The ten number surprise

Next time you have a big shop at your supermarket, record the two-digit cents part of the prices of any ten items. For example, the numbers might be: 89, 77, 42, 38, 12, 9, 64, 46, 53, 17. It turns out that you can always find two subsets of these numbers that do not overlap and have the same sum. In the example, we see that the sets $\{38\}$ and $\{12, 9, 17\}$ have the same sum.

How can we know for sure that, given *any* ten numbers less than 100, not necessarily all distinct, we will find two disjoint subsets with the same sum? One way is to write out all the possibilities and check each case. To be sure, if any number is repeated, then we have two sets with one element each for which the sums are the same. However, since there over three million possibilities for our choice of numbers, if they are all distinct, this is a long and tedious task.

However, there is another way in to the result which depends on a very simple principle: if we put a collection of objects into categories, and there are more objects than categories, then some category must have at least two objects in it.

We begin by counting the number of subsets we can pick from ten numbers. We look at each number in turn and decide whether to include it in the subset. We have two choices – take it or leave it. For the first two numbers there four choices: reject both, take exactly one of them (two possibilities), take both. For the first three numbers, there are $8 = 2^3$ choices. For the ten numbers, there are $1024 = 2^{10}$ possibilities. This includes the case where we include none of the numbers, so we actually have 1023 different subsets of the ten numbers that contain at least one of them.

For each of these subsets, let us write down the sum of the numbers in it. What can we say about these sums? Since each number is less than 100 and there are no more than 10 numbers in each sets, all the sums must be less than 10×99; thus, there are fewer than 1000 possible sums.

We have a list of 1023 sums, each less than 1000. Therefore there must be a sum that is listed more than once, that belongs two distinct subsets.

We are not quite done. It may turn out that these two subsets overlap. This is easily remedied; just remove the numbers they have in common.

Next time you have a big family gathering, try this out with the ages of ten of the guests.

The surprisingly powerful mathematical tool that we used is called the *Pigeonhole Principle* or *Shubfachprinzip* to give it its German birthname. It was used by the mathematician Gustav Lejeune-Dirichlet (a brother-in-law of Felix Mendelssohn) in the early nineteenth century to prove a result in number theory. However, it is available for your own use. We can assert without fear of contradiction that there are two people in Ottawa with same number of hairs on their body. If your high school has more than 365 people, there will be two students with the same birthday. If there are more than 1500 students, there will be two born on the same day (assuming the students are all of the standard ages for their grades).