

## Billiard balls

When I was a kid, it was popular for the lads at school to pose riddles and problems to one another. One that I remember is the problem of the billiard balls: *You have twelve billiard balls that look identical. Eleven of them weigh the same, but the twelfth has a different weight. You do not know whether it is heavier or lighter than the rest. Using an equal-arms balance at most three times, determine which is the odd ball and whether it is heavier or lighter.* An equal-arms balance has two pans on either side of a fulcrum (like a see-saw); you can use it to determine whether a mass put in one pan is heavier than, lighter than, or weighs the same as the mass in the other pan. .

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Let us do some preliminary analysis. There are 24 possible situations: each of the 12 balls may be the odd one, and it may be either heavier or lighter. When you apply the balance, there are three outcomes, each of which will either support or rule out possibilities. Since we do not know the absolute weight in grams of any of the balls, in each comparison, we should put the same number of balls on each pan. Also, it makes sense that each outcome should support roughly the same number of possible situations so that we reduce the number of possibilities to about a third.

On the first use of the balance, put four balls on each pan. If the left pan goes down, then either one of the four balls on the left is heavy or one of the four balls on the right is light for a total of eight possibilities. Similarly, the descent of the right pan corresponds to eight other possibilities. Finally, if the pans balance, then one of the four balls not used is either heavy or light. In any case, after the first use of the balance, there are eight outstanding possibilities.

For the second use of the balance, we try for a 3-3-2 split of these possibilities, so that when we come to the third use, we can nail the answer. One way to solve the problem is to use the outcome of each of the first two comparisons to determine which balls you want to use for the next comparison. A more elegant solution assigns in advance which eight balls you want to use for the three comparisons.

Here is one solution. Label each of the balls with a different letter from the words SILENT COWARD. Let the balls placed on the left and right pans for the three weighings be given by (SCAN, WORD), (SCAR, LINE), (SLOT, RAID). If the pans balance, cross out all the letters of the balls involved. Otherwise, put a circle around any word corresponding to a pan that goes down. Any letter not crossed out that appears in all of the circled words represents a heavier ball; any letter not crossed out in all of the noncircled word represents a lighter ball.

For example, suppose the pans balance the first time, the left pan goes down the second time and the right pan goes down the third time. Then we cross out the letters S, N, C, O, W, A, R, D. We circle the words SCAR and RAID. For these words, the only letter that has not been eliminated is I, but this appears in only one of the circled words. The uncircled words are LINE and SLOT. The uncrossed letters are L, E, I, T, but only L is in both. Therefore the odd ball is labelled L and is lighter.

Another way of formulating a solution is to begin by numbering the balls from 1 to 12 in base 3 numeration: (001, 002, 010, 011, 012, 020, 021, 022, 100, 101, 102, 110). We keep the label 001 and those that begin with 01, and change the rest by replacing each 0 by 2 and each 2 by 0. Thus, the balls are now labelled: (001, 220, 010, 011, 012, 202, 201, 200, 122, 121, 120, 112). For the first application of the balance, put the four balls with 0 in the first position on the left pan and those with 2 in the first position on the right; for the second, those with 0 in the second position on the left and 2 in the second position on the right; finally, put those with 0 in the third position on the left and those with 2 in the third position on the right. Thus, we are comparing the quartets ((1, 3, 4, 5); (2, 6, 7, 8)); ((1, 6, 7, 8), (2, 9, 10, 11)); ((2, 3, 8, 11), (5, 6, 9, 12)). To identify the odd ball, write 0 when the left pan goes down, 2 when the right pan goes down and 1 otherwise.

If we get a code for one of the balls, then that ball is heavy. If we do not get a code for one of the balls, then replacing each 0 by 2 and vice versa will give a code for a ball, which will then be light. For example, if the left pan goes down first, then the pans balance and finally the right pan goes down, then we code 012 and the fifth ball is heavy. However, if the pans balance on the first two weighings and the left pan goes down on the third, then we code 110, which must be changed to 112 to tell us that the twelfth ball is light.