CHAPTER NINE

ALLEMANDS

§1. INTRODUCTION.

We begin with the recursion of the last chapter, given by

$$x_{n+1} = \frac{x_n + c}{x_{n-1}} ,$$

with c a nonnegative real parameter. This formula can be written to take the sequence backward by the same process:

$$x_{n-1} = \frac{x_n + c}{x_{n+1}} ,$$

so we can let the index n range over all of the integers. We suppose that the terms of the sequence are selected to avoid division by 0. When $c = 0$ and $c = 1$, and bilateral sequence satisfying the recursion is periodic with respective periods 6 and 5. For other values of c, certain but not all sequences are periodic.

When

$$f_c(x, y) = \frac{x^2y + xy^2 + x^2 + (c + 1)x + y^2 + (c + 1)y + c}{xy} ,$$

we have that

$$f(x, y) = f\left(y, \frac{y + c}{x}\right) ,$$

with the result that $f(x_n, x_{n+1})$ is constant with respect to the index n. As n varies, the points (x_n, x_{n+1}) ranges over the planar curve with equation $f(x, y) = k$ where $k = f(x_0, x_1)$. When c, x_0, x_1 are positive, the portion of the locus of $f(x, y) = k$ is a loop in the positive quadrant. The mapping

$$T_c \quad : \quad (x, y) \longrightarrow \left(y, \frac{y + c}{x}\right)$$

maps the loop into itself.

The equation $f_c(x, y) = k$ is equivalent to $h_{c,k}(x, y) = 0$ where

$$\begin{aligned} h_{c,k}(x, y) &= x^2y + xy^2 + x^2 + y^2 - kxy + (c + 1)(x + y) + c \\ &= xy(x + y) + (x + y)^2 - (k + 2)xy + (c + 1)(x + y) + c \\ &= (y + 1)x^2 + (y^2 - ky + (c + 1))x + (y + 1)(y + c) \quad . \end{aligned}$$

The function $h_{c,k}(x, y)$ is a symmetric polynomial quadratic in each variable. For each n, the terms x_{n-1} and x_{n+1} are the roots of the quadratic equation $h_{c,k}(x, x_n) = 0$, provided $x_n \neq 1$. Indeed, from the relation between the coefficients and the product of the roots of a quadratic, we corroborate the relation

$$x_{n-1}x_{n+1} = x_n + c \quad . \tag{9.1}$$

Moreover, we have

$$x_{n-1} + x_{n+1} = -\left(\frac{x_n^2 - kx_n + c + 1}{x_n + 1}\right) \quad . \tag{9.2}$$

This means that, if c and k are given, we can use (9.1) and (9.2) and a single seed x_0 to define the sequence. Choose x_{-1} and x_1 to be the roots of the quadratic $h_{c,k}(x, x_0) = 0$, x_2 (along with x_0) to be the roots of $h_{c,k}(x, x_1) = 0$, and so on.

This suggests the following formalization:

Definition. *An allemand is a bilateral sequence* $\{x_n : n \in \mathbf{Z}\}$ *formed from a seed* x_0 *and a symmetric polynomial* $h(x, y)$ *which is quadratic in each variable, for which, given any integer* n, x_{n-1} *and* x_{n+1} *are the two roots of the equation* $h(x, x_n) = 0$ *(or, equivalently* $h(x_n, x) = 0$*).*

Observe that, once x_{-1} and x_1 have been fixed, then the remainder of the sequence is uniquely determined.

We provide a preliminary exploration of this topic. There are three cases, according as the degree of $h(x, y)$ is 2, 3 or 4.

One way to understand the structure of allemands is to consider a dynamical system in the plane, as we have done for the Lyness example. Given a point (x, y) on the locus of $h(x, y) = 0$, we define the mapping $T(x, y) = (y, z)$, where z is the value other than x (except in the case of a double root) for which $h(x, y) = h(z, y) = 0$. Thus, T maps the locus to itself. The locus may be connected and may have several components, bounded or unbounded. If there is a component which is a loop (homeomorph of a circle), it is tempting to ask whether this loop is invariant under the action of T and whether the action of T on the loop is conjugate to a rotation on a circle.

We can track the path of an allemand geometrically. Starting with the point (x_0, x_1). Locating the intersection of the perpendicular to $y = x$ from this point and the locus of $h(x, y) = 0$, locate the point (x_1, x_0). The vertical line $x = x_1$ meets the locus of $h(x, y) = 0$ again at the point (x_1, x_2). We can continue in this way to obtain in turn each point (x_n, x_{n+1}).

§2. QUADRATIC ALLEMANDS.

Suppose that $h(x, y)$ has degree 2, so that

$$\begin{aligned} h(x, y) &= \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta \\ &= \alpha x^2 + (\beta y + \gamma)x + (\alpha y^2 + \gamma y + \delta) \end{aligned} .$$

We may assume that $\alpha = 1$, since dividing $h(x, y)$ by a constant does not change the allemand associated with $h(x, y)$.

