Functional Analysis and Its Applications, Vol. 35, No. 1, pp. 51–59, 2001

On the Continuability of Multivalued Analytic Functions to an Analytic Subset*

A. G. Khovanskii

UDC 517.9

In the paper it is shown that a germ of a many-valued analytic function can be continued analytically along the branching set at least until the topology of this set is changed. This result is needed to construct the many-dimensional topological version of the Galois theory. The proof heavily uses the Whitney stratification.

Introduction

In the topological version of Galois theory for functions of one variable (see [1–4]) it is proved that the character of location of the Riemann surface of a function over the complex line can prevent the representability of this function by quadratures. This not only explains why many differential equations cannot be solved by quadratures but also gives the sharpest known results on their nonsolvability. I always thought that there is no many-dimensional topological version of Galois theory of full value. The point is that, to construct such a version in the many-variable case, it would be necessary to have information not only on the continuability of germs of functions outside their branching sets but also along these sets, and it seemed that there is nowhere one can obtain this information from. Only in spring of 1999 did I suddenly understand that germs of functions are sometimes automatically continued along the branching set. Therefore, a many-dimensional topological version of the Galois theory does exist. I am going to publish it in forthcoming papers. In this paper, the property of continuability of functions along their branching sets is described, which is also of independent interest in my opinion.

Let M be an analytic manifold and let Σ be an analytic subset of M. Let $b \in M$ and let f_b be a germ of an analytic function at b that can be continued analytically along any curve $\gamma\colon [0,1] \to M$, $\gamma(0) = b$, such that γ intersect the set Σ at the initial instant only. What can be said about the continuability of the germ f_b along the curves that belong to Σ starting from some instant? This is just the subject of our investigation. In Sec. 1 we study the classical case in which it is additionally assumed that the continuations of the germ f_b define a single-valued analytic function on the set $M \setminus \Sigma$. In this case, the only obstruction to the continuability of the germ f_b Q1 is formed by the irreducible components of the set Σ that are of codimension one in M and whose closures do not contain the given point b. (See Proposition 3, which is a version of the Riemann and Hartogs theorems on the continuability of analytic functions.) The germ f_b can be continued to the complement of the union of these components, and it generally cannot be continued further. However, as is shown by the following simplest example, this result cannot be extended immediately to the case of multivalued functions.

Example. Consider the general cubic equation

$$y^3 + py + q = 0$$

with zero coefficient of y^2. In the complement of the discriminant curve Σ, this equation defines a three-valued analytic function $y(p,q)$. The discriminant curve of this equation is a semicubical parabola, that is, an irreducible curve that has a unique singular point, namely, at the origin, where

*The research was partially supported by the Russian Foundation for Basic Research under Grant No. 99-01-00245 and by the Canadian Grant No. 0GP0156833. The paper was completed during the author's stay at the Department of Algebra, Geometry, and Topology of the University of Valladolid. I am indebted to Spanish colleagues for hospitality.

Institute for System Analysis, Russian Academy of Sciences. Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 35, No. 1, pp. 62–73, January–March, 2001. Original article submitted June 5, 2000.

all three roots of the cubic equation coincide, and this is the only point of the plane p, q with this property. Over the set $\Sigma \setminus \{0\}$, exactly two roots of the equation merge. Let b be an arbitrary point of the plane belonging to the complement of the discriminant curve, let a be an arbitrary nonzero point that belongs to the discriminant curve, and let $\gamma\colon [0,1] \to \mathbb{C}^2$ be an arbitrary curve starting from the point b, ending at the point a, and intersecting Σ only at the last instant, which means that $\gamma(0) = b$, $\gamma(1) = a$, and $\gamma(t) \notin \Sigma$ for $t \neq 1$. Let us choose a germ of the function $y(p,q)$ over the point b that merges with none of the other germs under the continuation along the curve γ as the point a is approached. There is exactly one germ of this kind. Denote it by f_b. First, the germ f_b can be continued analytically along any curve not intersecting Σ. Second, this germ can be continued up to the point $a \in \Sigma$ along the curve γ. Third, the germ f_a obtained by this continuation can be continued analytically along any curve belonging to the set Σ and not passing through the origin. At the origin, there is no analytic germ of the function $y(p,q)$. In this example, the obstruction to the continuability of a germ along the curve Σ is the point 0. At this point, none of the other branches of the discriminant approaches the curve Σ, but the local topology of the curve Σ is changed (the semicubical parabola Σ has a singularity at zero and is smooth at the other points).

The above example suggests the following natural conjecture. Let B be a stratum (an analytic submanifold) belonging to the set Σ and let $a \in B$. Let f_a be a germ of an analytic function that can be continued analytically along any curve such that it can intersect the set Σ at the initial instant only. In this case, if the topology of the pair Σ, B is preserved along a curve $\gamma(t) \in B$, $\gamma(0) = a$, then the germ f_a can be continued analytically along this curve. This conjecture turns out to be true indeed. We first prove it in Sec. 3 for the functions f that are single-valued in $M \setminus \Sigma$. According to the result in Sec. 1, it suffices to show that the topology of the pair Σ, B changes if the stratum B intersects the closure of an irreducible component of Σ of codimension one. In the proof, we heavily use Whitney's results on the existence of analytic stratifications of analytic sets that are well compatible with the topology. These Whitney's results are recalled in Sec. 2. The case of a multivalued function f in $M \setminus \Sigma$ can be reduced to that of a single-valued function f by using a simple topological construction (see Sec. 6). This topological construction generalizes the classical construction of a locally trivial covering (see Sec. 4) and also heavily uses Whitney's stratification (see Sec. 5).

