NEWTONIAN POLYHEDRONS AND GROTHENDIECK RESIDUES

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Consider the system of equations $P_1 = \cdots = P_n = 0$ in $(\mathbb{C}\setminus 0)^n$, where $P_1, \dots, P_n$ are Laurent polynomials with the Newtonian polyhedrons $\Delta_1, \dots, \Delta_n$. Let us associate each Laurent polynomial Q with the n-form

$$\omega = Q/\left(P\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}\right),$$

where $z_1, \dots, z_n$ are independent variables and $P = P_1 \cdot \ldots \cdot P_n$. For general sets of the polyhedrons $\Delta_1, \dots, \Delta_n$ the sum of the Grothendieck residues of the form ω over all roots of the system of equations is evaluated. The results were reported at workshops conducted by Arnol'd and Gel'fand at the Mittag-Leffler Institute and at Maryland University in 1991–92. A detailed exposition is being prepared for publication.

1. COMBINATORIAL COEFFICIENT

Let $\Delta_1, \dots, \Delta_n$ be convex polyhedrons in $\mathbb{R}^n$ and Δ be their Minkowski sum. Each face of the polyhedron Δ is the sum of faces of Δ_i polyhedrons. A face Γ will be called locked if its terms include at least one vertex. A vertex $A \in \Delta$ will be called critical if all faces adjacent to this vertex are locked.

Consider a continuous map $F\colon \Delta \to \mathbb{R}^n$, $F = (f_1, \dots, f_n)$, such that each its components f_i is nonnegative and vanishes on those and only those faces $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ whose term Γ_i is a point-vertex of the polyhedron Δ_i. The restriction $\tilde{F}$ of the map F onto the boundary $\partial\Delta$ of the Δ polyhedron transfers a neighborhood of a critical vertex into a neighborhood of the zero point on the boundary $\partial\mathbb{R}^n_+$ of the positive octant.

The combinatorial coefficient k_A of a critical vertex $A \in \Delta$ is the local degree of the germ of the map $\tilde{F}\colon (\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$. The coefficient k_A is well-defined and depends only on the orientations of the polyhedron Δ and the positive octant $\mathbb{R}^n_+$.

The set of the polyhedrons $\Delta_1, \dots, \Delta_n$ is called developed if all faces of the sum polyhedron Δ are locked. Almost all sets of n polyhedrons in the space $\mathbb{R}^n$ are developed.

2. ORIENTATIONS

The sign of the form ω depends on the order of the independent variables $z_1, \dots, z_n$. This order also determines the orientation of the linear space $\mathbb{R}^n$ that contains me lattice of the monomials z^a and the Newton polyhedron $\Delta = \Delta_+ \cdots + \Delta_n$.

The order of the equations $P_1 = \cdots = P_n = 0$ (or their Newton polyhedrons $\Delta_1, \dots, \Delta_n$) determines the orientation of the space $\mathbb{R}^n_+$, which is involved in the definition of the combinatorial coefficient. The order of the equations also determines the sign of the Grothendieck residue in the roots of the system of equations.

Let us arbitrarily select the orders of the independent variables and the equations. These determine the signs of the form ω, the Grothendieck residue [1], and the signs of the combinatorial coefficients.

3. RESIDUE OF THE FORM IN A VERTEX OF A POLYHEDRON

For each vertex A of the Newton polyhedron $\Delta(P)$ of the Laurent polynomial P, we construct the Laurent series or the function Q/P, where Q is an arbitrary Laurent polynomial.

The monomial z^a corresponding to the vertex A of the polyhedron $\Delta(P)$ is included in P with some nonzero coefficient C_A; therefore, the free term of the Laurent polynomial $\tilde{P} = P/(C_A z^a)$ is equal to unity. Let us specify, the Laurent series for $1/\tilde{P}$ by the formula $1/\tilde{P} = 1 + (1-\tilde{P}) + (1-\tilde{P})^2 + \dots$. Each monomial z^b is included with nonzero coefficients in only a finite number of the terms $(1-\tilde{P})^k$. Therefore, the coefficient of each monomial z^b in this series is well-defined. The formal product of the series obtained and the Laurent polynomial $C_A z^a Q$ will be called the Laurent series of the rational function Q/P at the vertex A of the Newton polyhedron $\Delta(P)$.

