Toric Geometry and Grothendieck Residues[1]

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Abstract

We consider a system of n algebraic equations $P_1 = \cdots = P_n = 0$ in the space $(\mathbb{C}/0)^n$. It is assumed that the Newton polytopes of the equations are in a sufficiently general position with respect to one another. Let ω be any rational n-form which is regular on $(\mathbb{C} \setminus 0)^n$ outside the hypersurface $P_1 \dots P_n = 0$. Formerly we have announced an explicit formula for the sum of the Grothendieck residues of the form ω at all roots of the system of equations. In the present paper this formula is proved.

Key words and phrases: Grothendieck residues, Newton polytopes, toric varieties.

Consider a system of equations

$$P_1 = \cdots = P_n = 0$$

in $(\mathbb{C} \setminus 0)^n$, where $P_1, \dots, P_n$ are Laurent polynomials with Newton polytopes $\Delta_1, \dots, \Delta_n$. To each Laurent polynomial Q let us assign the n-form

$$\omega = \left(\frac{Q}{P}\right) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

where $z_1, \dots, z_n$ are the independent variables and $P = P_1 \dots P_n$. This paper contains the proof of the formula announced in [**1**] for the sum of the Grothendieck residues of the form ω at all roots of the system of equations, under the assumption that the Newton polytopes $\Delta_1, \dots, \Delta_n$ are in a sufficiently general position with respect to one another. The main result is stated in 1.5.

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1. Statements of the results and applications

1.1. The Grothendieck residue. Recall the definition of the Grothendieck residue. Let a point z be an isolated solution of a system of analytic equations $P_1 = \cdots = P_n = 0$ on an n-dimensional complex analytic variety M^n. Let us define the Grothendieck cycle γ_z in n-dimensional homology of the complement $(U \setminus \Gamma)$ of a small neighborhood U of the point z to the hypersurface Γ defined by the equation $P = P_1 \dots P_n = 0$. For almost all points $\epsilon \in \mathbb{R}^n_+$, $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, the set $\gamma_{z,\epsilon}$ defined by the system of equations $|P_1| = \epsilon_1, \dots, |P_n| = \epsilon_n$ is a smooth

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real submanifold in U. For ϵ small, all components ϵ_i being positive, the manifold $\gamma_{z,\epsilon}$ is a compact submanifold of $U \setminus \Gamma$. Define an orientation on the manifold $\gamma_{z,\epsilon}$ by means of the form $d(\arg P_1) \wedge \cdots \wedge d(\arg P_n)$. This orientation depends on the choice of order of the equations $P_1 = 0, \ldots, P_n = 0$. Let us call by the *Grothendieck cycle* γ_z related to the root z of the system of equations $P_1 = 0, \ldots, P_n = 0$ the oriented manifold $\gamma_{z,\epsilon}$ for sufficiently small ϵ. The homology class of the Grothendieck cycle in the set $U \setminus \Gamma$ does not depend on the choice of ϵ (but does depend on order of the equations $P_1 = 0, \ldots, P_n = 0$).

The *Grothendieck residue* at the root z of the system of equations $P_1 = \cdots = P_n = 0$ of n-form α holomorphic on the complement to the hypersurface Γ is defined as the number $\frac{1}{(2\pi i)^n} \int_{\gamma_z} \alpha$, where the integral of the form α is taken over the Grothendieck cycle γ_z.

The Grotendeick residue is defined correctly, because a holomorphic n-form is automatically closed and its integrals over homologous cycles are equal.

1.2. The combinatorial coefficients. Let $\Delta_1, \ldots, \Delta_n$ be convex polytopes in $\mathbb{R}^n$, and let Δ be their Minkowski sum. Each face of the polytope Δ is a sum of faces of the polytopes Δ_i. Let us call a face Γ *locked* if among its summands there is at least one vertex. Let us call a vertex $A \in \Delta$ *critical* if all faces adjacent to it are locked.

Consider a continuous map $F\colon \Delta \to \mathbb{R}^n$, $F = (f_1, \ldots, f_n)$, whose each component f_i is nonnegative and vanishes on exactly those faces $\Gamma = \Gamma_1 + \cdots + \Gamma_n$ for which the summand Γ_i is a point, a vertex of the polytope Γ_i. The restriction $\widetilde{F}$ of the map F to the boundary $\partial\Delta$ of the polytope Δ sends a neighborhood of a critical vertex to a neighborhood of the zero point on the boundary $\partial\mathbb{R}^n_+$ of the positive octant.

Let us call by the *combinatorial coefficient* k_A of a critical vertex $A \in \Delta$ the local degree of the germ of the map $\widetilde{F}\colon (\partial\Delta, A) \to (\partial\mathbb{R}^n_+, 0)$. The coefficient k_A does not depend on the choice of a map F and depends only on the choice of orientations of the polytope Δ and the positive octant $\mathbb{R}^n_+$.

Let us call an n-tuple of polytopes $\Delta_1, \ldots, \Delta_n$ *unfolded* if all the faces of the sum polytope Δ are locked. Almost all n-tuples of polytopes in $\mathbb{R}^n$ are unfolded. For an unfolded n-tuple of polytopes, each vertex of the polytope Δ is critical.

1.3. Orientations. The sign of the form ω depends on the choice of order of the independent variables $z_1, \ldots, z_n$. The choice of this order yields also an orientation of the space $\mathbb{R}^n$ containing the lattice of monomials z^a and the Newton polytope $\Delta = \Delta_1 + \cdots + \Delta_n$. The orientation of the Newton polytope Δ appearing in the definition of the combinatorial coefficient is induced from the orientation of $\mathbb{R}^n$.

The choice of order of the equations $P_1 = 0, \ldots, P_n = 0$ yields an orientation of the Grothendieck cycle and hence the sign of the Grothendieck residue. The choice of order of the equations $P_1 = 0, \ldots, P_n = 0$ (or their Newton polytopes $\Delta_1, \ldots, \Delta_n$) yields also an orientation of the space $\mathbb{R}^n_+$ appearing in the definition of the combinatorial coefficient.

Let us choose order of the independent variables and order of the equations in an arbitrary way. The sign of the form ω, the sign of the Grothendieck residue, and the signs of the combinatorial coefficients are defined by this choice.