Thus, let

$$h(x, y) = x^2 + y^2 + \beta xy + \gamma(x + y) + \delta = x^2 + (\beta y + \gamma)x + (y^2 + \gamma y + \delta) \quad .$$

Any allemand corresponding to this function must satisfy both of the recursions

$$x_{n+1} + x_{n-1} = -(\beta x_n + \gamma) \tag{9.3}$$

$$x_{n+1}x_{n-1} = x_n^2 + \gamma x_n + \delta \tag{9.4}$$

It turns out that sequences that satisfy either of the recursions (9.3) or (9.4) are allemands. If (9.3) is satisfied, then

$$x_{n+1}x_{n-1} - x_n^2 - \gamma x_n = -x_{n-1}(\beta x_n + \gamma + x_{n-1}) - x_n^2 - \gamma x_n$$

$$= -x_n(\beta x_{n-1} + x_n + \gamma) - \gamma x_{n-1} - x_{n-1}^2 = x_n x_{n-2} - x_{n-1}^2 - \gamma x_{n-1}$$

so that $x_{n+1}x_{n-1} - x_n^2 - \gamma x_{n-1}$ is an invariant for any recursion satisfying (9.3) alone. If we let δ be the value of this invariant, then we have (9.4) holding as well, so that any sequence (9.3) turns out to be an allemand.

Similarly, if (9.4) is satisfied, then

$$\begin{aligned} x_{n+1} + x_{n-1} + \gamma &= \frac{x_n^2 + \gamma x_n + \delta}{x_{n-1}} + x_{n-1} + \gamma \\ &= \frac{1}{x_{n-1}}(x_n^2 + \gamma x_n + \delta + x_{n-1}^2 + \gamma x_{n-1}) \\ &= \frac{x_n}{x_{n-1}}\left(x_n + \gamma + \frac{x_{n-1}^2 + \gamma x_{n-1} + \delta}{x_n}\right) \\ &= \frac{x_n}{x_{n-1}}(x_n + x_{n-2} + \gamma) \end{aligned}$$

so that $x_n^{-1}(x_{n+1} + x_{n-1} + \gamma)$ is an invariant β and $x_{n+1} + \beta x_n + x_{n-1} + \gamma = 0$. Thus, any sequence defined by (9.4) alone is in fact an allemand.

We note the relations:

$$h(y, -(\beta y + x + \gamma)) = h(x, y)$$

and

$$h\left(y, \frac{y^2 + \gamma y + \delta}{x}\right) = \frac{y^2 + \gamma y + \delta}{x^2} h(x, y)$$

with the result that $h(x, y)$ is an invariant for two consecutive terms of sequence (9.3) and $h(x, y)/xy$ is an invariant for two consecutive terms of (9.4).

Case 1: $\beta = -2$

If $\gamma = 0$, then $x_{n+1} + x_{n-1} = 2x_n$ and $x_{n+1}x_{n-1} = x_n^2 + \delta$, so that the allemand is an arithmetic progression and the common difference d satisfies $\delta = -d^2$. For a real allemand, $\delta < 0$ and the locus of $h(x, y) = 0$ is a parallel pair of lines given by $0 = (x - y + d)(x - y - d)$. If $\delta = 0$, then these lines coincide and the allemand must be a constant.

If $\gamma \neq 0$, then, the recursion (9.3) has the general term $x_n = \frac{1}{2}\gamma n^2 + \lambda n + \mu$, with (9.4) giving a relation between λ and μ and the coefficients of the function $h(x, y)$. Since $h(x, y) = (x - y)^2 + \gamma(x + y) + \delta$, the locus of $h(x, y) = 0$ is a parabola.

Case 2: $\beta = 0$.

Let $\tau^2 = \frac{1}{2}\gamma^2 - \delta$. Then

$$h(x, y) = \left(x + \frac{\gamma}{2}\right)^2 + \left(y + \frac{\gamma}{2}\right)^2 - \tau^2 \ .$$

Since $x_{n+1} + x_{n-1} = -\gamma$, it follows that the allemand is a sequence of period 4 with periodic section

$$-\frac{\gamma}{2} + u, -\frac{\gamma}{2} + v, -\frac{\gamma}{2} - u, -\frac{\gamma}{2} - v \ ,$$

where

$$(-\frac{\gamma}{2} + v)(-\frac{\gamma}{2} - v) = \delta - (-\frac{\gamma}{2} + u)(-\frac{\gamma}{2} - u)$$

or

$$u^2 + v^2 = \tau^2 \ .$$

The successive points (x_n, x_{n+1}) are displaced by a rotation of 90° around a circle of radius τ and centre $(-\frac{\gamma}{2}, -\frac{\gamma}{2})$.

Case 3: $\beta = 2$.

The recursion (9.3) has the general solution $x_n = (\rho n + \sigma)(-1)^n - \frac{1}{4}\gamma$ and we get an allemand with $\delta = (\gamma^2/4) - \rho^2$. The function $h(x, y)$ has the form

$$h(x, y) = \left[(x + y) + \frac{1}{2}\gamma\right]^2 + \left(\delta - \frac{1}{4}\gamma^2\right) \ .$$

When $\gamma^2 \geq 4\delta$, then $h(x,y)$ can be factored and any real seed will yield a real allemand. When $\gamma^2 < 4\delta$, there are no real allemands corresponding to $h(x,y)$. When $\gamma^2 = 4\delta$, then we get a period-2 allemand with repeating entries x_0 and $-(x_0 + \gamma/2)$.