1. Continuability of a Single-Valued Analytic Function to an Analytic Subset

Let us represent the space $\mathbb{C}^n$ as the direct product of an $(n-1)$-dimensional space $\mathbb{C}^{n-1}$ and a complex line $\mathbb{C}^1$. We identify $\mathbb{C}^{n-1}$ with the hyperplane $z = 0$, where z is a coordinate function in $\mathbb{C}^n$.

Lemma 1. *Let a neighborhood U of the origin in the space $\mathbb{C}^n$ be the direct product $U = U_1 \times U_2$ of a connected neighborhood U_1 in the space $\mathbb{C}^{n-1}$ and a connected neighborhood U_2 in the complex line $\mathbb{C}^1$. Then any function f that is analytic in the complement of the hyperplane $z = 0$ in the neighborhood U and is bounded in a neighborhood of the origin can be continued analytically to the entire neighborhood U.*

Proof. The lemma follows from the Cauchy integral formula. Indeed, let us define a function $\tilde{f}$ on the domain U by the Cauchy integral

$$\tilde{f}(x,z) = \frac{1}{2\pi i} \int_{\gamma(x,z)} \frac{f(x,u)\,du}{u - z},$$

where x and z are points in the domains U_1 and U_2, respectively, $f(x,u)$ is the given function, Q2 and $\gamma(x,z)$ is an integration contour that belongs to complex line $\{x\} \times \mathbb{C}^1$ in the the domain U, encloses the points (x,z) and $(x,0)$, and continuously depends on (x,z). The function $\tilde{f}(x,z)$ defines the desired analytic continuation. Indeed, the function $\tilde{f}$ is analytic in the entire domain U. According to the Riemann theorem on a removable singularity, this function coincides with the given function f in a neighborhood of the origin.

Proposition 2. *Let M be an n-dimensional complex analytic manifold, let Σ be an analytic subset of M, and let $a \in \Sigma$ be a point in Σ such that every $(n-1)$-dimensional irreducible component of Σ contains the point a. In this case, any function f that is analytic in the complement $M \setminus \Sigma$ of Σ in the manifold M and is bounded in a neighborhood of the point a can be continued analytically to the entire manifold M.*

Proof. Proposition 2 reduces to Lemma 1. Indeed, denote by Σ_H a subset of Σ defined by the following condition: in a neighborhood of any point of Σ_H, the analytic set Σ is a nonsingular $(n-1)$-dimensional analytic hypersurface in the manifold M. The intersection of any irreducible $(n-1)$-dimensional component D_i of Σ with Σ_H is a connected $(n-1)$-dimensional manifold. Let us prove that the function f can be continued analytically to the set $D_i \cap \Sigma_H$.

Denote by A_i the subset of $D_i \cap \Sigma_H$ to which the function f can be continued analytically. Obviously, A_i is open in the topology of the set $D_i \cap \Sigma_H$. The set A_i is nonempty because it contains all nonsingular points of the component D_i that are sufficiently close to a by the Riemann theorem on the continuation of a holomorphic function (see [5]). Let us show that A_i is closed in the topology of $D_i \cap \Sigma_H$. Indeed, let b be a limit point of this set. By the definition of $D_i \cap \Sigma_H$, one can choose a local system of coordinates in the vicinity of b in M in such a way that the set $D_i \cap \Sigma_H$ coincides with a coordinate hyperplane in the corresponding neighborhood. The desired property now follows from Lemma 1. Further, since $D_i \cap \Sigma_H$ is connected, it follows that the set A_i coincides with $D_i \cap \Sigma_H$, i.e., the function f can be continued analytically to the entire set A_i. Therefore, f can be continued to the entire set $\Sigma_H = \bigcup(D_i \cap \Sigma_H)$. However, the codimension of the set $\Sigma \setminus \Sigma_H$ in the manifold M is not less than two. By the Hartogs theorem (see [5]), this proves Proposition 2.

Proposition 3. *Let f be an analytic function in the complement of an analytic set Σ in an n-dimensional analytic manifold M. If f is bounded in a neighborhood of a point $a \in \Sigma$, then f can be continued analytically to the set $M \setminus D_a$, where D_a is the union of all $(n-1)$-dimensional irreducible components of Σ that do not contain the point a.*

Proof. Proposition 3 follows from Proposition 2 when applied to the manifold $M \setminus D_a$, to the analytic subset $\Sigma \setminus D_a$, and to the function f.

2. Admissible Stratifications

Let Σ be a proper analytic subset in a complex-analytic manifold M. By a *stratification* of the set Σ we mean its partition into disjoint submanifolds, the so-called *strata* (which in general have different dimensions), having the following properties:

1) any stratum Σ_i is a connected analytic manifold;

2) for any stratum Σ_i, its closure $\overline{\Sigma_i}$ is an analytic subset of M and its boundary $\overline{\Sigma_i} \setminus \Sigma_i$ can be represented as a union of some strata of lesser dimensions.

We say that a pair consisting of an analytic manifold M and its analytic subset Σ has *constant topology along a stratum* $B \subset \Sigma$ if the following two conditions hold.

