COMPLEX NUMBERS

Definitions and Notation

A complex number has the form $x + yi$ where x and y are real numbers and $i^2 = -1$. They can be added, subtracted, multiplied and divided following the rules of ordinary algebra with the simplification that i^2 can be replaced by -1.

For real numbers represented on a number line, we can think of addition in terms of a translation along the line. For example, to add 2 and 5, the sum of 2 and 5 is represented by the point obtained by translating the point 5 by 2 units in the positive direction (or equivalently the point 2 by 5 units in the positive direction). Multiplication by a positive real corresponds on the line to a dilatation whose centre is at the origin. Thus, the product of 2 and 5 is the place where 2 ends up when the line has been expanded by a dilatation with factor 5. Multiplication by -1 corresponds to a reflection in 0, and multiplication by a negative number corresponds to the composite of this reflection and a dilatation whose factor is its absolute value.

This leads to a geometric representation of the complex numbers (Argand diagram). Since $1 \times -1 = -1$, we can think of the position of -1 as the result of applying a reflection about 0, or, equivalently on the line, a rotation of $180°$ about the origin to the line. Since -1 is the result of multiplying 1 by i twice, it is reasonable to represent i be the point on the plane which is the image of 1 after a counterclockwise rotation of $90°$ about the origin. Thus 1 corresponds to the point $(1, 0)$, i to the point $(0, 1)$ and -1 to the point $(-1, 0)$.

In general, we represent the complex number $x + yi$ with x and y real, by the point (x, y) in the plane. Addition of complex numbers corresponds to vector addition in the plane. The *absolute value* $|x + yi|$ is equal to $\sqrt{x^2 + y^2}$, which geometrically is the distance from 0 to $x + yi$ in the Argand diagram. The angle θ measure counterclockwise from the positive real axis to the segment joining 0 to $x+yi$ is called the *argument* of $x + yi$ and is denoted by arg $(x + yi)$. It is given in radians and is determined up to a multiple of 2π.

For a complex number $x + yi$, x is called the *real part* and denoted by Re $(x + yi)$, and y is called the *imaginary part* and is denoted by Im $(x + yi)$. The number $x - yi$ is called the *complex conjugate* of $x + yi$ and is denoted by $\overline{x + yi}$.

If $r = \sqrt{x^2 + y^2}$ and $\theta = \arg(x+yi)$, then $x = r\cos\theta$ and $y = r\sin\theta$ and we get the polar representation of the complex number:

$$r\cos\theta + ir\sin\theta\ .$$

Exercises

In these exercises, we can use the standard notation $z = x + yi = r(\cos\theta + i\sin\theta)$ and $w = u + vi = s(\cos\phi + i\sin\phi)$ unless otherwise indicated.

1. Prove that $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\overline{w}$ and $\overline{\overline{z}} = z$. (Otherwise stated, this says that the operation of complex conjugation is an isomorphism and an involution, *i.e.*, it preserves the arithmetic operations of complex numbers and is a transformation of period 2.

2. Prove that $|z|^2 = z\overline{z}$ and deduce that, for any nonzero complex number z,

$$z^{-1} = \frac{\overline{z}}{|z|^2}\ .$$

   Provide a geometric interpretation of the mapping $z \longrightarrow z^{-1}$.

3. Prove that

$$\text{Re } z = \frac{1}{2}(z + \overline{z}) \le |z|$$

and

$$\text{Im } z = \frac{1}{2i}(z - \overline{z}) \leq |z| \ .$$

4. Prove that $|zw| = |z||w|$ and that $|z+w| \leq |z|+|w|$. The latter inequality can be obtained algebraically by expressing $|z+w|^2$ as $(z+w)(\overline{z}+\overline{w})$, multiplying out and observing tht $z\overline{w}$ is the complex conjugate of $\overline{z}w$.

5. Prove that

$$\frac{1}{\sqrt{2}}(|x| + |y|) \leq |z| \leq (|x| + |y|) \ .$$

6. Prove that

$$[r(\cos\theta + i\sin\theta)][s(\cos\phi + i\sin\phi)] = rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) \ .$$

Deduce that arg $zw \equiv$ arg $z +$ arg w modulo 2π and give a geometric interpretation in the complex plane of the product of two complex numbers z and w.

7. Prove, for integers n, de Moivre's theorem:

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n \ .$$

Use this result to obtain $\cos k\theta$ and $\sin k\theta$ as polynomials in $\cos\theta$ and $\sin\theta$ for $k = 2, 3, 4$.

8. Let n be a positive integer. Describe all the solutions of the equation $z^n = 1$. Such solutions are called *roots of unity*, and those that are not roots of unity of lower degree are called *primitive roots of unity.* Determine all the nth roots of unity for $1 \leq n \leq 8$. Use your results to find a complete factorization of $z^n - 1$ as a product of polynomials with rational coefficients.

