Reproducing squares

Look at these squares: $5^2 = 25$; $6^2 = 36$; $25^2 = 625$; $76^2 = 5776$; $376^2 = 141376$; $625^2 = 390625$; $9376^2 = 87909376$; $90625^2 = 8212890625$. In each case, the last few digits of the square reproduce the number being squared.

There are all sorts of such number oddities around, and it is legitimate to ask whether they occur just by chance or systematically. If we look as the numbers involved, the last digits seem to conclude in either 25 or 76, so there does seem to be some kind of pattern.

Look at an example. 376 is a three-digit number that appears at the end of its square, 141376. As a consequence, if we subtract it from its square, we must get a multiple of 1000: $141376 - 376 = 141000$. This means that the difference is divisible by the cube of 10. But anything divisible by the cube of 10 is divisible by the cube of 2, namely 8, and the cube of 5, namely 125 (since 2 and 5 are both divisors of 10).

Suppose we represent a three-digit number by N, and suppose that the last three-digits of its square N^2 are the digits of N. Then $N^2 - N$ must be a multiple of 1000, and therefore a multiple of 8 and of 125. We can write $N^2 - N$ as a product $N \times (N - 1)$ of two consecutive integers, one even and one odd. Since these two factor differ by 1, they cannot both be divisible by 8 nor can both be divisible by 125. Since each of N and $N - 1$ has three digits, neither of these number can be a multiple of both 8 and 125 (since it would then be divsible by 1000 which has more than three digits). So one of them is divisible by 8 and the other by 125. The one that is divisible by 125 must be odd (since the other is even). So the only choices for N are 125, 126, 375, 376, 625, 626, 875, 876. Since 126, 626 and 876 are not multiples of 8, we can rule them out. Since 124, 374, 874 are not multiples of 8, we can rule out 125, 375, 875. So the only numbers left for N are 376 and 625.

If we want a four-digit number for N, then we need $N \times (N - 1)$ to be divisible by 10000, which means that it has to be a multiple of $16 = 2^4$ and $625 = 5^4$. I will leave it to you to find reproducing squares of numbers with more than three digits.