Condition 1. For any point $a \in B$ and any analytic submanifold L of M that is transversal to the stratum B at the point a, there is a small neighborhood V_a of a in L such that the topology of the pair V_a, F_a, where $F_a = V_a \cap \Sigma$, depends neither on the choice of the point a nor on that of the section L and is defined only by the stratum B and by the subset Σ.

Condition 2. The stratum B has a neighborhood U in M and a projection $\pi\colon U \to B$ such that the restriction of π to $B \subset U$ is the identity mapping and the pair $\pi^{-1}(a)$, $\pi^{-1}(a) \cap \Sigma$ is homeomorphic to the pair V_a, F_a for any point $a \in B$. Moreover, any point $a \in B$ has a neighborhood K_a in B such that the pair $\pi^{-1}(K_a)$, $\pi^{-1}(K_a) \cap \Sigma$ is homeomorphic to the pair $V_a \times K_a$, $F_a \times K_a$, and the homeomorphism between these two pairs takes the projection π to the projection of the direct product $V_a \times K_a$ to the factor K_a, and the restriction of this homeomorphism to the set $K_a \subset \pi^{-1}(K_a)$ is the identity mapping (more exactly, this restriction sends any point $b \in K_a$ to the point $a \times b$ of the direct product $V_a \times K_a$).

We say that a stratification of an analytic set $\Sigma \subset M$ is *admissible* if the pair M, Σ has constant topology along any stratum Σ_i of this stratification.

As was discovered by Whitney, admissible stratifications exist for any complex-analytic set in any complex-analytic manifold (see [6]). We use this result below.

3. Modification of the Topology of an Analytic Set in the Vicinity of an Irreducible Component

According to Lemma 4 below, the number of connected components of a real topological submanifold of M that belongs to an analytic hypersurface Σ and differs from this hypersurface by a set of small dimension is equal to the number of irreducible $(n-1)$-dimensional components of the hypersurface Σ.

Lemma 4. *Let a subset T of an $(n-1)$-dimensional analytic set Σ belonging to an n-dimensional analytic manifold M have the following properties.*

1. *The set T is a real topological submanifold of M of codimension two, i.e., any point $a \in T$ has a neighborhood U_a in M such that the set $U_a \cap T$ is a topological submanifold in the domain U_a of real dimension $2n-2$.*

2. *The set $\Sigma \setminus T$ is a closed subset of Σ of real codimension $\geqslant 2$ (i.e., $\Sigma \setminus T$ is a union of finitely many real topological submanifolds of M of dimension $\leqslant 2n-4$).*

Then any $(n-1)$-dimensional irreducible component of Σ intersects exactly one connected component of the topological manifold T. Moreover, any connected component of T is dense in the corresponding irreducible $(n-1)$-dimensional component of the analytic set Σ.

Proof. Lemma 4 is a consequence of the following facts: a) a set of codimension two cannot separate a topological manifold, b) if all singular points are deleted from an irreducible component of an analytic set, then the remaining manifold is connected.

Let us first show that any connected component T^0 of the set T intersects exactly one irreducible component of the set Σ. Indeed, the set $\Sigma \setminus \Sigma_H$ is of real dimension $\leqslant 2n-4$; therefore, this set cannot separate the connected $(2n-2)$-dimensional real manifold T^0 into parts. Thus, the complement of the intersection of the sets T^0 and $\Sigma \setminus \Sigma_H$ in T^0 is covered by exactly one set $D_i \cap \Sigma_H$. Since the set $D_i \setminus \Sigma_H$ is dense in the component D_i and the set D_i is closed, it follows that T^0 is entirely contained in the irreducible component D_i of the set Σ. Suppose that a point a of T^0 belongs to another $(n-1)$-dimensional component D_j, $D_j \neq D_i$, of Σ. However, by assumption, the set T and hence its component T^0 are open in the topology of Σ. Since the set $D_j \cap \Sigma_H$ is dense in D_j, it follows that T^0 contains some points of the set $D_j \cap \Sigma_H$, which is impossible. A contradiction. This proves the desired assertion.

Let us now show that different connected components of the manifold T cannot belong to the same $(n-1)$-dimensional irreducible component of the set Σ. Indeed, if all singular points and all points not belonging to the manifold T are deleted from an irreducible $(n-1)$-dimensional component, then a connected manifold is obtained. Hence, it is covered by exactly one connected component of the manifold T. This proves the lemma.

Proposition 5. *Let a pair consisting of an n-dimensional analytic manifold and an analytic subset Σ of this manifold have constant topology along a connected stratum $B \subset \Sigma$ (see Sec. 2). Then any $(n-1)$-dimensional irreducible component of Σ is either disjoint with the stratum B or entirely contains it.*

Proof. We first consider the local case, i.e., assume that the manifold B coincides with the set K_a and the manifold M coincides with the neighborhood $\pi^{-1}(K_a)$ (the set and the neighborhood are those mentioned in Condition 2 of Sec. 2). Let us show that, in this case, the closure of every irreducible $(n-1)$-dimensional component of Σ coincides with the set K_a.

Let us use the notation of Sec. 2, and let $F_a^0 \subset F_a$ be the set consisting of the points of the analytic set F_a with the following property: each of these points has a neighborhood in which the set F_a is an analytic hypersurface in the manifold V_a. The set F_a^0 decomposes into the connected components $F_a^{0,i}$. The complement $F_a \setminus F_a^0$ has lesser complex dimension than the set F_a.

