimensional logarithmic functional reduces to the usual logarithmic function. It has the following multiplicativity property: for any pair of functions $f, g \in K(X)$ and any zero-dimensional cycle D the equality $\ln(fg, D) = \ln(f, D) + \ln(g, D)$ holds. For any fixed function

$f \in K(X)$ and any pair of zero-dimensional cycles D_1, D_2 the equality $\ln(f,(D_1+D_2)) = \ln(f,D_1)+\ln(f,D_2)$ holds. Another words, for fixed function f functional $\ln(f,D)$ is zero-dimensional co-chain with the values in group $\mathbb{C}/\mathbb{Z}$.

Let M be real manifold and $f : X \to \mathbb{C}^*$ be a continuous mapping. Let's associate with every zero-dimensional cycle $D = \sum k_j x_j$ ($k_j \in \mathbb{Z}$, $x_j \in M$) on the manifold M an element $\ln(f,D)$ of the group $\mathbb{C}/\mathbb{Z}$ defined by formula $\ln(f,D) = \frac{1}{2\pi i}\sum k_j \ln f(x_j)$. It is obvious that *co-chain* $\ln(f,D)$ *defines an element of zero-dimensional cohomology group of manifold* M *with coefficients in group* $\mathbb{C}/\mathbb{Z}$ *if and only if* $df \equiv 0$ (i.e. if and only if the function f is constant on each connected component of the manifold M).

7.2. *Properties of one-dimensional logarithmic functional*

In this subsection we define one-dimensional logarithmic functional and formulate its important properties. As before we will use the term "logarithmic functional" skipping word "one-dimensional".

Consider group $(\mathbb{C}^*)^2$ with co-ordinate functions z_1 and z_2. Group $(\mathbb{C}^*)^2$ is homotopy equivalent to the torus $T^2 \subset (\mathbb{C}^*)^2$, defined by equations $|z_1| = |z_2| = 1$. On group $(\mathbb{C}^*)^2$ there is remarkable 2-form

$$\omega = \frac{1}{(2\pi i)^2}\frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2}.$$

Restriction of form ω to the torus T^2 is real 2-form

$$\omega|_{T^2} = \frac{1}{4\pi^2} d(\arg z_1)\wedge d(\arg z_2),$$

that is not equal to zero nowhere. Integral of form ω over the torus T^2, oriented by the form $d(\arg z_1)\wedge d(\arg z_2)$, is equal to 1. Define by Id a subset of group $(\mathbb{C}^*)^2$ consisting of points (z_1, z_2) such, that one of co-ordinates is equal to 1, i.e. $Id = \{z_1 = 1\}\cup\{z_2 = 1\}$.

Let X be one-dimensional simplicial complex and let $\gamma = \sum k_j \Delta_j$ be integer linear combination of its oriented edges Δ_j which constitutes a cycle, i.e. $\partial\gamma = 0$. Let $(f,g) : X \to (\mathbb{C}^*)^2$ be a piecewise-smooth mapping of complex X into the group $(\mathbb{C}^*)^2$.

Logarithmic functional is a functor, that associates with the mapping $(f,g) : X \to (\mathbb{C}^*)^2$ and one-dimensional cycle γ on X an element $\ln(f,g,\gamma)$ of the group $\mathbb{C}/\mathbb{Z}$, defined by formula

$$\ln(f,g,\gamma) = \frac{1}{(2\pi i)^2}\int_\sigma \frac{dz_1}{z_1}\wedge\frac{dz_2}{z_2} = \int_\sigma \omega,$$

where σ is 2-chain in $(\mathbb{C}^*)^2$, with the boundary $\partial\sigma$ equal to the difference of the image $\mathbf{f}_*(\gamma)$ of cycle γ under mapping $\mathbf{f} = (f, g)$ and some cycle γ_1 lying in the set Id.

We list the major properties of the logarithmic functional:

1) The value of the logarithmic functional is correctly defined element of the group $\mathbb{C}/\mathbb{Z}$.