9. Let $p(z)$ be a polynomial in the complex variable z with real coefficients. Prove that

$$\overline{p(z)} = p(\overline{z})$$

and deduce that if r is a root of $p(z)$ then so is its complex conjugate $\overline{r}$. Explain why every polynomial with real coefficients and odd degree must have at least one real root. Provide an example to show that these assertions are not necessarily true when a polynomial has at least one nonreal coefficient.

10. The quadratic polynomial equation

$$az^2 + bz + c = 0$$

can be written as a system of two real equations in two unknowns by making the subsitution $z = x + yi$ and equating the real and imaginary parts to 0.

(a) Suppose that a, b, c are real numbers. Sketch the graphs of the two real equations obtained, indicate where the solutions of the equation are in the Argand plane and discuss the situation when the discriminant $b^2 - 4ac$ is positive, negative and zero.

(b) Make a similar analysis for the equation $z^2 - iz + 2 = 0$.

11. The function $f(\theta) = \cos\theta + i\sin\theta$ satisfies the equation $f(\theta+\phi) = f(\theta)f(\phi)$ and $f(0) = 1$, which makes it look like an exponential function. In fact, this is precisely what it is. Because this function satisfies the differential equaion $f''(\theta) = f(\theta)$ as well as the "initial" conditions $f(0) = 1$ and $f'(0) = i$, we can write it in the form $e^{i\theta}$, where e is the base of the natural logarithms, a number that lies between 2 and 3. However, for our purposes, we can leave this at the formal level.

Sum the geometric progression

$$\sum_{k=0}^{n} e^{(i\theta)k}$$

and equate the real and imaginary parts of this sum to the real and imaginary parts of the closed from of the sum that you get to obtain an expression for the sums of the following trigonometric series:

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta$$

and

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos n\theta \ .$$

Check that your expressions for these sums are correct when $n = 1, 2, 3$.

12. Establish the identity

$$\cos 7\theta = (\cos\theta + 1)(8\cos^3\theta - 4\cos^2\theta - 4\cos\theta + 1)^2 - 1$$

and deduce that the three roots of the polynomial

$$8z^3 - 4z^2 - 4z + 1$$

are $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$. By using the relationship between the coefficients and roots of a polynomial, obtain three equations satisfied by these roots.

Problems on Complex Numbers

These problems can be solved by complex techniques and you should do so. However, if you can solve them some other way, compare your solution with the complex one with respect to naturalness, ease of understanding and the insight it gives into the situation.

1. Using complex multiplication, show that the product of two integers that are equal to the sum of two squares is also equal to the sum of two squares. Use this to write 85 as the sum of two squares in two different ways.

2. Using complex numbers, prove that the angle subtended at the circumference of a circle by a diameter is right.

3. Some pirates wish to bury their treasure on an island. They find a tree T and two rocks U and V. Starting at T, they pace off the distance from T to U, then turn right and pace off an equal distance from U to a point P, which they mark. Returning to T, they pace off the distance from T to V, then turn left and pace off an equal distance from V (to TV) to a point Q which they mark. The treasure is buried at the midpoint of the line segment PQ.

Years later, they return to the island and discover to their dismay that the tree T is missing. One of them decides just to assume any position for the tree and carry out the procedure. Is this strategy likely to succeed?

4. Let ABC be a triangle and P any point in its plane. Let P_1 be the reflection of P in A, P_2 be the reflection of P_1 in B and P_3 be the reflection of P_2 in C. Suppose that I is the midpoint of the segement PP_3.

(a) How does the position of I depend on P?

(b) Is it possible for the points P and P_3 to coincide? Justify your answer.

5. For nonzero complex numbers z and w, show that

$$(|z| + |w|)\left|\frac{z}{|z|} + \frac{w}{|w|}\right| \le 2(|z + w|) \ .$$

6. Determine the set of complex numbers z that satisfy each of the following equations:

(a) Re $(wz) = c$, where w is a fixed nonzero complex number and c is a fixed real number.

(b) $|z| = k|z+1|$ where k is a fixed positive real number.

(c) $|z-u| + |z-v| \leq k$ where u and v are fixed distinct complex numbers and k is a positive real number.

(d) Im $(z^4) = (\text{Re } (z^2))^2$.

7. Describe those triangles with vertices at the points z_1, z_2, z_3 in the complex plane for which

$$(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_1)^2 = 0 \ .$$

8. Evaluate

(a) $\cos 5° + \cos 77° + \cos 149° + \cos 221° + \cos 293°$.

(b) $\sin 10° \sin 50° \sin 70°$.

9. (a) A regular pentagon has side length a and diagonal length b. Prove that

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} = 3 \ .$$

(b) A regular heptagon (seven equal sides and equal angles) has diagonals of two different lengths. Let a be the length of a side, b the length of a shorter diagonal and c the length of a longer diagonal of a regular heptagon. Prove that

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} = 6$$

and

$$\frac{b^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{c^2} = 5 \ .$$

(c) Can the results of (a) and (b) be generalized?

10. Suppose that z_1, z_2, z_3, z_4 are four distinct complex numbers for which there exists a real number t not equal to 1 such that

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| \ .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.