The homeomorphism mentioned in Condition 2 takes the set F_a^0 to the set Σ. It follows from Lemma 4 that this homeomorphism takes the sets $F_0^{0,i} \times K_a$ to diverse irreducible $(n-1)$-dimensional components of the set Σ. Moreover, the image of each of the sets $F_0^{0,i} \times K_a$ is dense in the corresponding irreducible component of the set Σ, and, for a given $(n-1)$-dimensional component of the set Σ, there is a set of the form $F_a^{0,i} \times K_a$ mapped into this component.

Furthermore, for each connected component $F_a^{0,i}$, the point a is a limit point (the components for which this is not the case are disjoint with a sufficiently small neighborhood of a and do not belong to the set F_a^0). Therefore, the closure of each of the sets $F_0^{0,i} \times K_a$ contains K_a. Hence, any irreducible $(n-1)$-dimensional component of the set Σ contains K_a. (The homeomorphism mentioned in Condition 2 of Sec. 2 is the identity on the base K_a.)

The local case is thus investigated. We now assume that the manifold M belongs to a small neighborhood of the stratum B. Namely, let the manifold M coincide with a neighborhood U of the stratum B mentioned in Condition 2. In this case, the stratum B is covered by domains K_{a_j}. The above argument can be applied to each of these domains. Therefore, if an irreducible $(n-1)$-dimensional component D_i of the set Σ intersects the set $\pi^{-1}(K_{a_j})$, then the closure of D_i contains the entire neighborhood K_{a_j}. Thus, the set of limit points of the component D_i that belong to the stratum B is open in the topology of B. However, this set is obviously closed in the topology of the stratum B. Therefore, since the stratum B is connected, it must be contained in the closure of D_i.

Let us now proceed with the general case. If an irreducible $(n-1)$-dimensional component of the set Σ is disjoint with the neighborhood U of the stratum B mentioned in Condition 2, then B contains no limit points of this component. However, if this component intersects the domain U, then the above argument can be used, which proves that the closure of the component contains the entire stratum B.

The proposition is proved.

Theorem 6. *Let a pair consisting of an n-dimensional manifold M and an analytic subset Σ of M have constant topology along a connected stratum $B \subset \Sigma$. Then any function f that is analytic in the complement $M \setminus \Sigma$ of Σ in M and is bounded in a neighborhood of a point $a \in B$ can be continued analytically to a neighborhood of B.*

Proof. Any $(n-1)$-dimensional irreducible component D_i of Σ that does not contain the point a is disjoint with the stratum B (see Proposition 5). Therefore, the union D_a of the irreducible $(n-1)$-dimensional components of Σ that do not contain the point a is a closed set disjoint with the stratum B. The theorem now follows from Proposition 3.

4. Coverings over the Complement in the Entire Manifold of a Subset of Hausdorff Codimension Higher than One

In the topological version of Galois theory, the role of fields is played by Riemann surfaces, and the role of the Galois groups by the monodromy groups of the surfaces. In this case, one needs to assume that the Riemann surfaces have reasonable topological properties. Riemann surfaces that are locally trivial coverings do have these properties. However, the class of locally trivial coverings is too narrow and insufficient for our purposes. In this section we describe a class of *covering manifolds over* $M \setminus \Sigma$, where M is a manifold in which a subset Σ is distinguished that is small, in a sense. In the one-dimensional topological version of Galois theory [1–4], the functions are discussed whose Riemann surfaces are coverings over the complex line in which a countable (and possibly dense) set Σ is distinguished. In the present paper, the key role is played by covering manifolds over a complex manifold M in which an analytic subset Σ is distinguished (see Sec. 5).

Let M, Σ be a pair consisting of a connected real manifold M and a subset $\Sigma \subset M$ such that the complement $M \setminus \Sigma$ is locally arcwise connected and dense in M. For an example of such a subset Σ one can take any subset of M whose Hausdorff codimension is strictly greater than one. Let us distinguish a point b in the complement of Σ.

Definition 7. A connected manifold R together with a distinguished point c and a projection $\pi\colon R \to M$ is called a *covering manifold over* $M\setminus\Sigma$ with distinguished point b if, first, the mapping π is a local homeomorphism, second, π takes the distinguished point c to the distinguished point b, $\pi(c) = b$, and, third, for any curve γ in $M \setminus \Sigma$ starting from b, $\gamma\colon [0,1] \to M \setminus \Sigma$, $\gamma(0) = b$, there is a lifting $\tilde\gamma\colon [0,1] \to R$, $\pi \circ \tilde\gamma = \gamma$, of this curve starting from c, $\tilde\gamma(0) = c$.

For the convenience of our exposition, we assume that the manifold M is endowed with a Riemannian metric.

Definition 8. A subgroup Γ of the fundamental group $\pi_1(M \setminus \Sigma, b)$ is said to be *open* if, for any curve $\gamma\colon [0,1] \to M \setminus \Sigma$, $\gamma(0) = \gamma(1) = b$, belonging to Γ, there is a number $\varepsilon > 0$ such that any curve $\tilde\gamma\colon [0,1] \to M \setminus \Sigma$, $\tilde\gamma(0) = \tilde\gamma(1) = b$, for which the distance between the points $\gamma(t)$ and $\tilde\gamma(t)$ does not exceed ε for any t, $0 \leqslant t \leqslant 1$, also belongs to Γ, $\tilde\gamma \in \Gamma$.