2) Logarithmic functional skew-symmetrically depends on components f and g, i.e. $\ln(f, g, \gamma) = -\ln(g, f, \gamma)$.

3) Logarithmic functional multiplicatively depends on components f and g, i.e.

$$\ln(f\phi, g, \gamma) = \ln(f, g, \gamma) + \ln(\phi, g, \gamma), \ln(f, g\phi, \gamma) = \ln(f, g, \gamma) + \ln(f, \phi, \gamma).$$

4) Logarithmic functional has the following topological property. Let M be a real manifold and $(f, g) : M \to \mathbb{C}^*$ be a smooth mapping. One-dimensional cycle $\tilde{\gamma}$ on manifold M can be viewed as an image under some piecewise-smooth mapping $\phi : X \to M$ of some one-dimensional cycle $\gamma = \sum k_j \Delta_j$ on some one-dimensional complex X, where $k_j \in \mathbb{Z}$ and Δ_j are edges of complex X. Associate with the cycle $\tilde{\gamma}$ on manifold M an element $\ln(f \circ \phi, g \circ \phi, \gamma)$ of group $\mathbb{C}/\mathbb{Z}$. Obtained with this association one-dimensional co-chain gives a class of one-dimensional cohomology group of manifold M with coefficients in group $\mathbb{C}/\mathbb{Z}$ if and only if $df \wedge dg \equiv 0$ (i.e. if and only if the differential of the mapping $(f, g) : M \to (\mathbb{C}^*)^2$ degenerates at each point of manifold M).

5) The equality $\ln(f, g, \gamma) = \mathrm{LB}_\gamma(f, g)$ holds. One one hand, this equality gives a formula for logarithmic functional that does not use auxiliary 2-chain σ. On the other hand, this equality gives geometric sense to LB-functional and shows that LB-functional is analog of logarithm.

7.3. *Prove of properties of logarithmic functional*

This subsection will give prove of properties 1)–4). Property 5) will be proved in next section.

It is easy to see that sets Id (defined in previous subsection) and $(\mathbb{C}^*)^2$ have the following topological properties:

1) The set Id is homotopy equivalent to the boucquet of two circles, $\pi_1(Id)$ – free group with two generators and $H_1(Id, \mathbb{Z}) \approx \mathbb{Z} + \mathbb{Z}$.

2) The set $(\mathbb{C}^*)^2$ is homotopy equivalent to the real torus T^2 and $\pi_1((\mathbb{C}^*)^2) \approx H_1((\mathbb{C}^*)^2, \mathbb{Z}) \approx \mathbb{Z} + \mathbb{Z}$.

3) Enclosure $\pi : Id \to (\mathbb{C}^*)^2$ induces isomorphism of one-dimensional homologies and mapping "onto" of the fundamental groups of sets Id and $(\mathbb{C}^*)^2$.
It follows from item 3) that *any continuous mapping $f : X \to (\mathbb{C}^*)^2$ of one-dimensional simplicial complex X into group $(\mathbb{C}^*)^2$ is homotopic to the mapping of complex X into the set $Id \subset (\mathbb{C}^*)^2$*. We now prove more precise statement.
Let V be a set of vertices of one-dimensional complex X and E be the set of its edges. Fix inside each edge $\tau_j \in E$ two different points A_j and B_j. Points A_j, B_j split edge τ_j into three segments, which intersect at the ends only — *central* segment with endpoints A_j and B_j and two *boundary* segments. Fix also in each edge τ_j point O_j, laying inside central segment $[A_j, B_j]$.
Denote by X_1 the union of all central segments on all edges τ_j, and by X_0 – the union of all extreme segments on all edges τ_j. As it is seen from this construction, every connected component of the set X_1 is a central segment of one of the edges τ_j. Every connected component of the set X_0 contains one of the vertices of complex X and is the union of all extreme segments, containing this vertex. So, every connected component of the sets X_0 and X_1 is contractible.