The residue $res_A \omega$ of the rational form $\omega = \frac{Q}{P}\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ at the vertex A of the Newton polyhedron $\Delta(P)$ is the free term of the Laurent series of the function Q/P at the vertex A. The residue $\operatorname{res}_A \omega$ can explicitly be written as a polynomial of C_A^{-1} and the coefficients of the Laurent polynomials P and Q.

4. ADMISSIBLE FORMS

Let $\Delta_1, \dots, \Delta_n$ be polyhedrons in $\mathbb{R}^n$ such that the dimension of the polyhedron $\Delta = \Delta_1 + \cdots + \Delta_n$ is n. Each nonlocked face $\Gamma \subset \Delta$ of highest dimension is associated with a half-space L_Γ containing the polyhedron Δ and such that $\partial L_\Gamma \supset \Gamma$. The extended sum of the polyhedrons $\Delta_1, \dots, \Delta_n$, is a (possibly, unbounded) polyhedron $\tilde{\Delta}$ equal to the intersection of the half-spaces L_Γ over all nonlocked faces F of highest dimension of the polyhedron Δ.

Examples. 1. If all polyhedrons Δ_i coincide, then $\tilde{\Delta} = \Delta$.

2. If the polyhedrons $\Delta_1, \dots, \Delta_n$ are developed, then $\tilde{\Delta} = \mathbb{R}^n$.

Let $P_1 = \cdots = P_n = 0$ be a Δ-regular [2] system of equations in $(\mathbb{C} \setminus 0)^n$ with Newton polyhedrons $\Delta_1, \dots, \Delta_n$. The form $\omega = \frac{P}{Q}\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ is called admissible for this system if the support of the Laurent polynomial Q lies strictly inside the extended sum $\tilde{\Delta}$ of the polyhedrons $\Delta_1, \dots, \Delta_n$.

5. MAIN THEOREM

Theorem. *The sum of the Grothendieck residues of an admissible form ω over all roots in $(\mathbb{C} \setminus 0)^n$ of a Δ-regular system of equations $P_1 = \cdots = P_n = 0$ is equal to $(-1)^n \sum k_A \operatorname{res}_A \omega$, where the summation is over all critical vertices A of the polyhedron Δ.*

The proof of this theorem uses the technique of torus compacting [3]. After compacting, the main theorem reduces to a special torus version of the theorem on the equality to zero of the sum of residues on a compact manifold (cf. [4, 1]).

Corollary 1. (generalized Euler-Jacobi formula from [4]). *If the support of the Laurent polynomial Q lies strictly inside the polyhedron $\Delta(P)$, then the sum of the Grothendieck residues of the form ω is 0.*

Indeed, all numbers $\operatorname{res}_A \omega = 0$ then vanish. (All numbers $\operatorname{res}_A \omega = 0$ also vanish for Laurent polynomials Q with supports lying in a somewhat larger domain depending on Δ_i, which gives a generalization of Corollary 1.)

6. TORUS VERSION

Theorem. *If the Newton polyhedrons of the equations in the system are developed, then the sum of the Grothendieck residues can be evaluated for a form ω with any Laurent polynomial Q.*

Indeed, a system of equations with developed Newton polyhedrons is Δ_i-regular. The extended sum $\tilde{\Delta}$ of developed polyhedrons is $\mathbb{R}^n$; therefore, the form ω with an arbitrary Laurent polynomial Q is admissible.

Corollary 2. *The sum $\sum R(a)\mu(a)$ of values of an arbitrary Laurent polynomial R over roots a of a system of equations with developed Newton polyhedrons, where the roots are evaluated while taking into account their multiplicities $\mu(a)$ is $(-1)^n \sum k_A \operatorname{res}_A \omega$ where*

$$\omega = R\frac{dP_1}{P_1} \wedge \cdots \wedge \frac{dP_n}{P_n} = \left[R z_1 \ldots z_n \det\left(\frac{dP}{dz}\right) / (P_1 \ldots P_n) \right] \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

7. GEOMETRIC APPLICATION

For each vertex A of the polyhedron $\Delta = \Delta_1 + \cdots + \Delta_n$, a set of vertices $A_i \in \Delta_i$ such that $A = A_1 + \cdots + A_n$ is determined. Put $\det_A$ to be equal to the determinant of the matrix formed by the vectors $A_1, \ldots, A_n$.