To any covering manifold $\pi\colon (R,c) \to (M,b)$ over the set $M \setminus \Sigma$ we assign a subgroup of the fundamental group of the set $(M \setminus \Sigma, b)$. A curve $\gamma\colon [0,1] \to M \setminus \Sigma$, $\gamma(0) = \gamma(1) = b$, is said to be *admissible* for the covering manifold (R,c) if the lifting $\tilde\gamma\colon [0,1] \to R$ of this curve $(\pi \circ \tilde\gamma = \gamma)$ starting from the point c, $\tilde\gamma(0) = c$, is a closed curve, i.e., if $\tilde\gamma(1) = c$. It is clear that all admissible curves for the covering manifold form a subgroup of the fundamental group $\pi_1(M \setminus \Sigma, b)$. We say that this subgroup *corresponds* to the covering manifold (R,c).

Definition 9. A covering manifold $\pi\colon (R,c) \to (M,b)$ over a set $M \setminus \Sigma$ is said to be *maximal* if it cannot be embedded in any larger covering manifold. In other words, the existence of another covering manifold $\pi^1\colon (R_1,c_1) \to (M,b)$ over the set $M\setminus\Sigma$ and of an embedding $i\colon (R,c) \to (R_1,c_1)$ commuting with the projections $\pi = \pi^1 \circ i$ implies that the embedding i is a homeomorphism.

Theorem 10. 1. *If a subgroup Γ of the fundamental group of the set $M\setminus\Sigma$ with distinguished point b corresponds to a covering manifold $\pi\colon (R,c) \to (M,b)$ over $M \setminus \Sigma$ with distinguished point c, where $\pi(c) = b$, then the subgroup Γ is open in $\pi_1(M \setminus \Sigma, b)$.*

2. *For any open subgroup Γ of the group $\pi_1(M \setminus \Sigma, b)$, there is a unique maximal covering manifold $\tilde\pi(\Gamma)\colon (\widetilde R(\Gamma), c) \to (M,b)$ over the set $M \setminus \Sigma$ to which the subgroup Γ corresponds.*

3. *In the manifold $\widetilde R(\Gamma)$, an arbitrary open set U, that contains the preimage of the set $M \setminus \Sigma$ under the mapping $\tilde\pi(\Gamma)$ and is endowed with the restriction of the projection $\tilde\pi(\Gamma)\colon U \to M$ is a covering manifold over $M \setminus \Sigma$ corresponding to the group $\Gamma \subset \pi_1(M \setminus \Sigma, b)$. Conversely, every covering manifold over $M \setminus \Sigma$ corresponding to the subgroup Γ can be obtained in this way.*

Let us outline the proof of the theorem. We first prove assertion 1. Let a curve γ in the set $M \setminus \Sigma$ be lifted to R as a closed curve starting and ending at the point c. The mapping $\pi\colon R \to M$ is a local homeomorphism. Therefore, all closed curves $\tilde\gamma$ that are sufficiently close to the curve γ and belonging to the manifold M can also be lifted to R as closed curves starting and ending at the point c. (This holds even if a close curve $\tilde\gamma$ intersects the set Σ.) Therefore, the subgroup Γ corresponding to the covering manifold over the set $M \setminus \Sigma$ is an open subgroup of $\pi_1(M \setminus \Sigma, b)$.

To prove assertion 2, one must first construct a maximal covering manifold $\tilde\pi(\Gamma)\colon (\widetilde R(\Gamma), c) \to (M,b)$ over $M \setminus \Sigma$ corresponding to an open subgroup Γ in the group $\pi_1(M \setminus \Sigma, b)$.

Definition 11. Let Γ be an open subgroup of $\pi_1(M \setminus \Sigma, b)$. A closed curve γ on M starting and ending at the point b, $\gamma\colon [0,1] \to M$, $\gamma(0) = \gamma(1) = b$, is said to be Γ-*admissible* if it has the following property. There is a number $\varepsilon > 0$ such that any closed curve in the set $M \setminus \Sigma$ starting and ending at the point b, $\tilde\gamma\colon [0,1] \to M \setminus \Sigma$, $\tilde\gamma(0) = \tilde\gamma(1) = b$, belongs to the group Γ if the distance between the points $\gamma(t)$ and $\tilde\gamma(t)$ does not exceed ε for any t, $0 \leqslant t \leqslant 1$.

Associated with any curve $\gamma\colon [0,1] \to M$ (in general nonclosed) such that $\gamma(0) = b$ is a closed *twice passed* curve that is the composition of the curves γ and γ^{-1}.

Definition 12. We say that a nonclosed curve $\gamma\colon [0,1] \to M$ is Γ-*nice* if

1) the curve γ starts from the distinguished point, $\gamma(0) = b$;

2) the twice passed curve $\gamma\gamma^{-1}$ is Γ-admissible.

Denote by $\Pi(\Gamma, b)$ the set of all Γ-nice curves and introduce the following equivalence relation on the set $\Pi(\Gamma, b)$. Two Γ-nice curves γ_1 and γ_2 are said to be Γ-*equivalent* if

1) the curves γ_1 and γ_2 have the same right endpoints, $\gamma_1(1) = \gamma_2(1)$;

2) the composition γ of the curves γ_1 and γ_2^{-1} is Γ-admissible.

