MATHEMATICS IN THE MODERN WORLD

Monday, November 4, 2013

THE TAKEAWAY

The role that mathematics plays in society.

1. Exact answers in situations where mathematics is directly involved in defining its terms. This includes commercial transactions (buying, selling, mortgages, bonds and investment certificates where the terms and interest rates are defined precisely), measurements.

2. Modelling of situations that involve external factors that cannot be described in mathematical terms. In this case, we need to describe how the mathematics to be used is related to the elements of the situation, and often involves defining concepts that help us crystallize what is at stake. This includes physics, chemistry, economics, biology, for example. The model is subject to continual updating as we check how it relates to reality.

3. As a tool to deal with uncertainty and risk. Probability and statistics are the areas of mathematics involved here. Applications include finanacial instruments, pension plans and annuities, insurance and polling.

4. As a means of isolating the characteristics of various alternatives and clarifying the criteria for choosing among these. This role is played in political science where we have to decide on means of electing parliaments or where we have to evaluate some policy alternatives. Another recent area of study is that of *Fair division.* The important point here is that mathematics opens the discussion, but the final decision will likely rely on nonmathematical considerations which may be ethical, political, cultural or efficiency.

Introduction. There is hardly any area of human endeavour that mathematics does not have any sort of role. Apart from commerce and engineering, and applications is sciences such as physics and chemistry, mathematics is also involved in political matters, biology, the making of decisions and fair division of assets.

Mathematics is generally applied for its precision, but this is a bit misleading. Broadly speaking, there are two ways in which mathematics is applied. In situations where mathematics is the basis for defining the situation, it does give exact answers. However, in situations where we cannot exert control over the context, mathematics serves as a kind of metaphor for reality. We make what is known as a mathematical model, where we make assumptions that will allow the mathematics mirror reality sufficiently closely as to be useful.

Suppose that you have a sum of money to be invested. If you invest this in a bond or a guaranteed investment certificate, then the entire transaction can be defined in mathematical terms; the interest and term of investment are assigned, and we can say exactly what the proceeds of the investment will be.

However, if we wish to invest in a mutual fund or the stock market, how that investment fares depends on external factors that are not known to us. To decide whether the investment is wise, we need to rely on assumptions and calculate a range of risk. The basic assumption that is that the future will be to some extent similar to the past, so we obtain whatever information we can about how the market has behaved in similar circumstances, and assess the quality of the current operation of the mutual fund or firms we wish to buy stock in. As new information becomes available, we may revise our model and our corresponding behaviour.

I wish to look at a couple of situations to illustrate the role mathematics plays in modern society.

Issues of politics. The US Constitution provides that the States should be represented prortionately (with respect to their populations) in the House of Representatives, with the proviso that each State should have at least one congressman. On the face of it, this should be an easy problem to solve. Simply multiply the number of seats available in the House by the population of the State divided by the total population of the whole country to get the apportionment for each State.

However, when we do this we generally get numbers that are not integers, and the question arises as to what to do with the fractional amounts. Let us look at an example:

We have five wards, A, B, C, D, E, whose populations are indicated and we have to assign 25, 26 or 27 seats on the town council. The total population of the five wards is 26000. We must agree on a systematic method of how many seats are assigned to each ward. Consider the following: *first assign the integer past of the proportion to each ward; this will not fill up every seat, so we look at the fractional part and assign seats available the ward with the largest fractional part, then the next largest, and so on, until the available seats are exhausted.*

Ward	Population	25 seats	26 seats	27 seats
A	9061	8.713 (9)	9.061 (9)	9.410 (9)
B	7179	6.903 (7)	7.179 (7)	7.455 (8)
C	5259	5.507 (5)	5.259 (5)	5.461 (6)
D	3318	3.191 (3)	3.319 (4)	3.447 (3)
E	1182	1.137 (1)	1.182 (1)	1.227 (1)

This example illustrates a problem with this method of apportionment. One would expect that if we increase the number of seats in the council, while the population figures remain unchanged, that each ward would have at least as many seats as before. But if we look at Ward D, an increase of one seat in the council chamber from 26 to 27 leads to this ward getting one seat fewer. This is a phenomenon known as *the Alabama paradox*.