Example 9.1. Suppose $\gamma = \delta = 1$ and $\beta = 2$. Then

$$h(x,y) = (x+y)^2 + (x+y) + 1 = \frac{(x+y)^3 - 1}{x+y-1} \ .$$

This function has no allemands that are entirely real. However, for any seed $x_0 = u$, we can generate an allemand with $x_{-1} = \omega^2 - u$ and $x_1 = \omega - u$, where ω is an imaginary cube root of 1.

Example 9.2. Suppose $\beta = \delta = 2$ and $\gamma = -3$. Then

$$h(x,y) = (x+y)^2 - 3(x+y) + 2 = (x+y-2)(x+y-1) \ .$$

The seed u gives rise to the allemand

$$\{\cdots, -u+2, u, -u+1, u+1, -u, u+2, -u-1, u+3, \cdots\} \ .$$

Example 9.3. *Geometric progressions.* From (4), we get a geometric progression if and only if $\gamma = \delta = 0$. Since $x_{n+1} + \beta x_n + x_{n-1} = 0$, the common ratio must satisfy $r^2 + \beta r + 1 = 0$. ♣

Suppose that $\alpha = 1$ and $\beta \neq -2$. For any real κ,

$$h(x+\kappa, y+\kappa) = x^2 + y^2 + \beta xy + [\kappa(2+\beta) + \gamma](x+y) + h(\kappa,\kappa)$$

and one can select κ so that the coefficients of the linear term vanishes. This has the effect of translating the allemand by a constant and not changing its essential character. We can thus suppose without loss of generality that

$$h(x,y) = x^2 + y^2 + \beta xy + \delta \ .$$

Since $h(x,y)$ is the sum of a homogeneous polynomial and a constant, we can change scale and assume that δ is equal to 0, 1 or -1.

Example 9.4. Let

$$h_u(x,y) = x^2 + y^2 + \beta xy - 1 \ .$$

Since x_{n-1} and x_{n+1} are the two roots of the quadratic equation

$$t^2 + \beta x_n t + (x_n^2 - 1) = 0 \ ,$$

the theory of the quadratic informs us that, for each integer n,

$$x_{n+1} + x_{n-1} = -\beta x_n$$

$$x_{n+1}x_{n-1} = x_n^2 - 1 \ .$$

The function

$$f(x,y) = \frac{x^2 + y^2 - 1}{xy}$$

is an invariant of the transformation $T : (x,y) \rightarrow (y, (y^2-1)/x)$.

Since the sequence $\{x_n\}$ in particular satisfies the second order recursion $x_{n+1} = -\beta x_n - x_{n-1}$, it will or will not be periodic according to the character of the zeros of the characteristic polynomial $t^2 + \beta t + 1$, regardless of what its values for x_0 and x_1 are.

We also observe that the form $x_{n+1}x_{n-1} - x_n^2$ is independent of n for each second order recursion of the type $x_{n+1} = -\beta x_n - x_{n-1}$, so that such a recursion is an allemand corresponding to the polynomial $x^2+y^2+\beta xy+\delta$, where δ is the common value of the form. In 1993, Problem A-2 on the Putnam Competition asked students to prove that the condition $x_n^2 - x_{n-1}x_{n+1} = 1$ on a sequence entailed that the same sequence also satisfied the linear recursion for some β.

Example 9.5. Suppose that $\beta = -2\cos\theta$ with $0 < \theta < \pi$, $\gamma = 0$ and $\delta = -1$, so that

$$h(x, y) = x^2 + y^2 - (2\cos\theta)xy - 1 \ .$$

Since $x_{n+1} - (2\cos\theta)x_n + x_{n-1} = 0$, x_n must have the form $x_n = a\cos n\theta + b\sin n\theta$ for some parameters a and b. Since $1 = x_n^2 - x_{n+1}x_{n-1}$, we must have $(a^2 + b^2)(1 - \cos 2\theta) = 2$. If θ is a rational multiple of π, then $\{x_n\}$ is periodic for each choice of a and b.

The locus of $h(x, y) = 0$ is an ellipse. Let us make the index n continuous and calibrate it so that $x_0 = 0$ and $x_1 = 1$. Then

$$x_t = \frac{\sin t\theta}{\sin\theta} \quad .$$

The points on the ellipse have the form

$$\left(\frac{\sin t\theta}{\sin\theta}, \frac{\sin(t+1)\theta}{\sin\theta}\right)$$

for $t \in \mathbf{R}$, and the ellipse is homeomorphic to the interval $[0, 2\pi/\theta]$ with the ends identified. The mapping T described in the introduction is conjugate to the rotation $t \longrightarrow t + 1 \pmod{2\pi/\theta}$.

