Impossible? Surprising Solutions to Counterintuitive Conundrums

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Reviewed by **Edward Barbeau**, *University of Toronto, Toronto, ON*

Julian Havil, a master at Winchester College in England, is a type of mathematics teacher that is all too rare, a well-read connoisseur of his subject who is eager to explore its byways and share his discoveries with his pupils and the public at large. (Another such was F.J. Budden, whose fine book *Fascination of Groups*, published by Cambridge, dates back to 1972.) This is Havil's third book, after *Nonplussed! Mathematical Proof of Implausible Ideas* and *Gamma: Exploring Euler's Constant* (2003), both published by Princeton. There is a lot to savour, and Havel has both the understanding and technical proficiency to convey it to his readers.

This book cannot be read casually. It is especially suitable for mathematics undergraduates, but only for the most capable secondary students and teachers. Professional mathematicians will be familiar with many of the topics, but will undoubtedly find something they have not yet encountered, particularly the puzzles that have recently come onstream.

While most of the topics date from the last century, he does reach back in time for some of the discussion. The infinitely long "trumpet" of Torricelli, with its infinite volume and finite surface area provides the occasion for a discussion of the divergence of the harmonic series and the convergence and evaluation by Euler of the sum of the series of square reciprocals. The study of the length of a run of heads in a sequence of coin tosses includes an account of the work of DeMoivre, an early investigator of the problem. A discussion of the frequency of poker hands is extended to include the effect of wild cards.

There are three enjoyable chapters for those with an affinity for numbers, especially large ones. A treatment of the occurrence of sequences of digits in powers of 2 leads smoothly into Benford's Law, to wit that in a "natural" set of numerical data, the numbers start with the smaller digits more frequently than the larger. A chapter on Goodstein Sequences packs a surprise. We start the sequence with a number written in the *complete base* 2 representation; we start with the standard base 2 representation and write all exponents that occur in base 2. Thus

$$2136 = 2^{2^{2+1}+2+1} + 2^{2^2+2} + 2^{2^2} + 2^{2+1} \ .$$

The Goodstein sequence starting with n is defined for $r \geq 2$ by $G_2(n) = n$ (written in complete base 2). For $r \geq 3$, $G_r(n)$ is obtained by taking $G_{r-1}(n)$, a number written in complete base $r-1$, changing all the occurrences of $r-1$ to r, subtracting 1 and adjusting the result to get a proper complete base r representation. Thus

$$G_3(2136) = 3^{3^{3+1}+3+1} + 3^{3^3+3} + 3^{3^3} + 2 \times 3^3 + 2 \times 3^2 + 2 \times 3 + 2 \ .$$

With this continual bumping up of the base, one might expect the terms of the sequence to keep increasing rapidly, which indeed they do to begin with. But eventually, they crash to 0 in finite time. This result turns out to be unprovable in ordinary arithmetic, and, indeed, Goodstein established it in 1944 by appealing to the well-ordering of the transfinite ordinals.

Geometry has a small but interesting niche. A brief look at Euclid's axioms leads to a discussion of Cantor cardinality and situations where the whole need not be greater than the part, such as the equipollence of the closed unit interval and $n-$dimensional Euclidean space. A chapter treats in detail the solution of the Kakeya problem where a needle can be reversed in direction within an arbitrarily small area. The particularly fine last chapter conveys the essence of the argument behind the Banach-Tarski Paradox.

Other paradoxes make an appearance. Within a discussion of the complex numbers, we learn about the reality of i^i and determine logarithms of negative numbers. Simpson's paradox is analyzed, although the insight offered by the observation that a mediant of two positive rationals (obtained by adding the numerators and the denominators) depends on their particular representations and can lie anywhere in the intercal between them, would have helped. Probably less familiar to the reader would be Braess' paradox that the addition of an additional road in a town may actually worsen the congestion.

Problems of recent notoriety are treated in detail: car and goats (Monty Hall), placement of coloured hats, interactions between two individuals to identify a number pair, chance of getting an elevator in your direction. These raise delicate issues in logic, probability and coding theory that are competently handled. The reader will want to check out two card tricks that exemplify the Kruskal and Gilbreath principles, the latter based on the structure preserved in a deck of cards by an imperfect riffle shuffle.

The author analyzes the phenomena in some depth without being tedious. The book is lightened by anecdote and historical asides; it is well-referenced, so that the reader can go to the literature. An interesting feature of the book is the set of illustrations that accompany each of the eighteen chapter headings. There are seventeen different symmetry groups for wallpaper patterns; the basic pattern is presented with Chapter 1 and each of the remaining chapters gives a fundamental region for one of these groups.

While the author dedicates the book to Martin Gardner, the treatment is more overtly mathematical than what Gardner would provide. A better comparator is Sherman Stein's *How the other half thinks; adventures in mathematical reasoning* (McGraw-Hill, 2001) or the essays of Ross Honsberger. In any case, this book makes a pleasant and absorbing read.