CHALLENGE WITH DIGITS

§**1. The challenge.** Look at the number 1583. There are four digits all different. We can write this number as the sum of 1331 and 252. These two numbers together have four distinct digits, 1, 2, 3 and 5.

Now look at the number 13940. It has has five digits, all different, and is the sum of 8585 and 5355. Together they have only three distinct digits, 3, 5 and 8.

Here is your challenge. Find the largest number you can which has all of its digits different and is the sum of two numbers which together have no more than four distinct digits. If you do it with fewer than four digits in the two numbers you add together, so much the better!

§**2. Responses from readers.** The problem posed in the March 27 issue of the *Frontenac News* was to find the largest number with all its digits different and is the sum of two numbers that involve at most four distinct digits. Readers were invited to try to find fewer digits in the summands.

The most insightful solution came from Doug Nuttall of Elphin who shared his strategy for arriving at an answer. The largest number with all digits distinct is 9876543210, so this is the sum that we strive for. He builds up his answer step by step. To save on digits, we make the last digit of each summand 5: 5 + 5 = 10. Then 55 + 55 = 110 which makes at least the last two digits distinct. We want the hundreds digit of the sum to be 2, which we can achieve by upping one of the 5s to a 6 in this place of the summands: 655 + 555 = 1210. Continuing on, we next have: 6655 + 6555 = 13210. Only two digits have been used in the summands so far, so we can now strategically introduce the digit 7. Eventually, we come to 7766555 + 776655 = 78543210. Note that, at this stage, the two summands involve only three distinct digits. Next up is 8887766555 + 88776655 = 8976543210. This gives a ten-digit sum, but it is not large enough. So we need to tinker a little to get 9777766555 + 99776655 = 9877543210. However the sum has two sevens and no sixes. But this is easily fixed by taking one million from the first summand:

$$9776766555 + 99776655 = 9876543210.$$

Brian Sutton got an result where one of the summands used only two digits:

$$6556555 + 9869986655 = 9876543210.$$

Sutton also obtained an example where only three distinct digits were used in the summands, with the first using only two of them:

$$767666 + 97997766 = 98765432.$$

Julian Rice-Laprise a Grade 12 student living in Brooke Valley obtained a ten-digit sum where the summands required only three digits:

$$816161186 + 818816866 = 1634978052.$$

J. Kerry Skipper of did a computer search and found that there were 1712 pairs of summands adding up to 9876543210, none of which involved fewer than four distinct digits. Doug Angle of ??? also did a computer search of possibilities and in addition looked at numbers written to bases larger than 10.

§**3. Extension of the challenge.** In the equation 8887766555 + 88776655 = 8976543210, all the digits of the sum are different, and the two summands involve only the four digits 5, 6, 7, 8. This is not the largest possible sum. The largest number with all its digits different is 9876543210, and this can be written as the sum of two numbers involving only four different digits in 1712 ways, one of which is

$$9776766555 + 99776655 = 9876543210.$$

A different example provides a summand that has only two distinct digits:

$$6556555 + 9869986655 = 9876543210.$$

However, if we allow only three different digits in the summands, we can find several examples, such as $77766555 + 776655 = 78543210$, $767666 + 97997766 = 98765432$ and

$$816161186 + 818816866 = 1634978052.$$

There are still outstanding questions. Below are a number of questions that you might consider. You can focus on one or two of them, or give a more comprehensive survey. In all cases, the digits of the sum all have to be different.

(1) What are the largest sums that can be written where the two summands involve only one distinct digit? two distinct digits? three distinct digits?

(2) Suppose that each of the summands is required to have only two distinct digits, and together have at most two, three or four distinct digits? What now is the largest sum in each instance?

(3) In the various cases indicated above, determine that largest number of the digits the sum can have, and for this number of digits, we want to make the total number of digits (counting repetitions) in the three numbers as small as possible.

(4) Find examples of sums that are interesting in some way. For example, the digits might appear in each number in increasing, rather than decreasing order; some of the numbers could be palindromes (reads the same from left to right or from right to left); you could have summands with two distinct digits that alternate. The possibilities are left to your imagination.

(5) You might consider similar problems involving subtraction or multiplication of two numbers. Here are the rules: There should be no reference to books or the internet, but a calculator or computer can be used to cut down on the tedium of searches and calculations. If you use a computer, give an informal description of how you set up the program.