§3. CUBIC ALLEMANDS

Let

$$\begin{aligned} h(x, y) &= x^2y + xy^2 + \alpha(x^2 + y^2) + \beta xy + \gamma(x + y) + \delta \\ &= (y + \alpha)x^2 + (y^2 + \beta y + \gamma)x + (\alpha y^2 + \gamma y + \delta) \quad . \end{aligned}$$

There is a singular case. Suppose we try to seed the allemand with $x_0 = -\alpha$. If $\alpha^2 - \beta\alpha + \gamma = 0$, then $h(x, -\alpha)$ is constant and the situation degenerates. If $\alpha^2 - \beta\alpha + \gamma \neq 0$, then $h(x, -\alpha) = 0$ has single root, namely

$$x = -\left[\frac{\alpha^2 - \gamma\alpha + \delta}{\alpha^2 - \beta\alpha + \gamma}\right]$$

and we could take x_1 to be this value and then define x_n for $n \geq 0$, at least until we arrive at an index n for which $x_n = -\alpha$.

Example 9.6. Let $\alpha = \gamma = \delta = 0$, $\beta \neq 0$, so that $h(x, y) = xy(x + y + \beta)$. Start with the seed u. Then $h(x, u) = 0$ if and only if $x = 0$ or $x = -u - \beta$. We cannot proceed further in the $x = 0$ direction, but $h(x, -u - \beta) = x(-u - \beta)(x - u) = 0$ leads to $x = 0$ in the other direction. So we have a closed sequential segment $\{0, u, -u - \beta, 0\}$ which cannot be extended in either direction.

Example 9.7. Let $\alpha = \beta = \gamma = \delta = 1$ so that $h(x, y) = (y + 1)x^2 + (y^2 + y + 1)(x + 1)$. Starting with the seed -1, we get the closed sequential segment $\{-1, -1\}$. ♣

We now consider allemands that avoid the term $-\alpha$. Any such allemand corresponding to function $h(x, y)$ satisfies both of the recursions

$$x_{n+1} + x_{n-1} = -\left[\frac{x_n^2 + \beta x_n + \gamma}{x_n + \alpha}\right] \tag{9.5}$$

$$x_{n+1}x_{n-1} = \frac{\alpha x_n^2 + \gamma x_n + \delta}{x_n + \alpha} \tag{9.6}$$

Proposition 9.1. *We have*

$$h\left(\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)}, y\right) = \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)}h(x,y)$$

and

$$h\left(-\left[\frac{\alpha y^2+\beta y+\gamma}{y+\alpha}\right]-x, y\right) = h(x,y)$$

with the result that $\frac{h(x,y)}{xy}$ *is invariant along any recursion satisfying (6) and* $h(x,y)$ *is invariant along any recursion satisfying (5).*

It follows that, given a sequence $\{x_n\}$ *satisfying (9.6), we can select* β *so that* $h(x_n, x_{n+1}) = 0$ *and so it is an allemand.*

Similarly, given a sequence $\{x_n\}$ *satisfying (9.5), we can select* δ *so that* $h(x_n, x_{n+1}) = 0$ *and so it is an allemand.*

Any recursion satisfying either (5) or (6) will satisfy the other with a suitable choice of parameters.

Proof.

$$\begin{aligned}
h\left(\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)}, y\right) &= (y+\alpha)\frac{(\alpha y^2+\gamma y+\delta)^2}{x^2(y+\alpha)^2} + (y^2+\beta y+\gamma)\frac{\alpha y^2+\gamma y+\delta}{x(y+\alpha)} + (\alpha y^2+\gamma y+\delta) \\
&= \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)}\Big[(\alpha y^2+\gamma y+\delta) + x(y^2+\beta y+\gamma) + x^2(y+\alpha)\Big] \\
&= \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)}\Big[x^2y + y^2x + \alpha(x^2+y^2) + \beta xy + \gamma(x+y) + \delta\Big] \\
&= \frac{\alpha y^2+\gamma y+\delta}{x^2(y+\alpha)}h(x,y) \quad .
\end{aligned}$$

$$\begin{aligned}
h\left(-\left[\frac{y^2+\beta y+\gamma}{y+\alpha}\right]-x, y\right) &= (y+\alpha)\left[x^2 + \frac{2x(y^2+\beta y+\gamma)}{y+\alpha} + \frac{(y^2+\beta y+\gamma)^2}{(y+\alpha)^2}\right] \\
&\quad - (y^2+\beta y+\gamma)\left[\frac{y^2+\beta y+\gamma}{y+\alpha} + x\right] + (\alpha y^2+\gamma y+\delta) \\
&= x^2y + \alpha x^2 + 2xy^2 + 2\beta xy + 2\gamma x - xy^2 - \beta xy - \gamma x + \alpha y^2 + \gamma y + \delta \\
&= h(x,y) \quad .
\end{aligned}$$

The remaining statements of the proposition follow from these relations. If (9.6) holds, we select β so that $h(x_0, x_1) = 0$; it then follows by induction that $h(x_n, x_{n+1}) = 0$ and so the recursion satisfies (9.5). A similar argument applies if we assume that (9.5) holds. ♠