Lemma 7.1. *Any continuous (piecewise-smooth) mapping $(f, g) : X \to (\mathbb{C}^*)^2$ of one-dimensional complex X into group $(\mathbb{C}^*)^2$ is homotopic (piecewise-smoothly homotopic) to the mapping $(f_1, g_1) : X \to (\mathbb{C}^*)^2$ such, that function f_1 is equal to 1 on the set X_1 and function g_1 is equal to 1 on the set X_0.*

Proof. First we will homotopically change function f without changing function g. Since the set $\mathbb{C}^*$ is connected we can assume that the following condition is satisfied: $f(O_j) = 1$ at the chosen point O_j on every edge τ_j. If this is not so, then we can homotopically change function f in such a way, that in the process of homotopy f_t values $f_t(O_j)$ move along the curves connecting points O_j with 1. Let this condition holds. It is possible to construct a homotopy $\phi_t : X \to X$ of the identity mapping of the complex X into itself such, that it leaves intact every vertex of the complex, translates every edge τ_j into itself and contracts every central segment $[A_j, B_j]$ into the point O_j. Homotopy $f_t = f \circ \phi_t$ translates function f into function f_1 which is equal to 1 on the set X_1. Analogously, without changing function f, we can homotopically change function g into the function g_1 having required properties. We can assume, that function g is equal to 1 at every vertex (otherwise it can be homotopically changed into the function having this property). It is possible to construct homotopy $\psi_t : X \to X$ of the identity mapping of the complex X into itself such, that it leaves intact every vertex of the complex, translates every edge τ_j into itself and contracts each of boundary segments into the vertex that is contained in this segment. Homotopy $g_t = g \circ \psi_t$ translates function g into function g_1 which is equal to 1 on the set X_0. □

On group $(\mathbb{C}^*)^2$ there is remarkable 2-form

$$\omega = \frac{1}{(2\pi i)^2} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

Lemma 7.2. *For any 2-cycle σ in $(\mathbb{C}^*)^2$ integral $\int_\sigma \omega$ is an integer number.*

Proof. Consider projection $\rho : (\mathbb{C}^*)^2 \to T^2$ of the group $(\mathbb{C}^*)^2$ into torus T^2, given by formula $\rho(z_1, z_2) = (\frac{z_1}{|z_1|}, \frac{z_2}{|z_2|})$. It is clear, that integral $\int_\sigma \omega$ is equal to the degree of the mapping $\rho : \sigma \to T^2$, which is restriction of projection ρ onto cycle σ. Thus, this integral is an integer number. □

Theorem 7.1. *The value $\ln(f, g, \gamma)$ of the logarithmic functional is correctly defined element of the group $\mathbb{C}/\mathbb{Z}$ (i.e. when 2-chain σ in the definition of functional changes, the value can be incremented by integer number only).*

Proof. Since embedding $\pi : Id \to (\mathbb{C}^*)^2$ induces isomorphism π_* of groups $H_1(Id, \mathbb{Z})$ and $H_1((\mathbb{C}^*)^2, \mathbb{Z})$, for the image $(f, g)_*\gamma$ of cycle γ there exists homological cycle γ_1 laying in the set Id, i.e. there exists 2-chain σ_1 such, that $(f, g)_*\gamma - \gamma_1 = \partial\sigma_1$. Let γ_2 be another cycle in Id that is homological to the cycle $(f, g)_*\gamma$ and σ_2 be such 2-chain, that $(f, g)_*\gamma - \gamma_2 = \partial\sigma_2$. Since isomorphism π_* has no kernel, cycles γ_1 and γ_2 are homological in the set Id, i.e. there exists a chain σ_3 laying in Id such, that $\gamma_1 - \gamma_2 = \partial\sigma_3$. By the construction the chain $\sigma = \sigma_2 - \sigma_1 - \sigma_3$ has zero boundary and therefore is a 2-cycle in $(\mathbb{C}^*)^2$. By Lemma 7.2 integral $\int_\sigma \omega$ is an integer number. On the other hand $\int_\sigma \omega = \int_{\sigma_2} \omega - \int_{\sigma_1} \omega - \int_{\sigma_3} \omega$. The chain σ_3 is laying in the set Id. Restriction of the form ω on Id is identically equal to zero, therefore $\int_{\sigma_3} \omega = 0$. Thus, the difference $\int_{\sigma_2} \omega - \int_{\sigma_1} \omega$ is an integer number. □

