THE BEER MUG THEOREM

Suppose you have a mug of beer, so cold that the outside of the mug is covered with condensation. When you place it on the bar, it leaves a wet circle. Take a swig and replace the mug on the bar, so that the new circle it makes intersects the first circle. Take another swig, and replace the mug so that you have three circles all passing through a common point. Then you can have a final swig, and replace the mug so that the circumference of its base rests on the points where the first three circles intersect in pairs.

There are three circles with equal radii all passing through the point P. Each of the points A, B, C are the intersections of exactly two of the circles. The assertion is that the unique circle that passes through A, B and C has the same radius as the three given circles.

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There is an interesting way to visualize why this happens. Let U, V, W be the three centres of the respective circles BPC, CPA, APB. For definiteness, assume that P is an interior point of triangle ABC. Join each centre to the three points known to be on its circumference. That is, join UP, UB, UC, VP, VA, VC, WP, WA, WB. All these segments are all equal to the common radius of the three circles. They also appear to form a two-dimensional skeleton of a cube. So complete the cube as indicated with the dotted lines; the remaining vertex of the "cube", Q, is the centre of the fourth circle.

We can make an honest proof out of this by noting that the sides of quadrilateral $AWPV$ are all equal, so that $AWPV$ is a rhombus. So are $BUPW$ and $CVPU$. Therefore the segments AW, PV and CU are parallel. Similarly, AVm WP and BU are parallel, as are BW, UP and CV.

Now draw a line through B parallel to WA and UC, and a line through A parallel to WB and VC. let them meet at Q. Since $AWBQ$ is a parallelogram with adjacent sides equal, it is a rhombus and $AQ = BQ = BW = AW$, the common radius of the three circles. Also AQ is equal and parallel to VC, so that $AQCV$ is a parallelogram and so $QC = AV$, the common radius. Thus, the distance from Q to each of the points A, B and C is equal to this radius and so is the centre of the circle through A, B and C.

We need to note however that there is more than one possible configuration. It could happen that the points B and C fall on the same side of the line AP. In this case, the construction of our "cube" is slightly different.