Example 9.8. *Constant allemands.* Suppose $h(x,y) = x^2y + xy^2 + \alpha(x^2+y^2) + \beta xy + \gamma(x+y) + \delta$ has a constant allemand for which $x_n = \kappa$ for all n. Then $t = \kappa$ must be a double root of the quadratic equation $h(t,\kappa) = 0$. In other words, not only

$$h(t,\kappa) = (\kappa+\alpha)t^2 + (\kappa^2+\beta\kappa+\gamma)t + (\alpha\kappa^2+\gamma\kappa+\delta)$$

but also its derivative

$$2(\kappa+\alpha)t + (\kappa^2+\beta\kappa+\gamma)$$

must vanish when $t = \kappa$, *i.e.*, $h(\kappa,\kappa) = 2\kappa^3 + (2\alpha+\beta)\kappa^2 + 2\gamma\kappa + \delta = 0$ and $3\kappa^2 + (2\alpha+\beta)\kappa + \gamma = 0$. Subtracting κ times the second equation from the first yields the necessary condition $\kappa^3 = \gamma\kappa + \delta$ for there to be a constant allemand $\{\kappa\}$. Conversely, for any γ and δ with $\delta \neq 0$, we choose real κ for which $\kappa^3 = \gamma\kappa + \delta$

and then select α and β so that $2\alpha + \beta = -3\kappa - \gamma/\kappa$ to obtain a constant allemand. To obtain the zero sequence as a constant allemand, it is necessary and sufficient to take $\gamma = \delta = 0$.

Example 9.9. $h(x, y) = x^2y + xy^2 + \alpha(x^2 + y^2) + 2\alpha xy = (y + \alpha)x^2 + (y^2 + 2\alpha y)x + \alpha y^2 = [xy + \alpha(x + y)](x + y)$. 0 seeds the constant allemand. In general, any term $\{x_n\}$ in an allemand has neighbours $-x_n$ and $-\alpha x_n/(x_n + \alpha)$. If we seed with $x_0 = -\alpha$, we get the unilateral sequence

$$\{-\alpha, \alpha, -\alpha/2, \alpha/2, -\alpha/3, \alpha/3, \cdots, -\alpha/m, \alpha/m, \cdots .\}$$

Example 9.10. $h(x, y) = x^2y + xy^2 + \alpha(x^2 + y^2 + xy)$

Any allemand satisfies $x_{n+1} + x_n + x_{n-1} = 0$ and so is periodic with period 3, the periodic segment being of the form $\{u, v, -u - v\}$ with u, v, α related by $(u^2 + uv + v^2)\alpha + (u + v)uv = 0$. ♣

If $H(x, y) = h(x + \kappa, y + \kappa)$, and if $\{x_n\}$ is an allemand for h, then $\{x_n - \kappa\}$ is an allemand for H; thus there is a one-one correspondence between the allemands for the two functions. Since

$$H(x, y) = x^2y + xy^2 + (\alpha + \kappa)(x^2 + y^2) + (\beta + 4\kappa)xy + (3\kappa^2 + 2\alpha\kappa + \beta\kappa + \gamma)(x + y) + h(\kappa, \kappa)$$

we can select κ to make a desired coefficient vanish.

Replacing (x, y) by $(x - \alpha, y - \alpha)$ in a given allemand function transforms it to the form

$$h(x, y) = x^2y + xy^2 + \beta xy + \gamma(x + y) + \delta = yx^2 + (y^2 + \beta y + \gamma)x + (\gamma y + \delta)$$

and yields the recursion relations

$$x_{n+1} + x_{n-1} = -\left(\frac{x_n^2 + \beta x + \gamma}{x_n}\right) = -\left(x_n + \beta + \frac{\gamma}{x_n}\right)$$

$$x_{n+1}x_{n-1} = \frac{\gamma x_n + \delta}{x_n} = \gamma + \frac{\delta}{x_n} \quad .$$

Example 9.11. $h(x, y) = x^2y + xy^2 + \beta xy + \gamma(x + y)$

This yields an allemand of period 4, regardless of the starting value. If u, v are consecutive terms, then the next two terms are $\gamma/u, \gamma/v$ and $-\beta$ is the sum of four consecutive terms.

Example 9.12. $h(x, y) = x^2y + xy^2 + \beta xy + \delta$

In this case, we find that $x_{n+1} + x_n + x_{n-1} = -\beta$ and $x_{n+1}x_nx_{n-1} = \delta$ so that the allemand must have period 3. If $x_n = a\omega^n + b\omega^{2n} - \beta/3$ (where ω is an imaginary cube root of unity), then $\delta = a^3 + b^3 - ab\beta - \beta^3/27$.

Proposition 9.2. *An allemand generated by the function $h(x, y) = x^2y + xy^2 + \beta xy + \gamma(x + y) + \delta$ is periodic of period dividing 5 regardless of seed if and only if $\gamma^3 + \delta^2 = \beta\gamma\delta$.*

Proof. We can work from either recurrence satisfied by the allemand. From the first, we have

$$x_{n+2} + x_{n+1} + x_n = -\beta - \frac{\gamma}{x_{n+1}}$$

$$x_n + x_{n-1} + x_{n-2} = -\beta - \frac{\gamma}{x_{n-1}}$$

whence

$$\begin{aligned} x_{n+2} + x_{n+1} + x_n + x_{n-1} + x_{n-2} &= -2\beta - \gamma\left(\frac{x_{n+1} + x_{n-1}}{x_{n+1}x_{n-1}}\right) - x_n \\ &= -2\beta + \gamma\left(\frac{x_n^2 + \beta x_n + \gamma}{\gamma x_n + \delta}\right) - x_n \\ &= \frac{-(\beta\gamma + \delta)x_n + (\gamma^2 - 2\beta\delta)}{\gamma x_n + \delta} \\ &= -\left(\beta + \frac{\delta}{\gamma}\right) + \left(\frac{\gamma^3 + \delta^2 - \beta\gamma\delta}{\gamma^2 x_n + \gamma\delta}\right) \end{aligned} .$$

The allemand has period 5 if and only if the sum of any five consecutive terms is constant. This will occur if and only if the term involving x_n vanishes identically, *i.e.*, if and only if the required condition holds.

