**Rates; Percentages; Ratio and proportion**

**Problems for elemtnary teachers and their pupils**

R.1. Two old women started walking at sunrise and each proceeded at a constant speed. One went from Town A to Town B, while the other went from Town B to Town A. They passed each other at noon, and continued on their ways without stopping. The first arrived at B at 4 in the afternoon while the second arrived at A at 9 in the evening. At what time was sunrise on this day? (Russian problem)

R.2. The following problem was given to Chinese elementary students:

*Farmhands need to hoe up weeds in two fields. The larger field is twice the size of the smaller one. All of the farmhands work in the larger field until noon, when they are divided up into two groups. Half the hands remain in the larger field and just have sufficient time to finish it, while the other half move to the smaller field and leave a small area unfinished at the end of the day. One farmhand returns to the field the next day and just finishes it in the day. How many farmhands are there on the farm?*

(a) The posing of this problem may leave some ambiguities or make some unstated assumptions. If you feel that this is so, point out where you think the statement might be unclear, discuss possible interpretations and then repose the problem so that all of the ambiguities are resolved in a reasonably way and all unstated assumptions are made explicitly. (You might ask yourself what the proposer intended.)

(b) Solve the given problem (in its revised form if you have changed it). Make sure that you check that your solution is correct.

(c) Discuss possible ways in which this problem might be solved. For example, is it necessary to use algebra? Would manipulatives be of any use?

(d) For what age of pupil would this problem be suitable? (You might experiment on the youth within your orbit and see what happens.)

(e) What do you think of this as a school problem? Does the "real world" criterion have any relevance in your judgment?

R.3. The following three problems are taken from *The Treviso Arithmetic* published in 1478. Note that there are 24 grossi in a ducat.

(a) Three merchants have invested their money in a partnership, whom to make the problem clearer I will mention by name. The first was called Piero, the second Polo, and the third Zuanne. Piero put in 112 ducats, Polo 200 ducats, and Zuanne 142 ducats. At the end of a certain period they found that they had gained 563 ducats. Required is to know how much falls to each man so that no one shall be cheated.

(b) Two merchants, Sebastiano and Jacomo, have invested their money for gain in a partnership. Sebastiano put in 350 ducats on the first day of January, 1472, and Jacomo 500 ducats, 14 grossi on the first day of July, 1472; and on the first day of January, 1474, they found they had gained 622 ducats. Required is the share of each.

(c) Three men, Tomasso, Domenego and Nicolo, entered into partnership. Tomaso put in 760 ducats on the first day of January, 1472, and on the first day of April took out 200 ducats. Domenego put in 616 ducats on the first day of February, 1472, and on the first day of June took out 96 ducats. Nicolo put in 892 ducats on the first day of February, 1472 and on the first day of March took out 252 ducats. And on the first day of January, 1475, they found that they had gained 3168 ducats, 13 grossi and 1/2. Required is the share of each so that no one shall be cheated.

R.4. A man wishes to buy a coat normally priced at \$135. However, the store he patronizes has it on sale for 25% off. The clerk takes the normal price of \$135, adds on the 15% sales tax, and then applies the 25% discount to the whole amount.

However, the customer objects that he is being overcharged. He tells the clerk that she should apply the discount first to the \$135, and then apply the sales tax to the discounted price. Is the customer correct?

R.5. Consider the following problems:

*(i) How long does in take for sunlight to reach the earth? (If the sun were suddenly to blow up, this is how long it would take us to find out about it.)*

*(ii) An electronic signal sent from the earth to the moon is reflected back. How long a time will elapse between the sending and receiving of the signal.*

*(iii) An electronic command is send to a small remote-controlled vehicle trundling along the surface of Mars. How long will it take for the command to reach the vehicle?*

(a) What information is necessary in order to answer these three questions? How do you handle the fact that the distances between various pairs of celestial bodies is not constant? Does this have any implications on the form in which your answer should be given?

(b) Look up and record all the information you need to answer these questions.

(c) Answer the questions.

(d) Comment on the possible use of such questions in school.

R.6. The following problem is from a textbook for the fifth and sixth grades in Singapore. Explain how the problem can be solved without recourse to algebra.

*Susan had 2/3 as much money as Mary at first. After Mary gave half of her money to Susan, Susan had $175. How much money did Susan have at first?*

R.7. The ratio of the number of Peter's stamps to the number of Henry's stamps was 2:3. The ratio becomes 5:6 when Peter bought another 8 stamps. How many stamps did Peter and Henry have at first?