1.4. The residue of a form at a vertex of the polytope. For each vertex A of the Newton polytope $\Delta(P)$ of a Laurent polynomial P, let us construct the Laurent series of the function Q/P, where Q is an arbitrary Laurent polynomial.

The monomial corresponding to the vertex A of the polytope $\Delta(P)$ occurs in P with some nonzero coefficient q_A, and hence the constant term of the Laurent polynomial $\widetilde{P} = P/(q_A z^a)$ equals one. Let us define the Laurent series of $1/\widetilde{P}$ by the formula $1/\widetilde{P} = 1 + (1-\widetilde{P}) + (1-\widetilde{P})^2 + \dots$. Each monomial z^b has a nonzero coefficient in only a finite number of summands $(1-\widetilde{P})^k$. Therefore the coefficient of each monomial z^b in this series is defined correctly. Let us call the formal product of the obtained series with the Laurent polynomial $q_A \cdot z^a \cdot Q$ the *Laurent series of the rational function Q/P at the vertex A* of the Newton polytope $\Delta(P)$.

By the *residue* $\operatorname{res}_A \omega$ *of a rational form* $\omega = \left(\frac{Q}{P}\right) \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$ *at the vertex A* of the Newton polytope $\Delta(P)$ let us call the constant term of the Laurent series of the function Q/P at the vertex A. *The residue* $\operatorname{res}_A \omega$ *is an explicitly calculated polynomial in q_A^{-1} and the coefficients of the Laurent polynomial P and Q.*

1.5. Main theorem.

The main theorem. *The sum of the Grothendieck residues of the form*

$$\omega = \left(\frac{Q}{P}\right) \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$$

at all the roots in $(\mathbb{C} \setminus 0)^n$ of the system of equations $P_1 = \dots = P_n = 0$ with unfolded Newton polytopes $\Delta_1, \dots, \Delta_n$ is equal to $(-1)^n \sum k_A \operatorname{res}_A \omega$, where the summation is taken over all vertices A of the polytope $\Delta = \Delta_1 + \dots + \Delta_n$.

The signs of the quantities appearing in the main theorem depend on the choice of order of the independent variables $z_1, \dots, z_n$ and on the choice of order of the equations $P_1 = 0, \dots, P_n = 0$. In the statement of the main theorem, orders of the independent variables and of the equations are fixed in an arbitrary way. A change of order of the independent variables yields a change of the sign of the form ω and hence a change of the signs of all Grothendieck residues. Simultaneously, the signs of all combinatorial coefficients change also. A change of order of the equations also simultaneously changes the signs of all Grothendieck residues and all combinatorial coefficients.

The proof of the main theorem (see 4.3) uses the technique of toric compactifications [**2**] and is based on the topological theorem stated in 1.10.

1.6. An algebraic application.

Corollary. *The sum $\sum R(z)\mu(z)$ of the values of any Laurent polynomial R over all roots z of a system of equations with unfolded Newton polytopes, counted with multiplicities $\mu(z)$, is equal to $(-1)^n \sum k_A \operatorname{res}_A \omega$, where*

$$\omega = R\frac{dP_1}{P_1} \wedge \dots \wedge \frac{dP_n}{P_n} = \left[R z_1 \dots z_n \det\left(\frac{\partial P}{\partial z}\right) \Big/ (P_1 \dots P_n)\right] \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}.$$

The corollary follows immediately from the main theorem, since the Grothendieck residue of the form ω at a root z equals $R(z)\mu(z)$. ∎

1.7. A geometric application. Each vertex A of the polytope $\Delta = \Delta_1 + \cdots + \Delta_n$ defines an n-tuple of vertices $A_i \in \Delta_i$ such that $A = A_1 + \cdots + A_n$. Put $\det_A$ equal to the determinant of the matrix constituted by the vectors $A_1, \ldots, A_n$.

Theorem. *The following formula holds for the mixed volume V of unfolded polytopes $\Delta_1, \ldots, \Delta_n$ with rational vertices: $n!V = (-1)^n \sum k_A \det_A$.*

Homothetically increasing the polytopes $\Delta_1, \ldots, \Delta_n$, one can assume that their vertices be integral. For polytopes with integral vertices, the theorem follows by comparing the Bernstein formula for the number of roots of a system of equations [**3**] with the corollary from 1.6 for $R \equiv 1$.

1.8. An application to the theory of elimination. The corollary from 1.6 allows one to construct an explicit theory of elimination for a system of equations in $(\mathbb{C} \setminus 0)^n$ with unfolded Newton polytopes. For example, let us explain how to obtain an equation for the first coordinate z_1 of the roots of the system. To this end, it suffices to calculate the sums $\sum R(z)\mu(z)$ for polynomials R equal to 1, $z_1, \ldots, z_1^N$, where $N = n!V - 1$, and apply the Newton formulas expressing the coefficients of the equation via the sums of powers of its roots. Equations for the other coordinates $z_2, \ldots, z_n$ are obtained similarly.

1.9. The cycle related to a vertex of the Newton polytope. Consider a hypersurface Γ in the torus $(\mathbb{C} \setminus 0)^n$ given by the equation $P = 0$, where P is a Laurent polynomial with the polytope Δ. To each vertex A of Δ let us assign an n-dimensional cycle T_A^n in the complement $(\mathbb{C} \setminus 0)^n \setminus \Gamma$ of the torus $(\mathbb{C} \setminus 0)^n$ to the hypersurface Γ. Denote by T^n the oriented torus given by the equations $|z_1| = \cdots = |z_n| = 1$, whose orientation is given by the form

$$\left(\frac{1}{2\pi i}\right)^n \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

Let $\xi = (\xi_1, \ldots, \xi_n)$ be an integral covector such that the maximum of the linear function $\langle \xi, x \rangle$ is achieved at the vertex A of the polytope Δ. Consider the one-parametric group $\lambda(t) = (t^{\xi_1}, \ldots, t^{\xi_n})$. Let us act by this one-parametric group on the torus T^n. For the absolute value of t sufficiently large, the cycle $\lambda(t)T^n$ does not intersect the hypersurface Γ. Denote this cycle by T_A. It is easy to check that the homology class of the cycle T_A in $(\mathbb{C} \setminus 0)^n$ is defined correctly, i. e., does not depend on the choice of an integral covector ξ and the parameter t, provided $|t|$ is sufficiently large.