Let us now describe the set $\widetilde{R}(\Gamma)$ and the mapping $\tilde{\pi}(\Gamma)\colon \widetilde{R}(\Gamma) \to M$ at the set-theoretic level. The set $\widetilde{R}(\Gamma)$ is the quotient of the set $\Pi(\Gamma, b)$ of all Γ-nice curves by the above equivalence relation. The mapping $\tilde{\pi}(\Gamma)$ assigns to any curve $\gamma \in \pi(\Gamma, b)$ its right endpoint $\gamma(1)$. The distinguished point $\tilde{c}$ of the set $\widetilde{R}(\Gamma)$ is the equivalence class of the constant curve $\gamma(t) \equiv b$.

Let us now introduce a topology in the set $\widetilde{R}(\Gamma)$ as follows: $\widetilde{R}(\Gamma)$ is endowed with the weakest topology for which the mapping $\tilde{\pi}(\Gamma)\colon \widetilde{R}(\Gamma) \to M$ is continuous.

We can readily see that the resulting manifold $\widetilde{R}(\Gamma)$, together with the distinguished point $\tilde{c}$ and the projection $\tilde{\pi}(\Gamma)$, is actually a covering manifold over $M \setminus \Sigma$ corresponding to the subgroup Γ.

Let us prove that $\widetilde{R}(\Gamma)$ is an extension of any other covering manifold $\pi\colon (R, c) \to (M, b)$ over $M \setminus \Sigma$ corresponding to the subgroup Γ. Let $\gamma\colon [0, 1] \to R$ be an arbitrary curve on R starting from the point c. Obviously, the projection $\pi \circ \gamma$ of this curve is Γ-nice in the manifold M.

To any point d of the manifold R we assign the set $\Pi(c, d, R)$ of all curves $\gamma\colon [0, 1] \to R$ on R starting from the point c and ending at the point d, $\gamma(0) = c$, $\gamma(1) = d$. It is clear that the projections $\pi \circ \gamma$ of all curves γ in the set $\Pi(c, d, R)$ are $\tilde{\Gamma}$-equivalent curves. Therefore, the mapping that assigns to any point d of the manifold R the set of projections $\pi \circ \gamma$ of all curves γ in $\Pi(c, d, R)$ is an embedding of the manifold R in the above manifold $\widetilde{R}(\Gamma)$.

The verification of the other assertions of the theorem is quite easy, and we do not dwell on it.

5. Covering over the Complement of an Analytic Subset in a Manifold

Proposition 13. *Let M be an analytic manifold, let Σ be an analytic subset of M, and let $b \in M \setminus \Sigma$ be a distinguished point. Let us choose a subgroup Γ of the fundamental group $\pi_1(M \setminus \Sigma, b)$. Assume that $\gamma_1\colon [0, 1] \to M$, $\gamma_1(0) = b$, is a Γ-nice curve* (*see Definition* 12) *such that it belongs to $M \setminus \Sigma$ for $0 \leqslant t < 1$ and its right end $a = \gamma_1(1)$ belongs to Σ. Consider an arbitrary admissible stratification of the set Σ* (*see* Sec. 2). *Let B be the stratum of this stratification that contains the point a and let $\gamma_2\colon [0, 1] \to B$ be an arbitrary curve in B starting from the point a, $\gamma_2(0) = a$. Then the composition of the curves γ_1 and γ_2 is a Γ-nice curve.*

Proof. Let U be a sufficiently small neighborhood of the stratum B and let $\pi\colon U \to B$ be the projections mentioned in Condition 2 (see Sec. 2). Denote by $\pi(\Gamma)\colon (R(\Gamma), c) \to (M \setminus \Sigma, b)$ a Q3
locally trivial covering corresponding to the subgroup $\Gamma \subset \pi_1(M \setminus \Sigma, b)$. By the definition of the covering, the curve $\gamma_1\colon [0, t_1] \to M \setminus \Sigma$, where t_1 is any number strictly less than 1, can be lifted to the manifold $R(\Gamma)$ in such a way that the lifting of this curve starts from the distinguished point $c \in R(\Gamma)$. Let us choose a value of the parameter t_1 which is so close to 1 that the point $b_1 = \gamma(t_1)$ belongs to the set U. Denote by c_1 the point on the lifting that corresponds to the parameter t_1, $\pi(c_1) = b_1$. Let R_1 be a connected component of the preimage of the set U under the mapping $\pi(\Gamma)\colon R(\Gamma) \to M \setminus \Sigma$. The restriction ρ of the mapping $\pi(\Gamma)$ to the manifold R_1 defines a locally trivial covering $\rho\colon (R_1, c_1) \to (U \setminus \Sigma, b_1)$. We denote by Γ_1 the subgroup of $\pi_1(U \setminus \Sigma, b_1)$ corresponding to this covering.

Lemma 14. *The group Γ_1 contains the kernel of the homomorphism of the fundamental group of the space $U \setminus \Sigma$ into the fundamental group of the stratum B induced by the projection $\pi\colon U \to B$.*

Proof. The restriction of the mapping $\pi\colon U \to B$ to the domain $U \setminus \Sigma$ is a locally trivial fibration (see Condition 2 in Sec. 2). Denote by a_1 the image of the point b_1 under the projection π and by $V \setminus F$ the fiber of the fibration over the point a_1. It follows from the segment

$$\cdots \to \pi_1(V \setminus F, b_1) \to \pi_1(U \setminus \Sigma, b_1) \to \pi_1(B, a_1) \to \cdots$$

of the exact homotopy sequence of this fibration that the kernel of the homomorphism under consideration coincides with the image of the fundamental group $\pi_1(V \setminus F, b_1)$ of the above fiber. Therefore, we must show that the group Γ_1 contains the image of the fundamental group of this fiber. Let $\overline{\gamma}\colon [0, 1] \to V \setminus F$, where $\overline{\gamma}(0) = \overline{\gamma}(1) = b_1$, be an arbitrary closed loop in the fiber.