Theorem 7.2. *Logarithmic functional is skew-symmetrical, i.e. $\ln(f, g, \gamma) = -\ln(g, f, \gamma)$.*

Proof. Let $R : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ be a mapping that swaps co-ordinates, i.e. let $R(z_1, z_2) = (z_2, z_1)$. Under this mapping the form ω changes the sign, i.e. $R^*\omega = -\omega$. Mapping R translates the set Id into itself. Let σ be a 2-chain in $(\mathbb{C}^*)^2$ such, that the cycle $(\partial\sigma) - (f,g)_*(\gamma)$ is contained in the set Id. Then $\ln(f,g,\gamma) = \int_\sigma \omega$. Further, $\ln(g,f,\gamma) = \int_{R(\sigma)} \omega = \int_\sigma R^*\omega = -\int_\sigma \omega = -\ln(f,g,\gamma)$. $\square$

Theorem 7.3. *Logarithmic functional has the following multiplicative properties:*

$$\ln(f\varphi, g, \gamma) = \ln(f,g,\gamma) + \ln(\varphi, g, \gamma), \quad \ln(f, g\varphi, \gamma) = \ln(f,g,\gamma) + \ln(f, \varphi, \gamma).$$

Proof. Since the functional is skew-symmetrical, it is sufficient to prove the first equality $\ln(f\varphi, g, \gamma) = \ln(f,g,\gamma) + \ln(\varphi, g, \gamma)$. For evaluation of $\ln(f,g,\gamma)$ and $\ln(\varphi,g,\gamma)$ we will choose specific 2-chains σ_1 and σ_2. Let $X = X_0 \cup X_1$ be a covering of the complex X by two closed sets with contractible connected components the same as in Lemma 7.1.

We describe the choice of the chain σ_1 first. Let $I = [0,1]$ be a unit segment, $W = I \times X$ and $(F,G) : W \to (\mathbb{C}^*)^2$ be a piecewise-smooth homotopy of the mapping (f,g) discussed in Lemma 7.1. Another words, let (F,G) be such mapping, that: 1) restriction $(F,G)|_{\{1\}\times X}$ of the mapping (F,G) on the set $\{1\} \times X$ coincides with the mapping (f,g) when sets X and $\{1\} \times X$ are identified, 2) restriction $F|_{\{0\}\times X_1}$ of function F on the set $\{0\} \times X_1$ identically equals to 1, 3) restriction $G|_{\{0\}\times X_0}$ of function G on the set $\{0\} \times X_0$ identically equals to 1. For the cycle $\gamma = \sum k_i \Delta_i$ take $\sigma_1 = \sum k_i (F,G)_*(I \times \Delta_i)$. By construction the boundary of the chain σ_1 is equal to $(f,g)_*(\gamma) - \gamma_1$, where γ_1 is a cycle laying in the set Id. Thus $\ln(f,g,\gamma) = \int\limits_{\sigma_1} \omega$.

Chain σ_2 is constructed analogously. Let $(\Phi, G) : W \to (\mathbb{C}^*)^2$ be piecewise-smooth mapping, with component G the same as in above described homotopy, and component Φ having the following properties: 1) restriction $\Phi|\{1\} \times X$ coincides with function φ, 2) restriction $\Phi|_{\{0\}\times X_1}$ of function Φ on the set $\{0\} \times X_1$ identically equals to 1. For cycle $\gamma = \sum k_i \Delta_i$ take $\sigma_2 = \sum k_i (\Phi, G)_*(I \times \Delta_i)$. By construction the boundary of the chain σ_2 is equal to $(\varphi, g)_*(\gamma) - \gamma_2$, where γ_2 is a cycle laying in the set Id. Thus $\ln(\varphi, g, \gamma) = \int\limits_{\sigma_2} \omega$.