An alternative argument uses the product of five consecutive terms. We have

$$\begin{aligned} x_{n+2}x_{n+1}x_n^2x_{n-1}x_{n-2} &= (\gamma x_{n+1}+\delta)(\gamma x_{n-1}+\delta) \\ &= \gamma^2\left(\frac{\gamma x_n+\delta}{x_n}\right) - \gamma\delta\left(x_n+\beta+\frac{\gamma}{x_n}\right)+\delta^2 \\ &= (\gamma^3+\delta^2-\beta\gamma\delta)-\gamma\delta x_n \end{aligned}$$

whence

$$x_{n+2}x_{n+1}x_nx_{n-1}x_{n-2} = -\gamma\delta + \left(\frac{\gamma^3+\delta^2-\beta\gamma\delta}{x_n}\right) \quad .$$

The result again follows. □

Example 9.13. If $h_{c,k}(x,y) = x^2y+xy^2+x^2+y^2-kxy+(c+1)(x+y)+c$, we compute

$$\begin{aligned} H_{c,k}(x,y) = h_{c,k}(x-1,y-1) &= x^2y+xy^2-(k+4)xy+(k+c+2)(x+y)-(k+c+2) \\ &= yx^2+[y^2-(k+4)y+(k+c+2)]x+(k+c+2)(y-1) \quad . \end{aligned}$$

This has the form under discussion, where $\beta = -(k+4)$, $\gamma = -\delta = k+c+2$. It is readily checked that

$$\gamma^3+\delta^2-\beta\gamma\delta = (k+c+2)^2(c-1) \quad .$$

If $k+c+2=0$, then the allemand degenerates. Regardless of the seed, 0 is a root of the quadratic equation, and if we then plug in 0 to get the neighbouring entries, the quadratic degenerates. Therefore, we should suppose that $k+c+2 \neq 0$. Thus, we see that every allemand from $h_{c,k}(x,y)$ is of period 5 if and only if $c=1$, confirming the observation made in **[2]**..

Proposition 9.3. *Suppose there is an allemand $\{\cdots,\lambda,\mu,\lambda,\mu,\cdots\}$ of prime period 2 corresponding to the function $h(x,y) = x^2y+xy^2+\beta xy+\gamma(x+y)+\delta$. Then we have*

$$\lambda+\mu = -\beta = -\frac{\delta}{\gamma}$$

and

$$\lambda\mu = -\gamma \quad .$$

Conversely, if the conditions $\beta^2+4\gamma\neq 0$ and $\beta\gamma=\delta$ holds, then there is an allemand of period 2 whose entries are given by the roots of the quadratic equation $t^2+\beta t-\gamma=0$.

Proof. Suppose that there is an allemand of period 2 as specified. Then

$$2\lambda = -\left(\frac{\mu^2+\beta\mu+\gamma}{\mu}\right) \quad \text{and} \quad 2\mu = -\left(\frac{\lambda^2+\beta\lambda+\gamma}{\lambda}\right)$$

whence

$$2\lambda\mu+\mu^2+\beta\mu+\gamma=0 \quad \text{and} \quad 2\lambda\mu+\lambda^2+\beta\lambda+\gamma=0 \quad .$$

Taking the difference of the two equations yields

$$(\lambda-\mu)(\lambda+\mu+\beta)=0$$

whence $\lambda+\mu=-\beta$. On the other hand, starting with

$$2\lambda = -\mu-\beta-\frac{\gamma}{\mu} \quad \text{and} \quad 2\mu = -\lambda-\beta-\frac{\gamma}{\lambda}$$

and taking the difference yields

$$(\lambda - \mu) = -\gamma\left(\frac{\lambda - \mu}{\lambda\mu}\right)$$

whence $\lambda\mu = \gamma$.

Similarly, $\lambda^2\mu = \gamma\mu + \delta$ and $\lambda\mu^2 = \gamma\lambda + \delta$ yield $\lambda\mu(\lambda - \mu) = -\gamma(\lambda - \mu)$ whence $\lambda\mu = -\gamma$ (as before). Also $\lambda^2 = \gamma + \frac{\delta}{\mu}$ and $\mu^2 = \gamma + \frac{\delta}{\lambda}$ yield after taking the difference and dividing out $\lambda - \mu$, $\lambda + \mu = \frac{\delta}{\lambda\mu} = -\frac{\delta}{\gamma}$.