R.8. The Holy Father sent a courier from Rome to Venice, commanding him that he should reach Venice in 7 days. And the most illustrious Signora of Venice also sent another courier to Rome, who should reach Rome in 9 days. And from Rome to Venice is 250 miles. It happened by order of these lords that the couriers started their journeys at the same time. It is required to find in how many days they will meet, and how many miles each will have travelled. (*The Treviso Arithmetic*, 1478)

R.9. If a cardinal can pray a soul out of purgatory, by himself, in an hour, a bishop in three hours, a priest in five, and a friar in seven, how long would it take them to pray out three souls from purgatory, all praying together? (*The Scholar's Guide to Arithmetic*, 6th edition, Bonnycastle, 1795)

R.10. When my children were teenagers, they naturally regarded me as being somewhat old and decrepit. I told them, that in fact, they were getting older faster than I was. "Just look at it," I said, "every passing year makes you about 6% older. I only age about 2% with each passing year." Is there any merit to this point of view?

R.11. There is a saying that you might be familiar with, to wit that each dog-year is equivalent to about seven human-years. Some of you may have a pet dog (those without can substitute a cat, or use the animal provided below). Your dog will be aging at seven times your rate. Find out on which date your dog's age will be exactly equivalent to yours.

(If you do not have an animal, let me introduce you to Spot VII, who was born on April 1, 1996. Spot belongs to a long pedigree of animals, noted for their signal contribution to the cause of elementary education. Spot I, the founder of the line, used to hang out with a pair of siblings called Dick and Jane, who are in fact still alive and residing in the Grand Sunset Home. They continue to care for this line of distinguished animals; they have now learned to stoop-and-scoop.)

R.12. One day, at 9 am, Tom set out on a bike hike from his Saskatoon home, riding across the flat prairie at a uniform speed of 20 km per hr. Some time later, his mother observed that he had forgotten his lunch, and sent Dick after him with lunch; Dick rode his bike at a uniform rate of 30 km per hr. Shortly thereafter, dark clouds formed in the sky, so exactly a half hour after Dick left, their mother sent Harry after the two boys with raingear. Harry rode his bike at a uniform rate of 40 km per hr. It turned out that the three boys met at exactly the same time. What time was that?

R.13. Study the following passage and then answer these questions:

(a) Justify all of the figures given in the passage. Make a table or chart indicating the percentage of residents of Toronto across the two categories of citizenship and colour.

(b) What argument is counsel making in opposing the application presented by Rosenthal? Is this a valid argument? Comment.

**A case of black and white - but not so much black.**

Peter Rosenthal of the University of Toronto in Ontario is both a mathematician and a lawyer. Recently, he represented an applicant who made representation to Ontario Court of Justice concerning a constitutional challenge to a jury panel. In Ontario, only citizens can be selected for jury duty; it is not enough to be a permanent resident. Since the proportion of citizens that are black is less than the proportion of the population that is black, the citizenship requirement has a negative impact on the probability of choosing black persons as jurors. Rosenthal argued that the citizenship requirement was disciminatory within the meaning of the Canadian Charter of Rights and Freedoms. This application was opposed by the Her Majesty the Queen, in the form of a counsel for the Attorney-General of Canada, in the following lines:

> Furthermore, it is submitted that the citizenship requirement does not result in a pool of jurors in which blacks are differentially excluded to the extent that a representative jury cannot be obtained in Metropolitan Toronto. It is submitted that the difference in proportions as between black citizens and all blacks, and non-black citizens and all non-blacks (65.9% for blacks and 85.6% for non-blacks), is not such that the Applicants will be unable to have a realistic opportunity to have a panel which will include blacks. This becomes clear when one compares the percentage that blacks are of the total Metropolitan Toronto populations, 4.1%, with the percentage that black citizens are of the Metropolitan Toronto population who are citizens, 2.7/84.8 or 3.2%. In short, the citizenship requirement for jury service results in a pool for the array in which 3.2% of the available jurors are black, which is nearly the same proportion that blacks, citizens and non-citizens combined, are in the total population of Metropolitan Toronto, that is 4.1%. In fact, having regard to the difference in size of the black and non-black *non-citizen* groups, 1.4% and 13.8% respectively of the Metropolitan Toronto population, the inclusion of non-citizens in the array would probably result in *fewer* blacks being selected because a greater number of non-blacks would be available in the expected jury pool than blacks.

The application failed; the decision is being appealed to a higher court.

R.14. Consider a standard analogue (face) twelve-hour clock with an hour hand and a minute hand. For a properly set clock, as the hands turn, the relative positions of the two hands are related so that some positions are *attainable* and other positions are *not attainable*. For example, the hour hand pointing exactly at 3 and the minute hand at 12 is attainable, but a reversal the position of the hands is not attainable.

(a) For how many different attainable positions of the hands are they both pointing in the same direction? Give the exact time for three different positions of this type.

(b) For how many different attainable positions of the hands do you get an attainable position by reversing the hands? For two of such positions, give the exact time corresponding to the position and its reversal.

(c) Imagine that the positions of the numbers on the clock are given by dots. The clock is reflected in a mirror, so that it still looks like a proper clock. For how many attainable positions of the hands is the mirror reflection also attainable? For two of these, give the exact time corresponding to the position and its reflection. (There are clocks on the market that can be read by looking in a mirror.)