1.10. A topological theorem. Let $\Delta_1, \ldots, \Delta_n$ be an unfolded n-tuple of polytopes in $\mathbb{R}^n$, and let $P_1, \ldots, P_n$ be Laurent polynomials with the Newton polytopes $\Delta_1, \ldots, \Delta_n$. Denote by Γ the hypersurface in $(\mathbb{C} \setminus 0)^n$ given by the equation $P = 0$.

Theorem. *In the complement $(\mathbb{C} \setminus 0)^n \setminus \Gamma$ of the torus to the hypersurface Γ, the sum of the Grothendieck cycles γ_z over all roots z of the system $P_1 = \cdots = P_n = 0$ is homologous to the cycle $(-1)^n \sum k_A T_A^n$, where the summation is taken over all vertices A of the polytope Δ, and k_A is the combinatorial coefficient of the polytopes $\Delta_1, \ldots, \Delta_n$ at the vertex A.*

1.11. Remarks. In the paper [**1**] published 7 years ago, we have announced a more general theorem on sum of Grothendieck residues. Approximately at the same time we have prepared a text with proofs. A. G. Khovanskii delivered lectures on this subject at many universities. However, our text has not been submitted to a journal, and the proof has not been published. The proof of the more general theorem, including simultaneously toric, affine and local cases, turned out rather cumbersome. It obscured simple arguments necessary for the proof of the result in the toric case, with which we started and with which most of applications are concerned. In this paper we consider the toric case separately. More cumbersome general case is considered similarly. We intend to return to it later.

Since that time new results were obtained. In the paper [**4**] the product in the group $(\mathbb{C} \setminus 0)^n$ of all roots of a system of equations $P_1 = \cdots = P_n = 0$ with sufficiently general Newton polytopes is calculated. The formula for the product of roots uses Parshin symbols and surprisingly resembles the new formula for the mixed volume from 1.7. The new formula for the mixed volume can be proved geometrically and without the assumption of rationality of the vertices. Under some additional restrictions, such a proof was given in [**5**], and in the general case in [**4**].

Why the formula for the product of roots from [**4**] so much resembles the formula for the mixed volume? In [**6**] the answer to this question is given. That paper contains a simple construction of the cohomology class responsible for Parshin symbols. After that, the formula for the product of roots is proved by the reference to the topological theorem from 1.10. I. A. Soprunov (A. G. Khovanskii's graduate student) has proved the formula for the sum of Grothendieck residues and the formula for the product of roots using Parshin's reciprocity laws. His paper is in preparation.

1.12. Arrangement of the material. In section 2 some local characteristic classes are introduced and global relations between them are proved. In section 3, we show that the Grothendieck cycles and combinatorial coefficients are described via these local characteristic classes. In the same section 3, we compute local characteristic classes related to affine toric varieties. In section 4, the situation is described arising after the natural toric compactification of the group $(\mathbb{C} \setminus 0)^n$ and the hypersurfaces $P_i = 0$ lying in this group. Due to the condition of unfoldedness of the Newton polytopes $\Delta_1, \ldots, \Delta_n$, the points at infinity added in compactification of the group $(\mathbb{C} \setminus 0)^n$ can be covered by nice closed sets invariant with respect to $(\mathbb{C} \setminus 0)^n$. In 4.2, using, in this situation, global relations between local characteristic classes from section 2 and computation of these classes from section 3, we obtain the topological theorem from 1.10. In 4.3, using the topological theorem from 1.10 and the Stokes formula, we prove the main theorem from 1.5.

2. Global relations between local characteristic classes

2.1. The local characteristic class. Let X be a germ at the point a of a locally compact set, Y be a germ of its closed subset. We will assume that the complement $X \setminus Y$ to the closed set Y in X is a germs of a smooth oriented N-dimensional manifold. By a *covering* of the germ Y let us call its representation as a union of an ordered n-tuple of germs of closed subsets $Y_1, \ldots, Y_n$, numbered by indices $1, \ldots, n$, $\bigcup Y_i = Y$, intersecting at the only point a, $\bigcap Y_i = a$. To each covering of the set Y let us assign an $(N-n)$-dimensional homology class of the set $X \setminus Y$, which we will call the *local characteristic class of the covering* $Y_1, \dots, Y_n$.

Let us call a continuous function on X with values in nonnegative real numbers $\mathbb{R}_+$ a *lining function* of a closed subset A of X if the function f vanishes on the subset A and is positive on the complement to this subset. The existence of a lining function is guaranteed by Uryson's theorem.

Definition 1. For an ordered n-tuple of closed subsets $Y_1, \dots, Y_n$ of the set X, let us call by a *lining map* a map $F\colon X \to \mathbb{R}^n_+$, $F = (F_1, \dots, F_n)$, from the set X to the positive octant $\mathbb{R}^n_+$, whose i-th component F_i is a lining function of the i-th subset Y_i.

A lining map is defined not uniquely, but any two lining maps are homotopic in the class of lining maps. Indeed, if F_1 and F_2 are lining maps, then for any t subject to the inequalities $0 \le t \le 1$, the map $F_t = tF_1 + (1-t)F_2$ is lining.

A continuous lining map can be approximated by a smooth in $X \setminus Y$ lining map homotopic to the given one in the class of lining maps. Let us fix some lining map F smooth on $X \setminus Y$ for an ordered n-tuple of subsets $Y_1, \dots, Y_n$ which is a covering of the set Y. The preimage of zero under this map consists of the only point a of intersection of the sets Y_i. Hence the image of the boundary of a sufficiently small neighborhood U_a of the point a has a finite distance from zero in $\mathbb{R}^n_+$, and there exists a neighborhood V_0 of zero in the interior of the positive octant $\mathbb{R}^n_+$ whose preimage is contained in U_a. By the Sard theorem, the preimage of almost any point in V_0 is a smooth $(N-n)$-dimensional submanifold in $X \setminus Y$. This submanifold is endowed with a co-orientation induced by the standard orientation in $\mathbb{R}^n_+$. By assumption, the variety $X \setminus Y$ is oriented, hence a co-oriented submanifold is equipped with a natural orientation. Recall its definition. Consider a normal space at some point of a submanifold, choose a basis $e_1, \dots, e_n$ in it, whose image under the differential of the lining map is a positively oriented basis of $\mathbb{R}^n_+$. Let us take as a positively oriented basis such a basis $p_1, \dots, p_{N-n}$ in the tangent space to a submanifold that the basis $p_1, \dots, p_{N-n}, e_1, \dots, e_n$ of the tangent space to the manifold $X \setminus Y$ is positively oriented.