Conversely, let $h(x, y) = x^2y + xy^2 + \beta xy + \gamma(x + y) + \beta\gamma$ and suppose that λ and μ are the distinct roots of the quadratic equation $t^2 + \beta t - \gamma = 0$. Then $\lambda + \beta = -\mu$ and $\gamma = -\lambda\mu$, so that

$$\begin{aligned} h(x, \lambda) &= \lambda x^2 + (\lambda(\lambda + \beta) + \gamma)x + \gamma(\lambda + \beta) \\ &= \lambda(x^2 - 2\mu x + \mu^2) = \lambda(x - \mu)^2 \end{aligned}$$

so that both neighbours of the element λ are μ. Similarly, both neighbours of μ are λ and the result follows. ♠

Example 9.14. $h(x, y) = x^2y + xy^2 - 3xy - 2(x + y) + 6 = yx^2 + (y^2 - 3y - 2)x + (-2y + 6)$. We have that $h(x, 1) = (x - 2)^2$; $h(x, 2) = 2(x - 1)^2$; $h(x, 1/2) = (1/4)(x - 4)(2x - 5)$, $h(x, 4) = 2(2x - 1)(x + 1)$; $h(x, -1) = -(x - 4)(x + 2)$, $h(x, -2) = -2(x + 1)(x - 5)$, $h(x, 5) = (x + 2)(5x - 2)$, $h(x, 2/5) = (1/10)(x - 5)(5x - 13)$. This function generates the period 2 allemand $\{\cdots, 1, 2, 1, 2, \cdots\}$ as well as the (apparently) nonperiodic allemand $\{\cdots, 5/2, 1/2, 4, -1, -2, 5, 2/5, \cdots\}$.

§4. QUARTIC ALLEMANDS.

Let

$$\begin{aligned} h(x, y) &= x^2y^2 + \alpha xy(x + y) + \beta(x^2 + y^2) + \gamma xy + \delta(x + y) + \epsilon \\ &= (y^2 + \alpha y + \beta)x^2 + (\alpha y^2 + \gamma y + \delta)x + (\beta y^2 + \delta y + \epsilon) \quad . \end{aligned}$$

Then allemands avoiding the roots of $y^2 + \alpha y + \beta = 0$ satisfy the recursions

$$x_{n+1} + x_{n-1} = -\left[\frac{\alpha x_n^2 + \gamma x_n + \delta}{x_n^2 + \alpha x_n + \beta}\right]$$

$$x_{n+1}x_{n-1} = \frac{\beta x_n^2 + \delta x_n + \epsilon}{x_n^2 + \alpha x_n + \beta} \quad .$$

Example 9.15. As in the cubic case, the situation with the constant allemand $\{0\}$ requires that $\delta = \epsilon = 0$.

Since

$$h(\kappa x, \kappa y) = \kappa^4(x^2y^2 + \kappa^{-1}\alpha xy(x + y) + \kappa^{-2}\beta(x^2 + y^2) + \kappa^{-2}\gamma xy + \kappa^{-3}\delta(x + y) + \kappa^{-4}\epsilon)$$

and $h(x - \frac{1}{2}\alpha, y - \frac{1}{2}\alpha) = x^2y^2+$ (a quadratic in x and y), we can change scale and position and suppose without loss of generality that $\alpha = 0$ and $\beta = \pm 1$.

Example 9.16. $h(x, y) = x^2y^2 - (x^2 + y^2)$. If the allemand is seeded by $u > 1$, then it has period 4 with four successive entries u, $u(u^2 - 1)^{-1/2}$, $-u$ and $-u(u^2 - 1)^{-1/2}$. When $u = \sqrt{2}$, then the segment becomes $\sqrt{2}$, $\sqrt{2}$, $-\sqrt{2}$ and $-\sqrt{2}$.

More generally, let $h(x, y) = x^2y^2 - (x^2 + y^2) + \epsilon$ with $\epsilon \neq 1$. In this case, we obtain an allemand of period 4 for which $x_{n+1} + x_{n-1} = 0$ and $x_{n+1}x_{n-1} = (x_n^2 - \epsilon)/(x_n^2 - 1)$. When $\epsilon = 1$, the locus of $h(x, y) = 0$ is the union of four lines with equations $x = \pm 1$, $y = \pm 1$.

§5. PELL'S EQUATION.

A generalized (quadratic) Pell's equation has the form $x^2 - dy^2 = k$, where d is a positive nonsquare integer and k is an integer, and solutions are sought in integers x and y. The equation $x^2 - dy^2 = 1$ always has a *fundamental solution* $(x, y) = (u, v)$ for which the complete collection of solutions is given by $(x, y) = (u_n, v_n)$ where

$$u_n + v_n\sqrt{d} = (u + v\sqrt{d})^n$$

where n runs over all the integers. We suppose that (u, v) is such a fundamental solution and that $(x, y) = (r, s)$ is a particular solution of $x^2 - dy^2 = k$. We can get other solutions of $x^2 - dy^2 = k$ by transforming $(x, y) = (r, s)$ to $(x, y) = (ru + dsv, rv + su)$ and iterating this operation.

