The Monty Hall Problem: The Remarkable Story of Math's Most Contentious Brain Teaser, by Jason Rosenhouse. Oxford University Press, 2009, xii + 194 pp. ISBN 978-0-19-536789-8, $xx.xx

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Every once in a while, a problem captures the imagination and is widely circulated within the mathematical community. Occasionally, it breaks out of the professional circle into the public domain. A notorious example is the Monty Hall problem, inspired by the television game show *Let's make a deal*:

> *Monty Hall, a television host, shows a contestant three identical doors, behind one of which is a car and behind the other two are goats. He asks the contestant to select one of the doors and will give the contestant the prize behind it. After the contestant makes her selection, Monty opens one of the remaining doors to reveal a goat. He then offers the contestant the opportunity to switch her choice to the remaining unopened door. Should the contestant, preferring the car, stick with her first choice, or switch?*

This problem first came to my attention in a newsletter from Washington State University [**9**] and was reported on in this Journal in 1993 [**1**], with follow-up comments in [**2-6**]. Earlier, in 1990, Marilyn Vos Savant dealt with the problem in her weekly column, *Ask Marilyn*, in *Parade*. Public exposure resulted in a flurry of controversy in which opposing answers were offered with great passion and tenacity. Mathematicians began to produce papers purporting to sort things out; some of these came to the editor of the *College Mathematics Journal*. Before these were refereed, it seemed useful to find out what was already known about the problem, and with the help of several correspondents, I found several antecedents, notably a problem about three prisoners awaiting execution popularized by Martin Gardner in the 1950s: *Three prisoners, A, B, C are to be executed. The governor decides, at random, to pardon one of them and informs the prison warden. Prisoner A, aware of this, convinces the warden to tell him which of B and C will be executed (selecting at random if both are to be executed). What is the chance now that A will be pardoned?* In essence, the problem dated back to at least 1889 when Joseph Bertrand published a box problem in his book *Calcul des Probabilités*.

Some solvers of the Monty Hall problem were of the opinion the two unopened doors were equally likely to conceal the car. Others were equally sure that the door initially chosen gave a probability 1/3 of success and the remaining door 2/3; Monty just told you which of the doors might hide the car should you switch. The resolution you favour turns on whether you feel that Monty has provided any pertinent information, and if so, how this governs the eventual probability.

It is not surprising that this sort of probability problem engenders dispute. Since, in any given play of the game, the positions of the car and goats are given, probability is an artefact to handle the uncertainty. So we have to consider how this abstraction connects with reality. If we are to come up with useful advice as to whether one ought to stick or switch, a lot of spadework is needed to interpret the situation and decide what probability actually measures.

The book under review makes this analysis. The author delves into the problem with zest, considers it from different angles and treats generalizations. This book can be regarded as a treatise on probability in which the Monty Hall problem is a vehicle for introducing and clarifying the concepts. Indeed, the author tells us that "it is a recurring theme of this book that you can teach an entire course in probability theory using nothing more than variations on the Monty Hall problem". (p. 105) It is not quite selfcontained in this respect, although students in a first probability course could study it with profit.

The opening chapter sets the stage with a brief historical and philsophical sketch of probability, followed by an account of the origin and early renditions of the Monty Hall situation. Quite a bit of space is devoted to details of the controversy between Marilyn Vos Savant and her critics. Quotations from *Parade* and the *American Statistician* give a flavour of the acrimony in the exchange of opinions. The second chapter eases us into the problem. It opens with a statement of its "canonical version", where Monty is assumed always to open a door concealing a goat and does so with equal probability when he has a choice. There is an informal discussion of reasons that one might stick or switch. To work through the fog, we need a mathematical model, so the author provides a brief primer on probability before applying it to the canonical problem. A Monte Carlo simulation validates the conclusion that it is better to switch.

An important tool for treating the problem and its variants is Bayes' Theorem. This is the burden of the third chapter, where the reader is treated to a clear exposition of independent events, conditional probability, the law of total probability and the theorem itself. As an example, this result is applied to a concern of every air traveller: after I have waited for a long time at the baggage carousel, what are the chances that the airline has lost my case? (There is a small error in the middle of page 72, where one should consider the smallest integer x for which $P(A|B)$ is *larger* than $\frac{1}{2}$.) Now we can play with various scenarios and determine the best strategy when, for example, Monty opens one of the remaining doors at random (possibly revealing a car); Monty points to a door and says, possibly incorrectly, that it conceals a goat; the car is behind the doors with unequal probabilities; or Monty always reveals a goat, but opens the doors with unequal probabilities when given a choice.

The next two chapters introduce more variants and demand close attention from the reader to negotiate the mathematics. We can have more doors, more cars, more kinds of prize, more players or more hosts. For example, with many doors and one car, we can have Monty opening doors one by one until only the door chosen by the contestant and one other door remain closed. If the contestant is offered a chance to switch anywhere along the way, is it best to stick with the original choice, switch only at the very end, or follow some particular regime of sticking and switching? (At the bottom of page 103, the last inequality should read $P(\tilde{C}_i) \leq \frac{n-1}{n}$; at the top of page 116, two occurrences of the word "probability" should be replaced by "event"; on page 125, several occurrences of m should be replaced by n, the number of doors.)