Definition 2. Let us call by the *local characteristic cycle* C_a related to the point $a = \bigcap Y_i$ of a covering $Y_1, \dots, Y_n$ of a set Y the preimage $F^{-1}(c)$ of any sufficiently small regular value c of any lining map F smooth on $X \setminus Y$, endowed with the orientation described above.

General theorems of smooth topology [**7**] imply that the homology class of the characteristic cycle C_a in the set $X \setminus Y$ is defined correctly, i.e., depends neither on the choice of a lining map F nor on the choice of a sufficiently small regular value c.

Let us consider one particular case more closely. Let the number n of sets in a covering $Y_1, \dots, Y_n$ of a closed set Y be equal to the dimension n of the manifold $X \setminus Y$, and let the manifold $X \setminus Y$ be connected. In this case, the group $H_0(X \setminus Y)$ is one-dimensional.

The characteristic class of the covering $Y_1, \dots, Y_n$ of the set Y is an element of the one-dimensional group $H_0(X \setminus Y)$, i.e., equals $k_a e$, where k_a is an integer and e is the generator of the group.

Definition 3. The integer k_a is called the *degree* of the local characteristic class of the covering $Y_1, \dots, Y_n$.

Let us give an interpretation of the degree of the local characteristic class as the degree of a mapping. Consider any proper continuous map $F\colon X \to \mathbb{R}^n_+$ from the set X to the positive octant $\mathbb{R}^n_+$ for which the preimage of the i-th coordinate hyperplane $\Gamma_i \subset \mathbb{R}^n_+$ (defined by the equality $u_i = 0$ where u_i is the i-th coordinate function in the space $\mathbb{R}^n$) coincides with the set Y_i. The degree of the map F equals the degree k_a of the local characteristic class of the covering $Y_1, \dots, Y_n$.

2.2. Global relations. Let X be a connected compact topological space, Y be a closed subset of X. We will assume that the complement $X \setminus Y$ to the set Y in the set X is a smooth oriented N-dimensional manifold. Let Y be represented as the union of an n-tuple of closed subsets $Y_1, \dots, Y_n$ so that the intersection of the subsets Y_i consists of finitely many points. We will call the points from the finite set $\bigcap Y_i$ by the singular points of the n-tuple $Y_1, \dots, Y_n$. In a small neighborhood U_x of each singular point x of the n-tuple $Y_1, \dots, Y_n$ one defines the local characteristic cycle C_x of the covering $Y_1, \dots, Y_n$, belonging to the group $H_{N-n}((X \setminus Y) \bigcap U_x)$. Since the set $(X \setminus Y) \bigcap U_x$ is included into the manifold $X \setminus Y$, one can consider the cycle C_x as an element of the group $H_{N-n}(X \setminus Y)$.

THEOREM 1. *The sum of local characteristic classes over all singular points x of the n-tuple $Y_1, \dots, Y_n$ is homologous to zero in $X \setminus Y$.*

PROOF. Consider a lining map $F\colon X \to \mathbb{R}^n_+$ of the n-tuple of sets $Y_1, \dots, Y_n$ whose restriction to the submanifold $X \setminus Y$ is smooth. Then the preimage of an interior regular point of $\mathbb{R}^n_+$ sufficiently close to zero under the map F is the sum of local characteristic cycles. And the preimage of a point of $\mathbb{R}^n_+$ sufficiently far from zero is empty by compactness of X. By general theorems of smooth topology [**7**] the preimages of different regular values are homologous, whence the theorem follows. $\square$

Consider a somewhat more general situation. Let us call two ordered n-tuples $Z_1, \dots, Z_n, W_1, \dots, W_n$ of subsets of the set X *mutually normal* if for any $i = 1, \dots, n$ the intersection $Z_i \bigcap W_i$ is empty.

Let a closed subset Y be represented as the union of two subsets Z and W. Let the subsets Z and W be represented as the unions of two ordered mutually normal n-tuples of closed sets $Z_1, \dots, Z_n$ and $W_1, \dots, W_n$, $Z = \bigcup Z_i$ and $W = \bigcup W_i$, so that the sets $\bigcap Z_i$ and $\bigcap W_i$ consist of no more than a finite number of points. Then in a neighborhood of each singular point z of the n-tuple $Z_1, \dots, Z_n$ the local characteristic cycle C_z is defined, and in a neighborhood of each singular point w of the n-tuple $W_1, \dots, W_n$ the local characteristic cycle C_w is defined as well.

THEOREM 2. *The sum of local characteristic cycles C_z over all singular points z of the n-tuple $Z_1, \dots, Z_n$ in the manifold $X \setminus Y$ is homologous to the sum of local characteristic cycles C_w over all singular points w of the n-tuple $W_1, \dots, W_n$ taken with the coefficient $(-1)^n$:*

$$\sum_{z \in \bigcap Z_i} C_z \approx (-1)^n \sum_{w \in \bigcap W_i} C_w.$$

PROOF. Consider a continuous map $F\colon X \to \mathbb{R}^n_+$ whose each component F_i possesses the following properties. The function F_i satisfies the inequalities $0 \leq F_i \leq 1$, and $F_i^{-1}(0) = Z_i$ and $F_i^{-1}(1) = W_i$. The existence of the map F is guaranteed by Uryson's theorem. The map F can be chosen smooth on the submanifold $X \setminus Y$. Denote by I^n the cube in the space $\mathbb{R}^n_+$ defined by the inequalities $0 \le x_1 \le 1, \dots, 0 \le x_n \le 1$. By general theorems of smooth topology [**7**] the preimages of two regular values of the map F in the cube I^n are homologous to each other in the manifold $X \setminus Y$. This implies theorem 2. Indeed, the preimage of a regular value close to zero is the sum of local characteristic cycles C_z in the manifold $X \setminus Y$ over all singular points z of the n-tuple of sets $Z_1, \dots, Z_n$. The preimage of a regular value close to the vertex $(1, \dots, 1)$ is the sum of local characteristic cycles C_w over all singular points w of the n-tuple of sets $W_1, \dots, W_n$ with the coefficient $(-1)^n$. Indeed, the central symmetry τ of the cube I^n with respect to its center $\left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ changes the orientation of the cube for n odd and does not change it for n even. The preimage under the map F of a point a close to the vertex $(1, \dots, 1)$ of the cube I^n coincides with the preimage under the map τF of the point $\tau(a)$ close to the vertex $(0, \dots, 0)$. The map τF is lining for the n-tuple $W_1, \dots, W_n$, and its preimage equals the sum of local characteristic cycles C_w. □

