**The last three holiday problems**

The third holiday problem was to find a number $N$ for which $N$ and $N^2$ used the nine nonzero digits (from 1 to 9) exactly once. Before getting into trial and error, we try to reduce the field as much as possible.

We reject as possibilities any number whose square has the same last digit. Thus, $N$ does not end in 1, 2, 5 or 6. In a similar vein, we can rule out certain pairs for the last two digits. The last two digits of the square are determined by the last two digits of $N$, so we can check to see when we introduce duplicated digits or zeros. In this way, we can reject 12, 19, 22, 23, 28, 32, 33, 38, 42, 43, 44, 47, 48, 49, 52, 53, 62, 63, 64, 69, 77, 78, 82, 83, 88, 93, 97, 98, 99.

Since the square of a two-digit number has no more than four digits, and the square of a four-digit number has at least seven digits, we deduce that $N$ has three digits and $N^2$ has six. Since $347^2 = 120409$, any smaller square has either fewer than six digits, or contains a zero digit or two ones among its digits. Thus, the number is at least 354. Since $870^2 = 756908$ and $900^2 = 810000$, any number between 870 and 900 must be rejected as otherwise we get a duplication of either 7 or 8. Likewise, since $952^2 = 906304$, we have to reject any number bigger than 937.

Finally, since the sum of all the digits in the number and its square is 45, a multiple of 9, we see by casting out nines, that the sum of the number and its square must be a multiple of 9. This can happen only if the number itself is a multiple of 9 or one less than a muptiple of 9. Putting all this together leaves us with having to check 34 numbers with a pocket calculator. We find that $567^2 = 321489$ and $854^2 = 729316$.

The next problem was to find three whole numbers which together used each of the ten digits exactly once and for which the largest was the sum of the other two. Again, we try to reduce the field with a little preliminary analysis. Neither of the smaller numbers can have more than four digits (otherwise it and the sum would have to have too many digits). Also it is not possible for both smaller numbers to have fewer than three digits. So the possible scnarios is to have a four-digit plus a single digit adding up to a five-digit number (which cannot happen if all the digits are distinct), a four- and a two-digit number adding to a four-digit number and two three-digit numbers adding to a four-digit number.

Suppose that we have $ABCD + FG = PQRS$. The only way this can happen is for $B$ to be 9, the digit $Q$ to be 0 and $P$ to be one more than $A$. A little trial and error leads to $5978 + 34 = 6012$ and $4987 + 26 = 5013$.

The situation $UVW + XYA = KLMN$ forces $K$ to be 1 and the sum of $U$ and $X$ to be at least 9. $KLMN$ must be at least 1023. None of 1023, 1024 or 1025 is possible for the largest number, but we do have $589+437 = 1026$, $789+246 = 1035$, $756 + 342 = 1098$ and $675 + 423 = 1098$, for example.

The last problem concerning two people at the passport office was to determine two numbers each of which is the square of the sum of the digits of the other and find their difference (nonzero from the context of the problem). This just involves a little checking off possibilities. We can reject the pair $(81, 81)$. The pair $(169, 256)$ works, and the two patrons at the passport office have 86 people between them in line.