There are other apportionment schemes that might be tried. For example, for the 26-seat council, since the total population is 26000, there are 1000 citizens for every seat. Dividing the ward polulations by 1000 and taking the integer part gives as we see too few seats being filled. This time, we reduce the figure we divide the populations by until the integer parts give the number of total number of seats that we need.

Here is how the figures work out this time.

26000/(number of seats)		1040	1000	963
Divisor used		1006	906	897
Ward	Population	25 seats	26 seats	27 seats
A	9061	9.01 (9)	10.00 (10)	10.10 (10)
B	7179	7.14 (7)	7.92 (7)	8.00 (8)
C	5259	5.23 (5)	5.80 (5)	5.86 (5)
D	3318	3.30 (3)	3.36 (3)	3.70 (3)
E	1182	1.17 (1)	1.30 (1)	1.32 (1)

Notice that this method gives a different apportionment than before. It is not an issue whether one is right and the other wrong, because one cannot really be definitive about this. All that we ask is that the method be reasonable. How can we test for reasonableness? For once thing, we can ask that the Alabama Paradox should not occur. Another condition that is generally applied is called *quota*. This means that when we use proportion to work out the number of seats and get a fractional number, the system we uses gives either the largest integer below or the smallest integer above it.

It turns out that for any system, you cannot have both these desirable features at the same time. If you get quota, then there will be situations which give rise to the Alabama Paradox, and if you avoid the paradox, then there will be situations in which quota does not occur.

In an appendix, we give a further system. In this case, we assign seats one at a time. At each stage, we divide the ward populations by one more than the number of seats already assigned, and then assign the new set to the ward with the largest quotient.

A very similar system arise when we wish to elect members of parliament. The problem is to meld a plethora of individual choices into a collective one that authentically represents the preferences of the collective.

Making policy. About thirty years ago, Thomas Saaty invented the *Analytic Heirarchy Process.* This was designed to aid policy makers in deciding among a number of options. The idea was that while it is difficult to adjudicate among many alternatives, it is easier to compare just two possible courses of action. So the policy maker begins by looking at each pair of options and assigning a numerical measure of his subjective idea of how much better one is that the other. Then a little bit of linear algebra is applied to provide an overall ranking of the options. While it would likely to be foolish simply to rely on the mathematics to make the final decision, it does put something on the table to begin with and may help the discussion proceed more pointedly.

Matchmaking. In 2012, the Nobel Prize in Economics was awarded to Lloyd Shapley and Alvin Roth for their work in game theory. Economics and games may seem an odd juxtaposition, but both involve rational actors making decisions in order to maximize some outcome. We can use the simpler structure of games to crystallize ideas that have significance in modelling economics. One of the publications cited by the Swedish Royal Academy of Sciences was a 1962 paper by David Gale and Lloyd Shapley entitled "College Admissions and the Stability of Marriage", published in the *American Mathematical Monthly*, a widely read expository journal. This paper is nice illustration that mathematical progress does not always involve deep and technical work, but sometimes simple results where the achievement is recognition of its applicability in an important area and its ability to crystallize seminal ideas. It does not involve any equations nor require any background knowledge. It is therefore accessible to ordinary citizens.

Imagine a village in which there are equal numbers of men and women who have to be married off. The matchmaker has to ensure that the resulting marriages are stable; that is, there is never a situation in which a man prefers some woman over his assigned wife while at the same time this woman prefers him over her assigned husband. In other words, any attempts to defect from a marriage are rebuffed. Such a set of marriages is described as *stable.* (You can sede that college admissions involve a more complicated form of the same thing: candidatee apply to colleges and you want to end up with an assignment where no candidate will be able to turn down an offer to accept one from elsewhere.) Gale and Shapley at first wondered whether such an assignment was indeed possible, but were able to devise a procedure that would achieve it.