3. Computation of some local characteristic classes

3.1. The Grothendieck cycle. The class of the Grothendieck cycle of an isolated root z of a system of analytic equations $P_1 = \dots = P_n = 0$ on an n-dimensional complex analytic variety M^n can be described by means of a local characteristic class. Consider a small neighborhood of the point z in M^n as X, the hypersurface in X given by the equation $P_1 \dots P_n = 0$ as Y. The hypersurface Y is the union of closed hypersurfaces Y_i in X defined by the equations $P_i = 0$.

Statement. *In the group $H_n(X \setminus Y)$, the class of the Grothendieck cycle differs from the characteristic class of the covering $Y_1, \dots, Y_n$ of the hypersurface Y only by the sign $(-1)^{n(n+1)/2}$.*

Proof. Indeed, a lining map $F \colon X \to \mathbb{R}^n_+$ for the n-tuple of hypersurfaces $Y_1, \dots, Y_n$ can be chosen in the form $F = (|P_1|, \dots, |P_n|)$. Both the Grothendieck cycle and the characteristic class of the covering can be given by the equations $|P_1| = \epsilon_1, \dots, |P_n| = \epsilon_n$. In general, the orientation of the characteristic cycle of the covering differs from the orientation of the Grothendieck cycle: the form $d(\arg P_1) \wedge \dots \wedge d(\arg P_n) \wedge d|P_1| \wedge \dots \wedge d|P_n|$ yields an orientation of the complex variety, which in general differs from the standard one. To obtain the standard orientation of the complex variety, this form should be multiplied by $(-1)^{n(n+1)/2}$. □

3.2. The combinatorial coefficient of a covering of the boundary of a cone. An n-dimensional convex polyhedral cone X in $\mathbb{R}^n$ is called *sharpened* if it does not contain any affine subspace of positive dimension in $\mathbb{R}^n$. A sharpened cone has the unique vertice which we denote by A.

By a *combinatorial covering* $Y_1, \dots, Y_n$ of the boundary Y of a cone X let us call an ordered n-tuple of closed sets Y_i such that:

1) each set Y_i is a union of faces of the cone X;
2) the union $\bigcup Y_i$ of the sets Y_i coincides with the boundary Y of the cone X;
3) the intersection $\bigcap Y_i$ of the sets Y_i coincides with the vertex A of the cone X.

The interior $X \setminus Y$ of the cone X is a smooth manifold which inherits the standard orientation of the space $\mathbb{R}^n$.

Definition. The *combinatorial coefficient* of a combinatorial covering $Y_1, \dots, Y_n$ of the boundary of the cone X is the degree of the local characteristic class of this covering (see 2.1).

Note that the combinatorial coefficient F can be defined using only combinatorics of the covering $Y_1, \dots, Y_n$, thus explaining the name for this number.

Let $\Delta_1, \dots, \Delta_n$ be convex polytopes in $\mathbb{R}^n$, $\Delta = \Delta_1 + \dots + \Delta_n$, and A be a critical vertex of the polytope Δ (see 1.1). In 1.1 we have defined the combinatorial coefficient k_A at the critical vertex A. This combinatorial coefficient amounts to the combinatorial coefficient of a covering of the boundary of a cone. Indeed, near its vertex A the polytope Δ is a cone Δ_A. Let us assign a covering of the boundary of this cone Δ_A to the polytopes $\Delta_1, \dots, \Delta_n$. Each face P of the cone Δ_A in a neighborhood of the vertex A coincides with the unique face Γ of the polytope Δ. The face Γ can be uniquely represented as the Minkowski sum $\Gamma = \Gamma_1 + \dots + \Gamma_n$ of faces Γ_i of the polytopes Δ_i. Define the subset Y_i of the boundary of the cone Δ_A as follows: a face P is included into Y_i if and only if the summand Γ_i in the Minkowski sum for the face Γ of Δ corresponding to P is a vertex of the polytope Δ_i. The intersection of the sets Y_i contains only the vertex A. The union of the subsets Y_i near the point A coincides with the boundary of the cone Δ_A since the vertex A of Δ is critical.

Statement. *The combinatorial coefficient of the above covering of the boundary of the cone Δ_A coincides with the combinatorial coefficient of the vertex A defined in 1.1.*

The proof of the statement follows from comparing the definitions.

3.3. The local characteristic classes on affine toric varietes. Let X be an n-dimensional (singular) affine toric variety having a fixed point a with respect to the action of the group $(\mathbb{C}\setminus 0)^n$. Let Y be the union of all orbits of X whose complex dimension is less than n. Let $Y_1, \dots, Y_n$ be an n-tuple of closed sets invariant with respect to the action of the group $(\mathbb{C}\setminus 0)^n$ and such that $\bigcup_{i=1}^n Y_i = Y$ and $\bigcap_{i=1}^n Y_i = a$. In the present subsection the local characteristic class of the covering $Y_1, \dots, Y_n$ of Y near a is calculated. The set $X \setminus Y$ is a complex torus $(\mathbb{C}\setminus 0)^n$. The point a has arbitrarily small neighborhoods U_a such that the sets $(X\setminus Y)\cap U_a$ are homeomorphic to $(\mathbb{C}\setminus 0)^n$. The n-dimensional homology group of the complex torus $(\mathbb{C}\setminus 0)^n$ is one-dimensional and has a unique generator; as this generator one can take the real torus T^n defined by the equalities $|z_1| = \dots = |z_n| = 1$ whose orientation is given by the form $\omega = \frac{1}{(2\pi i)^n}\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_n}{z_n}$. The local characteristic class of the covering $Y_1, \dots, Y_n$ can be represented as $c(Y_1, \dots, Y_n)T_a^n$ where $c(Y_1, \dots, Y_n)$ is an integer and T_a^n is the cycle T^n moved to the neighborhood of the point a by the action of the group $(\mathbb{C}\setminus 0)^n$. It remains to calculate the integer $c(Y_1, \dots, Y_n)$.

