

It makes no difference

Suppose we take three of the first six whole numbers, say $(1, 3, 4)$, listed in increasing order. List the remaining three numbers in descending order: $(6, 5, 2)$. Now pair them off: $(1, 6)$, $(3, 5)$, $(4, 2)$. Finally, calculate the sum of the positive difference of the numbers in each pair: $5 + 2 + 2 = 9$.

Here is an experiment for you to try. Do the same thing with a different choice of three of the first six numbers and see what happens. Now replace six by any other even number. For example, look at the whole numbers from 1 to 20 inclusive. Arrange ten of them in increasing order, and arrange the remaining ten in descending order. Pair the corresponding entries in the two sets and sum the positive differences. What do you observe and how do you account for it?

Regardless of the partition, the sum is always the same, the square of half the largest whole number. For six, this sum is $9 = 3^2$. Let us look at the situation with twenty numbers. The important thing to notice is that each pair consists of one number no greater than 10 and one number that exceeds 10.

Let us understand, for example, why it is not possible for both numbers in the fourth pair to be greater than 10. For, this would mean that the first four numbers in the second set and the last seven numbers (from the fourth to the tenth) in the first set all exceed 10. But this is not possible, since there are only ten numbers that exceed 10. Similarly, it can be argued that both numbers in the fourth pair cannot both be at most 10.

Thus each pair consists of one number that is at least 11 and one that is at most 10. The required sum turns out to be:

$$\begin{aligned} (20 + 19 + 18 + 17 + \cdots + 11) - (10 + 9 + 8 + 7 + \cdots + 1) \\ &= (20 - 1) + (19 - 2) + (18 - 3) + (17 - 4) + \cdots + (11 - 10) \\ &= 19 + 17 + 15 + 13 + \cdots + 1 \\ &= (19 + 1) + (17 + 3) + (15 + 5) + (13 + 7) + (11 + 9) \\ &= 5 \times 20 = 10 \times 10 = 10^2 = 100. \end{aligned}$$