The matchmaker asks each man to list in strict order of preference all the women, and each woman to do likewise with the men. The matchmaking process proceeds in a number of rounds. In the first round, each man proposes to the woman at the top of his list. If every woman receives a proposal, the marriages are made and each man, having his first choice, will be faithful. However, if not every woman gets a proposal, then some will have more than one proposal. A woman receiving at least one proposal will keep on a string the one she prefers the most and reject all the others. The rejected men will participate in the second round. In this round, each one of them strikes from his list the woman who has rejected him and proposes to the next preferred. Upon receiving a proposal, a woman looks over all the prospects, including anyone that might be on her string, and accepts the most preferred, rejecting all the others. We go on to round three, in which all the loose men strike off their lists those who have rejected them and propose to their next choices. This process continues for as many rounds as necessary for there to be no further rejections.

We need to establish two things. First, we have to be sure that the process terminates. Secondly, we have to argue that it produces a set of stable marriages. For the first, note that there are finitely many names on all the lists. Every time we need a new round, it is because someone's name gets crossed off of a list. This cannot go on forever. When the last round is reached, each man has finally been accepted by a different woman.

Now we come to the meat of the situation. Suppose that, say, Al is married to Ann, and that Bob is married to Barb. Suppose that Al prefers Barb to his own wife. Then Barb would have been higher on Al's list than Ann, and so Al would have proposed to Barb and been rejected by her before proposing to Ann. Why would Barb have rejected Al? Either she would have already accepted Bob or Bob would have come along later, supplanting any other suitors. In either case, she would have preferred Bob to Al, and so has no incentive to defect.

That's all there is to it. The assignment is not necessarily the only one possible. For example, there is a symmetrical process in which the women do the proposing, and this will in general lead to a different assignment. Notice that the success of this assignment depends on the assumption that the preference orders, once made, are never alterned. Of course, in real life, things are not so cut and dried.

Fair division. Suppose that we wish to slice a unifrom cake (same consistency throughout, with no icing) among several people so that each is convinced that she is getting a fair share. For two people, the process is as follows. The first person slices at a place that he feels divides the cake fairly into two. The second person then selects one of the portions, leaving the other for the slicer. The first person should be happy, as he feels that both portions are of equal worth, the second, as he feels that if the portions are different, he has selected the larger one.

Around 1960, mathematicians began to wonder whether there was a method of dividing a cake among three or more people in such a way that there are fininely many slices and each is satisfied that she has at least a fair share. In fact, one might try to impose a stronger condition that the division be *envy-free* in that each recipient not only feels she has received at least a fair share, but does not believe that anyone else's share is preferable to her own.

This has given rise to an area of mathematics with all sorts of practical applications from divorce setlements to splitting of assets and distribution of inheritances. I will not go further into this; you can google it under "Fair division". However, you might think about how you would divide a cake among three people. This problem is solved by the "Selfridge-Conway Fair-Division Procedure" which you can also google.

Climate science. I will not go into detail about this, but because of the controversies surrounding this area, climate and ecological modelling raises some important issues. Here we have a system that is too complex to reflect in a set of mathematical equations, so we have to find s compromise between authenticity and tractability. We also need to consider what sort of information we want to get out of the model: do we want to try for predictability, or will we settle for trends and some indication of factors that are particularly significant?

Each model is based on a set of assumptions. In practice, there are many folks engaged in this area and so a number of competing models. This may not be a bad thing, because if people come at the problem from different directions, we can see where the models may be in agreement or in contradiction, and therefore have a richer discussion on the assumptions made and on how to proceed. Models can also be updated. One strategy is to make up a model using information up to a particular year, see what the model predicts for the immediate future and then check what actually happened.