For statement of the answer we need some facts from theory of toric varietes [**8**], [**9**]. An affine toric variety X corresponds to an n-dimansional sharpened cone in the real vector space spanned by the characters of the group $(\mathbb{C}\setminus 0)^n$. Denote this cone by X^σ. The faces of the cone X^σ are in one-to-one correspondence with the orbits of the toric variety X, so that a k-dimensional face corresponds to an orbit of complex dimension k. The cone is sharpened, because the affine toric variety X contains the zero dimensional orbit a.

By assumption each of the closed sets $Y_1, \dots, Y_n$ is a union of orbits. Hence the set Y_i corresponds to certain union of faces of X^σ which we denote by Y_i^σ.

By assumption the sets $Y_1, \dots, Y_n$ are closed and their union coincides with the set of all orbits of dimension less than n. Hence the union of the sets $Y_1^\sigma, \dots, Y_n^\sigma$ coincides with the boundary of the cone X^σ. The equality $\bigcap_i Y_i = a$ implies that $\bigcap_i Y_i^\sigma$ equals the vertex A of the cone X^σ.

The order of the independent variables $z_1, \dots, z_n$ in the torus $(\mathbb{C} \setminus 0)^n$ fixes the sign of the form $w = \left(\frac{1}{2\pi i}\right)^n \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$ responsible for the orientation of the torus T_a^n. The order of the independent variables $z_1, \dots, z_n$ also yields orientation in the space $\mathbb{R}^n$ spanned by the characters $z_1^{m_1} \dots z_n^{m_n}$ of the group $(\mathbb{C} \setminus 0)^n$, and orientation of the cone X^σ lying in this $\mathbb{R}^n$. So after fixing the form ω the cone X^σ is endowed with an orientation.

Theorem. *The local characteristic class corresponding to an invariant covering $Y_1, \dots, Y_n$ of the set Y in the affine toric variety X equals $(-1)^{n(n+1)/2} k_A T_a^n$ where k_A is the combinatorial coefficient of the covering $Y_1^\sigma, \dots, Y_n^\sigma$ of the boundary Y^σ of the cone X^σ.*

Proof. The group $(\mathbb{C} \setminus 0)^n$ is the product of its subgroups T^n and Π, where T^n is the real torus and the subgroup Π consists of the points $(z_1, \dots, z_n)$ whose all coordinates z_i are real. The group $(\mathbb{C} \setminus 0)^n$ is embedded into the toric variety X. Denote by X^Π the closure of the subgroup Π in X. Denote by $Y_1^\Pi, \dots, Y_n^\Pi, Y^\Pi$ respectively the intersections of X^Π with the sets $Y_1, \dots, Y_n, Y$, i.e., $Y_i^\Pi = Y_i \cap X^\Pi$, $Y^\Pi = Y \cap X^\Pi$. From theory of toric varietes it is known that the closure X^Π of the group Π intersects any orbit of the variety X, in particular, it intersects the orbit consisting of the point a. Hence $\bigcap Y_i^\Pi = a$. The complement $X^\Pi \setminus Y^\Pi$ coincides with the group Π and hence is a connected n-dimensional real manifold. Let us fix the orientation of this manifold by means of the form $\frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$.

We will need the following lemma.

Lemma. *The local characteristic class of an invariant covering $Y_1, \dots, Y_n$ of the set Y in the affine toric variety X equals $(-1)^{n(n+1)/2} k_a T_a^n$ where k_a is the degree of the local characteristic class of the covering $Y_1^\Pi, \dots, Y_n^\Pi$ of the set Y^Π inside X^Π.*

Proof. Consider a lining map $F \colon X \to \mathbb{R}_+^n$ for the sets $Y_1, \dots, Y_n$ whose restriction to the group $(\mathbb{C} \setminus 0)^n$ is smooth. Let us average this map by means of the action of the compact group $T^n \subset (\mathbb{C} \setminus 0)^n$ on the algebraic variety X. We will obtain a new lining map $G \colon X \to \mathbb{R}_+^n$ invariant with respect to the group T^n.

The restriction G^π of the map G to the set X^Π is a lining map for the n-tuple of sets $Y_1^\Pi, \dots, Y_n^\Pi$. Consider an interior point ϵ in the positive octant $\mathbb{R}_+^n$ whose all coordinates are sufficiently small, which is a regular value as for the map G, as for the map G^Π. The preimage of the point ϵ under the map G^Π consists of several points p_i. The set $G^{-1}(\epsilon)$ consists of orbits $T^n p_i$ of these points under the action of the froup T^n. The points p_i taken with signs of the corresponding Jacobians represent the local characteristic class of the covering $Y_1^\Pi, \dots, Y_n^\Pi$ of the set Y^Π in X^Π. The tori $T^n p_i$ taken with the corresponding orientations represent the local characteristic class of the covering $Y_1, \dots, Y_n$ of the set Y in X. Let a point p_i come into the local characteristic class of the covering $Y_1^\Pi, \dots, Y_n^\Pi$ of the set Y^Π in X^Π with the plus sign. Then the form giving an orientation of the shifted standard torus $T^n p_i$ yields an orientation of the torus $T^n p_i$ as a connected component of the characteristic cycle of the covering $Y_1, \dots, Y_n$ of the set Y, constructed via the map G and the point ϵ, after multiplication by $(-1)^{n(n+1)/2}$. Indeed, on the group Π

the orientation given by the form $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ coincides with the orientation given by the form $d|z_1| \wedge \cdots \wedge d|z_n|$. On the torus $T^n p_i$ the orientation given by the form $\left(\frac{1}{2\pi i}\right)^n \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$ coincides with the orientation given by the form $d(\arg z_1) \wedge \cdots \wedge d(\arg z_n)$. But the orientation on $(\mathbb{C} \setminus 0)^n$ given by the form $d(\arg z_1) \wedge \cdots \wedge d(\arg z_n) \wedge d|z_1| \wedge \cdots \wedge d|z_n|$ differs by the sign $(-1)^{n(n+1)/2}$ from the form giving the usual orientation of the complex manifold $(\mathbb{C} \setminus 0)^n$. The lemma is proved. □