Suppose that $z = rv + su$ is the second member of the transformed solution. We form a symmetric quadratic equation that involves s and z that will lead to a bilateral sequence for the second member of solutions of $x^2 - dy^2 = k$. From $r = (z - su)/v$, we get from $r^2 - ds^2 = k$ that

$$(z - su)^2 - dv^2s^2 = v^2k \ ,$$

which simplifies to

$$z^2 - 2usz + s^2 - v^2k = 0 \ .$$

Let $p(y, z) = y^2 - 2uyz + z^2 - v^2k$, and note that $p(y, z)$ is symmetric in y and z. We can define the allemand $\{s_n\}$ as follows.

Let $s_0 = s$, and let s_{-1} and s_1 be the two solutions of the quadratic equation $p(s_0, z) = 0$ with $s_{-1} \le s_1$. We can continue on to define the allemand with $p(s_i, s_{i+1}) = 0$.

We find that $ds_n^2 + k$ is always an integer square and that the recursions

$$s_{n+1} + s_{n-1} = 2us_n$$

and

$$s_{n+1}s_{n-1} = s_n^2 - v^2k$$

are both satisfied.

If $k < 0$ and the allemand has a positive term, then all of its terms are positive. If s_0 is the smallest term, then the allemand decreases to s_0 and increases thereafter. If $k > 0$ and the allemand contains at least one nonnegative term, with s_0 the smallest positive term, then the allemand is strictly increasing.

Example 9.17. The equation $x^2 - 2y^2 = 7$ is satisfied in particular by $(x, y) = (3, 1)$. The fundamental solution of $x^2 - 2y^2 = 1$ is $(x, y) = (3, 2)$. Creating the allemand with $s_0 = 1$ leads to

$$\{\cdots, -3771, -647, -111, -19, -3, 1, 9, 53, 309, 1801, 10497, \cdots\}$$

and the corresponding solutions that include $(x, y) = (27, -19), (5, -3), (3, 1), (13, 9)$. Another solution is $(x, y) = (5, 3)$, which leads to another allemand with $s_0 = 3$ and sequence of solutions.

Example 9.18. The equation $x^2 - 2y^2 = -7$ has the solution $(x, y) = (1, 2)$. The allemand with $s_0 = 2$ is

$$\{\cdots, 4348, 746, 128, 22, 4, 2, 8, 46, 268, 1562, \cdots .\}$$

More on this topic can be found in [**1**, 49 − 52].

§6. TANGENT CIRCLES AND SPHERES.

An interesting instance of this quadratic method of extending a sequence arises in the work of H.S.M. Coxeter on tangent spheres. [**4, 5**]. Suppose, in the plane, we have a sequence of circles such that any consecutive four of them are mutually tangent. Let ϵ_n be the reciprocal of the radius of the nth circle, signed to account for internal and external tangency. Then for each integer n,

$$(\epsilon_n + \epsilon_{n+1} + \epsilon_{n+2} + \epsilon_{n+3})^2 = 2(\epsilon_n^2 + \epsilon_{n+1}^2 + \epsilon_{n+2}^2 + \epsilon_{n+3}^2)$$

so that ϵ_n and ϵ_{n+4} are the two roots of the quadratic equation

$$x^2 - 2(\epsilon_{n+1} + \epsilon_{n+2} + \epsilon_{n+3})x + [\epsilon_{n+1}^2 + \epsilon_{n+2}^2 + \epsilon_{n+3}^2 - 2(\epsilon_{n+1}\epsilon_{n+2} + \epsilon_{n+1}\epsilon_{n+3} + \epsilon_{n+2}\epsilon_{n+3})] = 0$$

with the result that

$$\epsilon_n + \epsilon_{n+4} = 2(\epsilon_{n+1} + \epsilon_{n+2} + \epsilon_{n+3})$$

and

$$\epsilon_n\epsilon_{n+4} = \epsilon_{n+1}^2 + \epsilon_{n+2}^2 + \epsilon_{n+3}^2 - 2(\epsilon_{n+1}\epsilon_{n+2} + \epsilon_{n+1}\epsilon_{n+3} + \epsilon_{n+2}\epsilon_{n+3}) \ .$$

This can be generalized to $m-$dimensional Euclidean space. If $\{\epsilon_n\}$ is the sequence of radius reciprocals of spheres, any consecutive $m+2$ of which are pairwise tangent, then

$$m\sum_{i=0}^{m+1} \epsilon_i^2 = \Big(\sum_{i=0}^{m+1} \epsilon_i\Big)^2$$

which makes the sequences $\{\epsilon_n\}$ an allemand.

References

1. E.J. Barbeau, *Pell's equation* Springer, New York, 2003

2. E.J. Barbeau, B. Gelbord & S. Tanny, Periodicities of solutions of the generalized Lyness recursion, *Jour. Difference Equations and Applications* 1 (1995), 291-306

3. E.J. Barbeau & S. Tanny, Periodicities of solutions of certain recursions involving the maximum function, *Jour. Difference Equations and Applications* 2 (1996), 39-54

4. H.S.M. Coxeter, Loxodromic sequences of tangent spheres, *Aequationes Mathematicae* 1 (1968), 104-121

5. H.S.M. Coxeter, Numerical distances among the circles in a loxodromic sequence, *Mathematical Intelligencer* 19:4 (1997), 41-47

6. Daniel Pedoe, On a theorem in geometry, *American Mathematical Monthly* 74 (1967), 627-640