Let us show that $\overline{\gamma} \in \Gamma_1$. To this end, we must prove that the composition of the curves $\tilde{\gamma}_1$, $\overline{\gamma}$, and $\tilde{\gamma}_1^{-1}$, where $\tilde{\gamma}_1$ is the restriction of the curve γ_1 to the interval $[0, t_1]$, belongs to the group $\Gamma \subset \pi_1(M \setminus \Sigma, b)$. However, the composition of these curves can be regarded as a small perturbation of the twice passed curve γ_1. By assumption, the curve γ_1 is Γ-nice, which precisely means that if a small perturbation of the twice passed curve is disjoint with the set Σ, then it belongs to the group Γ. This proves the lemma.

We come back to the proof of Proposition 13. Let γ be the composition of the curves γ_1 and γ_2 mentioned in the statement of the proposition. We must show that the curve γ is Γ-nice, i.e., any small deformation of the twice passed curve γ belongs to the group Γ if it is disjoint with Σ. First, let us prove this fact for special small deformations disjoint with the set Σ that have the following form. The curve $\overline{\gamma}$ must be the composition of curves $\overline{\gamma}_1$, $\overline{\gamma}_2$, and $\overline{\gamma}_3$ that are small deformations of the curves γ_1, $\gamma_2\gamma_2^{-1}$, and γ_1^{-1}, respectively, and the curve $\overline{\gamma}_2$ must be closed. Certainly, we assume that the relations $\overline{\gamma}_1(1) = \overline{\gamma}_2(0) = \overline{\gamma}_2(1) = \overline{\gamma}_3(0)$ hold because otherwise the composition is not defined. Since $\overline{\gamma}_2$ is a closed curve that is close to the twice passed curve γ_2, it follows that the projection of this curve to the stratum B defines the trivial element of the fundamental group of the base. We consider the lifting of the curve $\overline{\gamma}_1$ to the space of the fibration $R(\Gamma)$; let this lifting start from a point c. Denote by c_1 the right end of the lifting. By Lemma 14, the lifting of $\overline{\gamma}_2$ to $R(\Gamma)$ that starts from the point c_1 ends at the same point c_1. Further, the lifting of $\overline{\gamma}_3$ to $R(\Gamma)$ that starts from the point c_1 must end at the point c. Indeed, the composition of the curves $\overline{\gamma}_1$ and $\overline{\gamma}_3$ is a small deformation of the twice passed curve γ_1. The curve γ_1 is Γ-nice. Therefore, the lifting of the composition of $\overline{\gamma}_1$ and $\overline{\gamma}_3$ to $R(\Gamma)$ that starts from the point c must end at the same point.

Thus, if a lifting of the composition of the curves $\overline{\gamma}_1$, $\overline{\gamma}_2$, and $\overline{\gamma}_3$ to $R(\Gamma)$ starts from a point c, then it ends at the same point c. In other words, the composition of these curves belongs to the group Γ. We thus proved the desired assertion for a specially perturbed curve that is the twice passed composition of γ_1 and γ_2. Obviously, any small perturbation of this curve, which belongs to the domain $M \setminus \Sigma$, is homotopic in this domain to some special perturbation of this curve.

(The twice passed composition of the curves γ_1 and γ_2 is the composition of the curves γ_1, $\gamma_2\gamma_2^{-1}$, and γ_1^{-1}. A perturbation of this composition is the composition of three curves l_1, l_2, and l_3, where the curve l_2 is close to the curve $\gamma_2\gamma_2^{-1}$ but is not necessarily closed. Obviously, this composition is homotopy equivalent to the composition of close curves $\tilde{l}_1$, $\tilde{l}_2$, and $\tilde{l}_3$ such that the curve $\tilde{l}_2$ is closed.)

This proves Proposition 13.

6. Main Theorem

We are now ready to state and prove the main theorem.

Theorem 15 (on the analytic continuation of a function along an analytic set). *Let M be a complex analytic manifold, let Σ be an analytic subset of M, and let f_b be a germ of an analytic function at a point $b \in M$. Assume that the germ f_b can be continued analytically along any curve $\gamma\colon [0,1] \to M$, $\gamma(0) = b$, disjoint with Σ for $t > 0$. Suppose that the germ f_b can be continued analytically along some curve $\gamma_1\colon [0,1] \to M$, $\gamma_1(0) = b$, whose right end a, $a = \gamma_1(1)$, belongs to the set Σ, $a \in \Sigma$. Consider an arbitrary admissible stratification of Σ (see* Sec. 2*). Let B be the stratum of this stratification whose closure contains the point a, and let $\gamma_2\colon [0,1] \to M$ be an arbitrary curve starting from a, $\gamma_2(0) = a$, such that $\gamma_2(t) \in B$ for $t > 0$. Then the germ f_b can be continued analytically along the composition of the curves γ_1 and γ_2.*

Proof. Suppose that the germ f_b of the analytic function can be continued along a curve γ. Consider an arbitrary point $\tilde{b}$ belonging to the domain of convergence of the Taylor series of the
Q4 germ f_b. The germ at $\tilde{b}$ of the sum of this Taylor series can be continued along any curve that is sufficiently close to γ outside the domain of convergence of the Taylor series. Therefore, *without loss of generality, in the statement of the main theorem we can assume that the point b belongs to the set $M \setminus \Sigma$, the point a belongs to the stratum B, and the curve $\gamma_1\colon [0,1] \to M$, where $\gamma_1(0) = b$*

and $\gamma_1(1) = a$*, is disjoint with the set* Σ *for* $0 \leqslant t < 1$. Let us prove the main theorem under these very assumptions.