Theorem. *For the mixed volume V of developed polyhedrons $\Delta_1, \ldots, \Delta_n$ with rational vertices,*

$$n!V = (-1)^n \sum k_A \det_A.$$

For polyhedrons with integer vertices, this theorem is proved by comparing the Bernstein formula for the number of the roots of the system of equations [5] with Corollary 2 for $R \equiv 1$. It would be interesting to prove the theorem geometrically and eliminate the condition that the vertices are rational.

8. ALGEBRAIC APPLICATION

Corollary 2 makes it possible to construct an explicit theory of elimination for a system of equations in $(\mathbb{C}\setminus 0)^n$ with developed Newton polyhedrons. Let us explain, for example, how an equation for the first coordinate z_1 of the roots of the system is obtained. For this purpose it is sufficient to evaluate the sums $\sum R(a)\mu(a)$, where R are polynomials equal to $1, z_1, \dots, z_1^N$, with $N = n!V - 1$, and use the Newton formulas expressing the coefficients in an equation in terms of the sums of powers of its roots.

9. APFINE VERSION

A Newton polyhedron $\Delta \subset \mathbb{R}^n_+$ will be called convenient if the set of its vertices contains one point on each positive coordinate ray and the point 0. A set of polyhedrons $\Delta_1, \dots, \Delta_n$ will be called affinely developed if all polyhedrons are convenient and each face of the polyhedron $\Delta = \Delta_1 + \dots + \Delta_n$, that does not lie in the coordinate subspace is locked.

Theorem. *All roots in $\mathbb{C}^n$ of a polynomial system $P_1 = \dots = P_n = 0$ with affinely developed Newton polyhedrons are isolated. The sum over all roots of the Grothendieck residues of the form ω corresponding to an arbitrary polynomial Q divisible by the monomial $z_1, \dots, z_n$ is equal to $(-1)^n \sum k_A \operatorname{res}_A \omega$, where the summation is over all critical vertices A of the polyhedron Δ.*

If all roots of the system lie in $(\mathbb{C} \setminus 0)^n$, then this statement follows from the main theorem. The general case reduces to the one considered by passing to the limit. A corollary to this theorem is an explicit theory of elimination for polynomial equations in $\mathbb{C}^n$ with affinely developed Newton polyhedrons (cf. Section 8). Only a very special case of this theorem (for one nontrivial vertex A and $k_A = 1$) has been known [6] before. Note that in the theorem, it is possible to eliminate conditions $0 \in \Delta_i$.

10. LOCAL VERSION

Consider a system of analytic equations $p_1 = \dots = p_n = 0$ in a neighborhood of the point $0 \in \mathbb{C}^n$. Let the Newton diagrams of $\Gamma_1, \dots, \Gamma_n$ of these equations be convenient [2]. The definitions of developed Newton diagrams and the combinatorial coefficients of the vertices of the diagram of the sum $\Gamma = \Gamma_1 + \dots + \Gamma_n$ and the residues $\operatorname{res}_A \omega$ almost literally repeat the definitions given above. The following theorem is valid.

Theorem. *For a system of analytic equations $p_1 = \dots = p_n = 0$ with developed Newton diagrams, the point 0 is an isolated solution. The Grothendieck residue at zero of the form $\omega = \dfrac{q}{p_1 \dots p_n} \dfrac{dz_1}{z_1} \wedge \dots \wedge \dfrac{dz_n}{z_n}$, where q is an arbitrary analytic function divisible by $z_l \cdot \ldots \cdot z_n$, is equal to $(-1)^{n-1} \sum k_A \operatorname{res}_A \omega$, where the summation is over all vertices A of the Newton diagram $\Gamma = \Gamma_1 + \dots + \Gamma_n$ that lie strictly inside the positive octant.*

The proof of this theorem is derived from the main theorem and the theorem from Section 9.

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