R.15. A Toronto-Dominion Bank ad compares the fortunes of Natalie and Ted who make a monthly payment of \$100 into an RRSP account for a period of years. The details are as follows:

Natalie invests \$100 per month from the beginning of her 23rd year until the end of her 31st year for a total investment of \$10800. With money accumulating at an interest rate of 8% per annum, she has an amount of \$236800 at the end of her 65th year.

Ted invests \$100 per month from the beginning of his 31st year until the end of his 65th year for total investment of \$42000 at a rate of 8% per annum. This accumulates to \$229400 at the end of his 65th year.

(a) Account for the amounts \$10800 and \$42000.

(b) Are you surprised at the amounts given for the accumulation at the end of the 65th year? Why?

(c) What does it mean for an investment to draw compound interest of 8% per annum? What would an investment of \$1 now be worth at the end of two years? at the end of three years?

(d) Actually, the ads do not make clear what the compounding is. For example, 8% compounded annually corresponds to an accumulation of an additional 8% of the principal at the end of each year. However, a rate of 8% per annum compounded semi-annually corresponds to a rate of 4% per six months compounded. A rate of 8% compounded monthly corresponds to a rate of $8/12\%$ per month compounded. If \$1 is invested at a rate of $8/12\% = 2/3\%$ per month compounded, what does it accumulate to at the end of 1 year.

(e) At the end of each month, \$100 is paid into an account with an interest rate of 8% per annum compounded monthly. Make up a table which indicates the total accumulation of the money at the end of the following periods: 1 month, 2 months, 3 months, 4 months, 5 months, 6 months, 1 year, 10 years, 35 years.

(f) Try to explain the accumulations in the ad. (*Note:* I was unable to reproduce the exact figures but came close with an assumption of interest compounded monthly.)

(g) Suppose that Natalie and Ted had their 22nd birthdays on the same day. Sketch a graph of the amounts that they would possess from their investments (including interest) at the end of each year up to the end of their 65th years if they had made the investments described in the ads. For comparison, sketch a graph of what they would possess if each of them had simply stitched the $100 into a mattress.

(h) Describe how you made your computations (tool: pencil and paper, type of calulator, computer, book of tables; method: mathematics or program used).

(i) Discuss any difficulties that you had in approaching this problem and how you handled them. Did you have a reasonable understanding of compound interest at the outset? Did you understanding evolve in doing this question? What questions in your mind are still outstanding?

(j) Discuss the mathematical background and level of maturity you feel is necessary in order for a school pupil to do this type of question? If you feel that a certain modification involving similar ideas could be introduced earlier, discuss how you would do this. What grade level would be appropriate?

R.15. The ideal (Pythagorean) standard major musical scale (doh, re, mi, fah, soh, la, ti, doh) is such that the pitches of the sounds are in the ratio:

$$24 : 27 : 30 : 32 : 36 : 40 : 45 : 48 \ .$$

The intervals "doh-fah" and "soh-doh" are called perfect fourths, in which the pitch of the upper sound divided by that of the lower term yields the ratio $4/3 = 32/24 = 48/36$. Each of these perfect fourths is the composite of three basic intervals, the whole tones with ratios 9:8 and 10:9, and the semitone with ratios 16:15. This can be expressed in terms of fractions by the following relationship:

$$\frac{4}{3} = \frac{9}{8} \times \frac{10}{9} \times \frac{16}{15} \ .$$

In Greek musicology, the basic intervals corresponded to ratios in which the antecedent was one more than the consequent, or, if you like, fractions (called *epimoric*) whose numerators were one more that their denominators.

These musical considerations motivate the following questions:

(a) In how many different ways can you express the fraction 4/3 as a product of three epimoric fractions; the heading of this problem gives you one of them.

(b) Make up a table expressing each of the following fractions as a product of two epimoric fractions in as many ways as you can find: 2/1, 3/2, 4/3, 5/4, 6/5, 7/6, 8/7, 9/8. Do you think that, in each case, there are only finitely many possible representations? Why?

R.16. In Dante's *Paradise*, the third book of his *Divine Comedy*, the poet encounters the shade of his great-great-grandfather, Cacciaguida, in the sphere of Mars. In the following two tercets, Cacciaguida tells Dante when he was born. "Ave" is a reference to the Annunication of the birth of Christ described in Luke 1:28; this would occur about 4 BCE. The fiery star referred to is Mars; it was known that the planet returned to the constellation Leo (Lion) every 687 days. Using this information, determine the approximate year of Cacciaguida's birth, and decide whether this is reasonable, knowing that the poem recounts Dante's journey as a pilgrim through Hell, Purgatory and Paradise in the year 1300 when he was 35 years old.

His light said, "From the day 'Ave' was said
to that on which my mother, now a saint,
heavy with child, gave birth to her son,

"to its own Lion this fiery star returned
five hundred and fifty times and thirty more
to be rekindled underneath its paw."

*Canto XVI, 34-39*