Let us return to the proof of the theorem. We will need some facts from toric geometry. The affine toric variety X can be embedded into a projective toric variety M. This can be done as follows. Let X^σ be the cone with the vertex A corresponding to the affine variety X. One can choose in different ways a polytope $\Delta \subset \mathbb{R}^n$ with integral vertices such that: 1) one of the vertices of the polytope Δ is the point A, 2) near the vertex A the polytope Δ coincides with the cone X^σ. The polytope Δ corresponds in the standard way to a toric variety M_Δ (see [**2**], [**9**]). This variety contains the affine toric variety X. To each polytope of the type $k\Delta$, where k is a natural number, one can assign a map from the variety M_Δ to the projective space. If the number k is sufficiently large, then the corresponding map will be an embedding. Changing, if necessary, the polytope Δ by $k\Delta$, one can assume that the map from M_Δ to the projective space corresponding to Δ is an embedding.

There is the remarkable moment map $Q\colon M_\Delta \to \Delta$ which maps the projective variety M_Δ onto the polytope Δ (see [**10**]).

The image under the moment map of an orbit O of the toric variety M_Δ is the interior $\Gamma \setminus \partial\Gamma$ of some face Γ of the polytope Δ (here the boundary $\partial\Gamma$ is taken in the topology of the minimal affine space containing the face Γ). Moreover, the restriction of the moment map Q to the intersection $O \cap \overline{\Pi}$ of the orbit O with the closure $\overline{\Pi}$ of the group Π in M_Δ yields a one-to-one smooth map from the manifold $O \cap \overline{\Pi}$ to the interior $\Gamma \setminus \partial\Gamma$ of the face Γ. The arising correspondence between the orbits of the variety M_Δ and the faces of the polytope Δ is one-to-one and preserves closures: if an orbit O_1 belongs to the closure of an orbit O_2, then the face Γ_1 corresponding to the orbit O_1 belongs to the face Γ_2 corresponding to the orbit O_2.

The image under the moment map of the affine toric variety X coincides with the union of all faces of the polytope Δ containing the vertex A (the polytope Δ itself is one of such faces). Denote by L_A the union of all faces of Δ which do not contain the vertex A. The set $\widetilde{X}^\sigma = \Delta \setminus L_A$ is an open neighborhood of the vertex A in the cone X^σ. On the other hand, the set $\Delta \setminus L_A$ is the image of the affine variety X under the map Q.

The covering $Y_1^\sigma, \dots, Y_n^\sigma$ of the boundary Y^σ of the cone X^σ induces the covering $\widetilde{Y}_1^\sigma, \dots, \widetilde{Y}_n^\sigma$ of the subset $\widetilde{Y}^\sigma = Y \cap \widetilde{X}^\sigma$ of the set $\widetilde{X}^\sigma$, where $\widetilde{Y}_i^\sigma = Y_i \cap \widetilde{X}^\sigma$. The degrees of the local characteristic classes of the covering $Y_1^\sigma, \dots, Y_n^\sigma$ of the set Y and the covering $\widetilde{Y}_1^\sigma, \dots, \widetilde{Y}_n^\sigma$ of the set $\widetilde{Y}^\sigma$ coincide by definition: the local characteristic class of a covering depends only on the germs of the sets arising in its definition. The restriction of the moment map Q to the set X^Π yields a one-to-one map from this set to the set $\widetilde{X}^\sigma$. Under this map the set Y_i^Π goes to the set $\widetilde{Y}_i^\sigma$. The restriction of the moment map to the group $\Pi \subset X^\Pi$ is smooth and preserves the orientation. Hence the degree of the local characteristic class of the covering

$Y_1^\Pi, \dots, Y_n^\Pi$ of the set Y^Π in X^Π equals the degree of the local characteristic class of the covering $\widetilde{Y}_1^\sigma, \dots, \widetilde{Y}_n^\sigma$ of the set $\widetilde{Y}^\sigma$ in $\widetilde{X}^\sigma$ which in its turn equals the combinatorial coefficient of the covering $Y_1^\sigma, \dots, Y_n^\sigma$ of the boundary Y^σ of the cone X^σ. Now the theorem follows from the lemma. □

4. Proof of the theorems

4.1. Construction. Let $\Delta_1, \dots, \Delta_n$ be an unfolded n-tuple of Newton polytopes in $\mathbb{R}^n$ and $\Delta = \Delta_1 + \dots + \Delta_n$ be their Minkowski sum. Each face Γ of the polytope Δ is representable as the sum $\Gamma = \Gamma_1 + \dots + \Gamma_n$ of faces Γ_i of the polytopes Δ_i. Denote by W_i^σ the union of all faces Γ of the polytope Δ for which the i-th summand Γ_i is some vertex of the polytope Δ_i. Since the Newton polytopes are unfolded, the union $\bigcup_{i=1}^n W_i^\sigma$ of the sets W_i^σ covers all the boundary $\partial\Delta$ of the polytope Δ. Let $P_1 = \dots = P_n = 0$ be a system of equations in $(\mathbb{C} \setminus 0)^n$ where P_i is a Laurent polynomial with the Newton polytope Δ_i.

Denote by M_Δ the toric compactification of the group $(\mathbb{C} \setminus 0)^n$ constructed for the polytope Δ. Let $Z_1, \dots, Z_n$ be the n-tuple of hypersurfaces in M_Δ where Z_i is the closure in M_Δ of the hypersurface in $(\mathbb{C} \setminus 0)^n$ given by the equation $P_i = 0$. The unfoldedness condition for $\Delta_1, \dots, \Delta_n$ implies that the intersection of the hypersurfaces $\bigcap_{i=1}^n Z_i$ lies in $(\mathbb{C} \setminus 0)^n$ and contains only a finite number of points. In other words, each singular point z for the n-tuple of sets $Z_1, \dots, Z_n$ is a root in $(\mathbb{C} \setminus 0)^n$ of the system of equations $P_1 = \dots = P_n = 0$.

