The problem of the large number

Students who are interested in mathematics have the opportunity to participate in all sorts of competitions. Some are local; some are national. But the premier competition is the International Mathematical Olympiad, in which about a hundred countries take part. It is held each July in a different country. Each country sends a team of six students to write two four-and-a-half hour examinations, each with three problems. This competition was founded in 1959 by several countries in Eastern Europe. Canada first sent a team in 1981 and hosted the event in Toronto in 1995.

The problems are indeed challenging and it is difficult for a student to write a perfect paper. But not all the problems involve a lot of technical mathematics or even require an extensive background. Sometimes, all that is needed is some divergent thinking and a good command of basics. An example of a problem whose solution can be followed by someone with a high school background in mathematics is this one:

Let A be equal to 4444^{4444}, that is, the product of 4444 factors, each equal to 4444. So A is a very large number. Suppose that this is written out; we sum its digits to get B. In turn, C is the sum of the digits of B, and finally D is the sum of the digits of C. What is D?

The size of A makes it impractical to simply work out what it is and add its digits; besides, we might make a mistake along the way. So let us do a little forensic investigation and see what circumstantial evidence we can pick up on the way.

The fact that each number is the sum of the digits of its predecessor reminds us of the principle of casting out nines. Each number upon division by 9 leaves the same remainder as the sum of its digits upon division by 9. This remainder is called the digital sum of the number. It is also true that the digital sum of the product of two numbers is equal to the digital sum of the product of the two digital sums of the numbers. For example, the digital sums of 137 and 412 are 2 and 7; $2 \times 7 = 14$, whose digital sum is 5, the same as the digital sum of $137 \times 412 = 56444$.

What is the digital sum of A? The digital sum of 4444 is 7, so the digital sum of A is the same as the digital sum of 7^{4444}. However, $7^3 = 343$ has digital sum 1, so $7^{4443} = (7^3)^{1481}$ also has digital sum 1. Since 7^{4444} is 7 times this, A has digital sum 7. From this we see that all of B, C and D leave a remainder 7 when divided by 9.

This bring the curtain down on Act I. Next, we get an upper estimate on how many digits A has. Since 4444 is less than 10^4, A is less than $10^{4^{4444}} = 10^{17776}$. So A has fewer than 20000 digits. Since each of these digits does not exceed 9, B has to be less than $9 \times 20000 = 180000$. By a similar reasoning, we find that C is less than $1 + 9 \times 5 = 46$. The largest sum of digits of any number less than 46 is realized by 39, whose sum of digits is 12. We conclude that D cannot be any bigger than 12.

Thus, we are looking for a number D that is no bigger than 12 and leaves a remainder 7 when divided by 9.