In the fundamental group of the domain $M \setminus \Sigma$ with the distinguished point b we introduce the subgroup Γ that consists of the loops in $M \setminus \Sigma$ which start and end at the distinguished point and for which the continuation of the germ f_b along these loops results in the same germ. Consider the maximal covering manifold $\tilde{\pi}(\Gamma)\colon (\widetilde{R}(\Gamma), c) \to (M, b)$ over $M \setminus \Sigma$ corresponding to this subgroup Γ (see Definition 9). The manifold $\widetilde{R}(\Gamma)$ is endowed with the natural structure of a complex-analytic manifold. This structure is inherited from the analytic structure on M under the mapping $\tilde{\pi}(\Gamma)$ (this mapping is a local homeomorphism). The set $\widetilde{\Sigma} = \tilde{\pi}^{-1}(\Gamma)(\Sigma)$ is an analytic subset of the manifold $\widetilde{R}(\Gamma)$. By assumption, $\tilde{f}_c = \pi^* f_b$ regarded as a germ of an analytic function at the point c on the analytic manifold $\widetilde{R}(\Gamma)$ can be continued analytically to the entire manifold $\widetilde{R}(\Gamma) \setminus \widetilde{\Sigma}$, and the germ $\tilde{f}_c$ defines there a single-valued analytic function $\tilde{f}$. Any curve $\gamma\colon [0,1] \to M$, $\gamma(0) = b$, along which the germ f_b can be continued analytically is a Γ-nice curve. Indeed, the analytic continuations of the germ f_b along the twice passed curve γ and along any closed curve $\tilde{\gamma}\colon [0,1] \to M$, $\tilde{\gamma}(0) = \tilde{\gamma}(1) = b$, that is close to $\gamma\gamma^{-1}$ obviously lead to the same original germ f_b.

In particular, the curve $\gamma_1\colon [0,1] \to M$, where $\gamma_1(0) = b$ and $\gamma_1(1) = a \in \Sigma$, which is mentioned in the main theorem as a curve along which the germ f_b is continued, is Γ-nice. Therefore, there is a lifting of γ_1 to $\widetilde{R}(\Gamma)$ that starts from the point c. Denote by $\tilde{a}$ the right end of the lifting. By Proposition 13, for any curve γ_2 starting from the point a and belonging to the stratum B, the composition of the curves γ_1 and γ_2 is a Γ-nice curve. Hence, there is a lifting of this composition to $\widetilde{R}(\Gamma)$ starting from the point c. In other words, this means that any curve belonging to the stratum B and starting from the point a can be lifted to $\widetilde{R}(\Gamma)$ as a curve starting from the point $\tilde{a}$. Let $\widetilde{B}$, $\widetilde{B} \ni \tilde{a}$, be the connected component of the preimage of the stratum B with respect to the projection $\tilde{\pi}(\Gamma)$. We have proved that the restriction of the mapping $\tilde{\pi}(\Gamma)$ to $\widetilde{B}$ defines a locally trivial covering over the stratum B. It is clear that the pair consisting of the manifold $\widetilde{R}(\Gamma)$ and the set $\widetilde{\Sigma}$, where $\widetilde{\Sigma}$ is the preimage of the set Σ under the projection $\tilde{\pi}(\Gamma)$, has constant topology along the stratum $\widetilde{B}$. Indeed, the triple $\widetilde{R}(\Gamma)$, $\widetilde{\Sigma}$, $\widetilde{B}$ is locally homeomorphic to the triple M, Σ, B, and, by assumption, the topology of the pair M, Σ is constant along the stratum B.

We can now apply Theorem 6 to the germ $\tilde{f}_c = \pi^* f_b$ of a single-valued analytic function on $\widetilde{R}(\Gamma) \setminus \widetilde{\Sigma}$ that can be continued to a neighborhood of the point $\tilde{a} \in \widetilde{B}$. This completes the proof of the main theorem.

References

1. A. G. Khovanskii, "The representability of functions by quadratures," Usp. Mat. Nauk, **26**, No. 4, 251–252 (1971).
2. A. G. Khovanskii, "Riemannian surfaces of functions that are representable by quadratures," in: Abstracts of the VI All-Union Topological Conference, Tbilisi, 1972, p. 125.
3. A. G. Khovanskii, On the representability of functions by quadratures, Candidate thesis, V. A. Steklov Mathematics Institute, Russian Academy of Sciences, 1973.
4. A. G. Khovanskii, "Topological obstructions to the representability of functions by quadratures," J. Dynam. Control Systems, **1**, No. 1, 91–123 (1995).
5. B. A. Fuks, Introduction to the Theory of Analytic Functions of Several Complex Variables [in Russian], Fizmatlit, Moscow, 1962.
6. M. Goresky and R. MacPherson, Stratified Morse Theory, Springer-Verlag, Berlin-New York, 1988.

Translated by A. I. Shtern

Edited by V. M. Volosov