Denote by W_i the union of orbits of the toric variety M_Δ corresponding to the faces of the polytope Δ belonging to the set W_i^σ. Since $\bigcup W_i^\sigma = \partial\Delta$, the union of sets W_i coincides with the union of all orbits of the toric variety M_Δ with dimension less than n. The intersection $\bigcap_{i=1}^n W_i$ of the sets W_i coincides with the set of fixed points w of the toric variety M_Δ. The set of the fixed points of the toric variwty M_Δ is in one-to-one correspondence with the set of vertices A of the polytope Δ. In other words, the singular points of the n-tuple of closed sets $W_1, \dots, W_n$ are numbered by the vertices of the polytope Δ.

Two n-tuples of constructed closed sets $Z_1, \dots, Z_n$ and $W_1, \dots, W_n$ in the toric variety M_Δ are mutually normal. Indeed, it is easy to see from the definition that the sets Z_i and W_i do not intersect each other.

4.2. Proof of the topological theorem from 1.10. Let us apply theorem 2 from 2.2 to the situation described in 4.1.

Consider the complement U of the complex torus $(\mathbb{C} \setminus 0)^n$ to the hypersurface Γ given by the equation $P_1 \dots P_n = 0$. By theorem 2 from 2.2, the sum of local characteristic cycles C_z over the singular points z of the n-tuple $Z_1, \dots Z_n$ is homologous in U to the sum of local characteristic cycles C_w over the singular points w of the n-tuple $W_1, \dots, W_n$ multiplied by $(-1)^n$.

$$\sum C_z \sim (-1)^n \sum C_w. \tag{1}$$

According to 3.1 the characteristic cycle C_z equals the Grothendieck cycle γ_z of the system $P_1 = \dots = P_n = 0$ at the root z multiplied by $(-1)^{n(n+1)/2}$.

Consider a singular point w of the n-tuple $W_1, \dots, W_n$ corresponding to a vertex A of the polytope Δ. Near the vertex A the polytope Δ coincides with the cone Δ_A. Near the point A the sets $W_1^\sigma, \dots, W_n^\sigma$ cover the boundary of this cone Δ_A. Denote by l_A the combinatorial coefficient of this covering (see 3.2). According to the theorem from § 3 the characteristic cycle C_w equals the cycle $l_A T_A$

multiplied by $(-1)^{n(n+1)/2}$. By the statement from 3.2 the number l_A coincides with the combinatorial coefficient k_A of the polytopes $\Delta_1, \dots, \Delta_n$ at he vertex A.

Substituting these quantities into the equivalence $(*)$, we obtain

$$\sum \gamma_z \sim (-1)^n \sum k_A T_A,$$

Q.E.D.

4.3. Proof of the main theorem from 1.5. Consider the n-form

$$\omega = \frac{Q}{P} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$$

on the variety M_Δ. This form is holomorphic on the complement U of the group $(\mathbb{C} \setminus 0)^n$ to the hypersurface $P = P_1 \dots P_n = 0$. Hence the form ω is closed in U. But according to the topological theorem from 1.10 the sum $\sum \gamma_z$ of the Grothendieck cycles over all roots in $(\mathbb{C} \setminus 0)^n$ of the system of equations $P_1 = \cdots = P_n = 0$ is homologous in U to the cycle $(-1)^n \sum k_A T_A$. Hence $\sum \operatorname{res}_z \omega = (-1)^n \sum k_A \frac{1}{(2\pi i)^n} \int_{T_A} \omega$, where the summation in the left-hand side is taken over all roots z of the system of equations $P_1 = \cdots = P_n = 0$ in $(\mathbb{C} \setminus 0)^n$ and the summation in the right hand side is taken over all vertices A of Δ. It remains to calculate $\int_{T_A} \omega$.

Each vertex A of the Newton polytope Δ yields a formal Laurent series decomposition of the rational function $\frac{Q}{P} = \sum q_{m_1,\dots,m_n} z_1^{m_1} \dots z_n^{m_n}$. This series is described in 1.4. Near the fixed point of the toric variety M_Δ corresponding to the vertex A this series absolutely converges. Hence in the neighborhood of this point we have

$$\omega = \sum q_{m_1,\dots,m_n} z_1^{m_1} \dots z_n^{m_n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

It remains to calculate the integral

$$\int_{T_A} z_1^{m_1} \dots z_n^{m_n} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

This integral is easily calculated explicitly. It vanishes for all $m = (m_1, \dots, m_n)$ except $m = (0, 0, \dots, 0)$. The integral for $m = (0, 0, \dots, 0)$ equals $(2\pi i)^n$. This implies the main theorem.

Bibliography

[1] Gelfond O. A, Khovanskii A. G., *Newton polytopes and Grothendieck residues*, Dokl. Russian Akad. Nauk **3** (1996), 350, 298–300; English transl. in Russian Acad. Sci. Dokl. Math.

[2] Khovanskii A. G., *Newton polyhedra and toroidal varietes*, Functional Analysis and its applications **11** (1977), 4, 56–64; English transl. in Funct. Anal. Appl. **11** (1978), 4, 289–296.

[3] Bernstein D., *The number of roots of a system of equations*, Funktsional. Anal. i Prilozhen. **9** (1975), 3, 1–4; English transl. in Functional Anal. Appl. **9** (1975), 3.

[4] Khovanskii A. G., *Newton polyhedra, a new formula for mixed volume, product of roots of a system of equations*, The Arnoldfest, Proceedings of a Conference in Honour of V. I. Arnold for his Sixtieth Birthday, vol. 24, Amer. Math. Soc., Fields Institute Communications, Providence, RI, 1999, pp. 325–364.

[5] Gelfond O. A., *Combinatorial coefficients and the mixed volume of polytopes*, Functional Anal. Appl. **30** (1996), 3, 207–208.

[6] Khovanskii A. G., *Toric varietes, multidimensional antiderivatives and Parshin's Symbols* (*in preparation*).

[7] Milnor J., *Topology from the differentiable viewpoint*, Univ. Press of Virginia, Charlottesville, 1965.

[8] Kempf G., Knudsen F., Mumford D., Saint-Donat B., *Toroidal embeddings, Lecture Notes in Math.*, vol. 339, Springer-Verlag, 1973.

[9] Danilov V. I., Geometry of toric varieties, UMN (Russian Math. Surveys) **3** (1978), 2, 85–135.

[10] Atyah M. F., *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), 1, 1–15.

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