Now we construct chain σ_3. Consider the mapping $(F\Phi, G) : W \to (\mathbb{C}^*)^2$ with first component being the product of functions F and Φ, and second component equal to function G. Take $\sigma_3 = \sum k_i (F\Phi, G)_*(I \times \Delta_i)$. By construction the boundary of the chain σ_3 is equal to the difference of the cycle $(f\varphi, g)_*(\gamma)$ and the cycle laying in the set Id. Therefore, $\ln(f\varphi, g, \gamma) = \int\limits_{\sigma_3} \omega$. Thus we have the following equalities

$$\ln(f, g, \gamma) = \frac{1}{2\pi i} \sum k_i \int\limits_{I \times \Delta_i} \frac{dF}{F} \wedge \frac{dG}{G},$$

$$\ln(\varphi, g, \gamma) = \frac{1}{2\pi i} \sum k_i \int\limits_{I \times \Delta_i} \frac{d\Phi}{\Phi} \wedge \frac{dG}{G},$$

$$\ln(f\varphi, g, \gamma) = \frac{1}{2\pi i} \sum k_i \int\limits_{I \times \Delta_i} \frac{d(F\Phi)}{(F\Phi)} \wedge \frac{dG}{G}.$$

Equality $\ln(f\varphi, g, \gamma) = \ln(f, g, \gamma) + \ln(\varphi, g, \gamma)$ now follows from the identity

$$\frac{d(F\Phi)}{(F\Phi)} = \frac{dF}{F} + \frac{d\Phi}{\Phi}.$$

□

Let real manifold M and smooth mapping $(f, g) : M \to (\mathbb{C}^*)^2$ be given. With each triple, consisting of one-dimensional complex X, one-dimensional cycle γ on X and piecewise-smooth mapping $\phi : X \to M$, we will associate an element $\ln(f \circ \phi, g \circ \phi, \gamma)$ of group $\mathbb{C}/\mathbb{Z}$. In order to prove topological property 4) of the logarithmic functional described in previous subsection it is sufficient to prove the following theorem.

Theorem 7.4. *Let form $df \wedge dg$ be identically equal to zero on the manifold M and let cycle $\phi_*\gamma$ homologically equals to zero. Then* $\ln(f \circ \phi, g \circ \phi, \gamma) = 0$. *Conversely, if from homological equality to zero of the cycle $\phi_*\gamma$ follows equality* $\ln(f \circ \phi, g \circ \phi, \gamma) = 0$ *then on the manifold M form $df \wedge dg$ is identically equal to zero.*

Proof. If cycle $\phi_*\gamma$ homologically equals to zero, then on the manifold M there exists piecewise-smooth 2-chain σ such that $\partial\sigma = \phi \circ \gamma$. By definition

$$\ln(f \circ \phi, g \circ \phi, \gamma) = \frac{1}{(2\pi i)^2} \int\limits_{\sigma} \frac{df}{f} \wedge \frac{dg}{g}.$$

If form $df \wedge dg$ is identically equal to zero, then integrand is equal to zero. Thus, $\ln(f \circ \phi, g \circ \phi, \gamma) = 0$.
Conversely, let on some bi-vector v at some point a of manifold M form $df \wedge dg$ is not equal to zero. Consider arbitrary smooth mapping $\phi : (\mathbb{R}^2, 0) \to$

(M, a) of the standard plane $\mathbb{R}^2$ into manifold M that translates point 0 into point $a \in M$, and differential $d\phi$ is nonsingular at point 0 and maps plane $\mathbb{R}^2$ into subspace of the tangent space TM_a at the point a, containing bi-vector v. Let S_ε be a square $-\varepsilon \le x \le \varepsilon, -\varepsilon \le y \le \varepsilon$ on the plane $\mathbb{R}^2$, and $X_\varepsilon = \partial S_\varepsilon$ be its boundary. Let γ be a cycle geometrically coinciding with X_ε and oriented as boundary of the square S_ε. Obviously, for small enough ε element $\ln(f\circ\phi, g\circ\phi, \gamma)$ of group $\mathbb{C}/\mathbb{Z}$ is not equal to zero, however cycle $\phi_*\gamma$ by construction is homologically equal to zero. □

8. Logarithmic functional and generalized LB-functional

In this section we prove the following theorem.