There is some confusion at the beginning of Section 5.3 due to a couple of minor mistakes that do not affect the overall correctness of the analysis. However, as the author uses an alternative to the Baysian approach that was unfamiliar to me, it is worth describing the situation in detail, correcting the mistakes, using Bayes theorem as an example of how he treated cases elsewhere in the book, and then describing his treatment.

The setting is similar to the standard problem, except that there are two hosts available, Coin-Toss Monty (CTM) and Three-Obsessed Monty (TOM). The presiding host is selected by a coin toss. The contestant starts by selecting Door 1. If the car is behind this door, Monty is assumed to open Door 3 with probability q; $q = \frac{1}{2}$ for CTM and $q = 1$ for TOM. We suppose that Monty opens Door 3 to reveal a goat. For $i = 1, 2, 3$, let M_i be the event that Monty opens Door i and C_i the event that the car is behind Door i. Suppose that $p = P(C_1|M_3)$ and $q = P(M_3|C_1)$. Since $P(C_1|M_3)P(M_3) = P(M_3|C_1)P(C_1)$ and $P(M_3) = qP(C_1) + P(C_2) = \frac{1}{3}(1 + q)$, we have that $p(1 + q) = q$, so that $(p, q) = (\frac{1}{3}, \frac{1}{2})$ for CTM and $(p, q) = (\frac{1}{2}, 1)$ for TOM.

Turning to the situation at hand, the chances are even that the presiding host is CTM and TOM. In any case, $P(M_1) = 0$. Since CTM will open Door 3 with probability $\frac{1}{2}$ and TOB will open Door 3 with probability $\frac{2}{3}$, then $P(M_3) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} = \frac{7}{12}$ and $P(M_2) = \frac{5}{12}$. Similarly, $P(M_3|C_1) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$. Then $p \cdot \frac{7}{12} = \frac{3}{4} \cdot \frac{1}{3}$ so $p = \frac{3}{7}$. Thus the probability of winning by switching is $\frac{4}{7}$.

The author's treatment relies on a "proportionality principle": *If various alternatives are equally likely, and then some event is observed, the updated probabilities for the alternatives are proportional to the probabilities that the observed event would have occurred under those alternatives*". (p. 82) Basically, it is argued that if we see the host open Door 3, then the host is more likely to be TOM than CTM. Since the respective values of $P(M_3)$ are $\frac{2}{3}$ and $\frac{1}{2}$ for the two hosts, we conclude that we are $\frac{4}{3}$ more likely to see the host open Door 3 if he is TOM rather than CTM. "Thus we conclude that we have drawn Coin-Toss Monty with probability $\frac{3}{7}$ and Three-Obsessed Monty with probability $\frac{4}{7}$. Our probability of winning by switching will therefore be the following weighted average $\frac{4}{7} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{2}{3} = \frac{4}{7}$." (p. 117-118)

The apparent irrationality of people tackling this problem has caught the attention of psychologists. Even if you believe that it does not matter whether you switch, why, according to some investigators, do fewer than 20% of players elect to switch? It seems that switching from a winning position weighs more heavily than failing to switch from a losing position. A study by Bar-Hillel and Falk [**8**] aims to understand why intuition may be an unreliable guide. In his next chapter, the author reports in detail on one of their examples: *Mr. Smith, a father of two, is seen walking along the street with a lad who turns out to be his son. What is the probability that Mr. Smith's other child is also a boy?* Before this question can be answered, according to Rosenhouse, one has to consider not just the data but "the precise statistical experiment that led you to the data". The answer depends on whether you learn that it is the elder who is the boy, that a child chosen at random is a boy, or that one of the children is a boy.

Using the work of Falk, and of Shimojo and Ichikawa [**10**], the author discusses how the solver's implicit subjective principles may militate against a cogent analysis. He concludes that "it seems that there is something in our cognitive architecture that leads us to make fools of ourselves when discussing problems of this sort". (p. 145) Whatever this is seems to cut across cultural boundaries, as this behaviour is common to people the world over. Other research on cognitive issues is briefly sketched.

Finally, it is the turn of the philosophers. Two issues are discussed. The first is whether, as philosophers Paul Moser and D. Hudson Mulder hold, you can justifiably follow one strategy in a single play and a quite different one over many repetitions. The second turns on a variant with two contestants, where philosopher Peter Baumann claims that the solution violates the principle that two rational people with the same information who fully determine their rational degree of belief in a proposition, should assign the same probability to the proposition. The author, convincingly in my view, argues against both positions.

The author has not only thought deeply about this problem. He has gone carefully through the literature and provides us with an entertaining, multifaceted and thoughtprovoking study of this challenging problem.

References

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3. *ibid.* **27** (1996), 46.

4. *ibid.* **27** (1996), 205.

5. *ibid.* **28** (1997), 44.

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