Theorem 8.1. *The equality* $\ln(f, g, \gamma) = LB_\gamma(f, g)$ *holds.*

We first prove auxiliary result.

Lemma 8.1. *Let $X_0, X_1 \subset X$ be subsets of complex X defined in Lemma 7.1, $(f, g) : X \to (\mathbb{C}^*)^2$ – piecewise-smooth mapping such, that restriction of function f on X_1 and restriction of function g on X_0 identically equal to 1. Then for any one-dimensional cycle γ in complex X the equality $LB_\gamma(f, g) = 0$ holds.*

Proof. Consider the following subpartition of complex X: every edge τ_j of complex X is split into three edges by points A_j, B_j belonging to the boundaries of sets X_0 and X_1. Choose collection ϕ of the branches of function $\ln f$ on the edges of sub-partitioned complex X that satisfies the following two conditions: 1) on every edge that is connected component of set X_1, the branch of function $\ln f$ is identically equal to zero; 2) branches $\ln f$ on the union of edges, that are connected components of the set X_0, define on this component continuous function. Condition 2) can be satisfied because every connected component of the set X_0 is contractible. By the definition of LB-functional we have

$$\mathrm{LB}_\gamma(f, g) = \frac{1}{(2\pi i)^2} \int_\gamma \phi \frac{dg}{g} - \frac{1}{2\pi i} \sum_{P \in \tilde{V}} m_{\phi,\gamma}(P) \ln g(P),$$

where summation is over set $\tilde{V}$ of all vertices of sub-partitioned complex X. Now, $\int_\gamma \phi \frac{dg}{g} = 0$, because integrand is equal to zero. Indeed, on every edge from set X_1 function ϕ is equal to zero. On every edge from set X_0 function g is identically equal to 1 and thus $dg/g = 0$. At the vertex P of the original complex X (before subpartition), coefficient $m_{\phi,\gamma}$ is equal to 0, because function ϕ is continuous at such point P. At every vertex P, belonging to the boundary of sets X_0 and X_1, the number $\ln g$ is equal to $2k\pi$, where $k \in \mathbb{Z}$, because function g is equal to 1 on the set X_0. Therefore, the number $\frac{1}{2\pi i} \sum\limits_{P \in \tilde{V}} m_{\phi,\gamma}(P) \ln g(P)$ is integer. From all this it follows that element $\mathrm{LB}_\gamma(f,g)$ of group $\mathbb{C}/\mathbb{Z}$ equals to zero. □

Proof. (of Theorem 8.1). By Lemma 7.1 the mapping $(f,g) : X \to (\mathbb{C}^*)^2$ is homotopic to the mapping $(f_1,g_1) : X \to (\mathbb{C}^*)^2$, for which restriction of function f_1 on X_1 and restriction of function g_1 on X_0 are identically equal to 1. Let $W = I \times X$ and $(F,G) : W \to (\mathbb{C}^*)^2$ be homotopy connecting mappings (f,g) and (f_1,g_1). By Theorem 6.1

$$\mathrm{LB}_\gamma(f,g) - \mathrm{LB}_\gamma(f_1,g_1) = \int_\sigma \omega,$$

where σ — 2-chain on W that is equal to $I \times \gamma$. By Lemma 8.1 $\mathrm{LB}_\gamma(f_1,g_1) = 0$. By construction $\partial\sigma = \gamma - \gamma_1$, where cycle γ_1 is laying in the set Id. Therefore, $\mathrm{LB}_\gamma(f,g) = \ln_\gamma(f,g)$. □

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