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283. (a) Determine all quadruples (a, b, c, d) of positive integers for which the greatest common divisor of its elements is 1,

$$\frac{a}{b} = \frac{c}{d}$$

and $a + b + c = d$.

(b) Of those quadruples found in (a), which also satisfy

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{a} \ ?$$

(c) For quadruples (a, b, c, d) of positive integers, do the conditions $a+b+c = d$ and $(1/b)+(1/c)+(1/d) = (1/a)$ together imply that $a/b = c/d$?

284. Suppose that $ABCDEF$ is a convex hexagon for which $\angle A + \angle C + \angle E = 360°$ and

$$\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1 \ .$$

Prove that

$$\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 \ .$$

285. (a) Solve the following system of equations:

$$(1 + 4^{2x-y})(5^{1-2x+y}) = 1 + 2^{2x-y+1} \ ;$$

$$y^2 + 4x = \log_2(y^2 + 2x + 1) \ .$$

(b) Solve for real values of x:

$$3^x \cdot 8^{x/(x+2)} = 6 \ .$$

Express your answers in a simple form.

286. Construct inside a triangle ABC a point P such that, if X, Y, Z are the respective feet of the perpendiculars from P to BC, CA, AB, then P is the centroid (intersection of the medians) of triangle XYZ.

287. Let M and N be the respective midpoints of the sides BC and AC of the triangle ABC. Prove that the centroid of the triangle ABC lies on the circumscribed circle of the triangle CMN if and only if

$$4 \cdot |AM| \cdot |BN| = 3 \cdot |AC| \cdot |BC| \ .$$

288. Suppose that $a_1 < a_2 < \cdots < a_n$. Prove that

$$a_1 a_2^4 + a_2 a_3^4 + \cdots + a_n a_1^4 \geq a_2 a_1^4 + a_3 a_2^4 + \cdots + a_1 a_n^4 \ .$$

289. Let $n(r)$ be the number of points with integer coordinates on the circumference of a circle of radius $r > 1$ in the cartesian plane. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2} \ .$$

290. The School of Architecture in the *Olymon* University proposed two projects for the new Housing Campus of the University. In each project, the campus is designed to have several identical dormitory buildings, with the same number of one-bedroom apartments in each building. In the first project, there are 12096 apartments in total. There are eight more buildings in the second project than in the first, and each building has more apartments, which raises the total of apartments in the project to 23625. How many buildings does the second project require?

291. The n-sided polygon $A_1, A_2, \cdots, A_n$ ($n \geq 4$) has the following property: The diagonals from each of its vertices divide the respective angle of the polygon into $n-2$ equal angles. Find all natural numbers n for which this implies that the polygon $A_1 A_2 \cdots A_n$ is regular.

292. 1200 different points are randomly chosen on the circumference of a circle with centre O. Prove that it is possible to find two points on the circumference, M and N, so that:

- M and N are different from the chosen 1200 points;
- $\angle MON = 30°$;
- there are *exactly* 100 of the 1200 points inside the angle MON.

293. Two players, Amanda and Brenda, play the following game: Given a number n, Amanda writes n different natural numbers. Then, Brenda is allowed to erase several (including none, but not all) of them, and to write either $+$ or $-$ in front of each of the remaining numbers, making them positive or negative, respectively, Then they calculate their sum. Brenda wins the game is the sum is a multiple of 2004. Otherwise the winner is Amanda. Determine which one of them has a winning strategy, for the different choices of n. Indicate your reasoning and describe the strategy.

294. The number $N = 10101\cdots0101$ is written using $n+1$ ones and n zeros. What is the least possible value of n for which the number N is a multiple of 9999?

295. In a triangle ABC, the angle bisectors AM and CK (with M and K on BC and AB respectively) intersect at the point O. It is known that

$$|AO| \div |OM| = \frac{\sqrt{6}+\sqrt{3}+1}{2}$$

and

$$|CO| \div |OK| = \frac{\sqrt{2}}{\sqrt{3}-1} \ .$$

Find the measures of the angles in triangle ABC.

296. Solve the equation

$$5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x} \ .$$

297. The point P lies on the side BC of triangle ABC so that $PC = 2BP$, $\angle ABC = 45°$ and $\angle APC = 60°$. Determine $\angle ACB$.

298. Let O be a point in the interior of a quadrilateral of area S, and suppose that

$$2S = |OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \ .$$

Prove that $ABCD$ is a square with centre O.

299. Let $\sigma(r)$ denote the sum of all the divisors of r, including r and 1. Prove that there are infinitely many natural numbers n for which

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k}$$

whenever $1 \leq k \leq n$.

300. Suppose that ABC is a right triangle with $\angle B < \angle C < \angle A = 90^\circ$, and let $\mathfrak{K}$ be its circumcircle. Suppose that the tangent to $\mathfrak{K}$ at A meets BC produced at D and that E is the reflection of A in the axis BC. Let X be the foot of the perpendicular for A to BE and Y the midpoint of AX. Suppose that BY meets $\mathfrak{K}$ again in Z. Prove that BD is tangent to the circumcircle of triangle ADZ.

301. Let $d = 1, 2, 3$. Suppose that M_d consists of the positive integers that *cannot* be expressed as the sum of two or more consecutive terms of an arithmetic progression consisting of positive integers with common difference d. Prove that, if $c \in M_3$, then there exist integers $a \in M_1$ and $b \in M_2$ for which $c = ab$.

302. In the following, $ABCD$ is an arbitrary convex quadrilateral. The notation $[\cdots]$ refers to the area.

(a) Prove that $ABCD$ is a trapezoid if and only if

$$[ABC] \cdot [ACD] = [ABD] \cdot [BCD] \ .$$

(b) Suppose that F is an interior point of the quadrilateral $ABCD$ such that $ABCF$ is a parallelogram. Prove that

$$[ABC] \cdot [ACD] + [AFD] \cdot [FCD] = [ABD] \cdot [BCD] \ .$$

303. Solve the equation

$$\tan^2 2x = 2 \tan 2x \tan 3x + 1 \ .$$

304. Prove that, for any complex numbers z and w,

$$(|z| + |w|) \left| \frac{z}{|z|} + \frac{w}{|w|} \right| \leq 2|z + w| \ .$$

305. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

306. The circumferences of three circles of radius r meet in a common point O. The meet also, pairwise, in the points P, Q and R. Determine the maximum and minimum values of the circumradius of triangle PQR.

307. Let p be a prime and m a positive integer for which $m < p$ and the greatest common divisor of m and p is equal to 1. Suppose that the decimal expansion of m/p has period $2k$ for some positive integer k, so that

$$\frac{m}{p} = .ABABABAB\ldots = (10^k A + B)(10^{-2k} + 10^{-4k} + \cdots)$$

where A and B are two distinct blocks of k digits. Prove that

$$A + B = 10^k - 1 \ .$$

(For example, $3/7 = 0.428571\ldots$ and $428 + 571 = 999$.)

308. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \geq 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x) \ .$$

(b) Determine a formula for the coefficient of x^k $(0 \leq k \leq n)$ in $f_n(x)$.

309. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

310. (a) Suppose that n is a positive integer. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1}(y-k)^{n-k} \ .$$

(b) Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k} \ .$$

311. Given a square with a side length 1, let P be a point in the plane such that the sum of the distances from P to the sides of the square (or their extensions) is equal to 4. Determine the set of all such points P.

312. Given ten arbitrary natural numbers. Consider the sum, the product, and the absolute value of the difference calculated for any two of these numbers. At most how many of all these calculated numbers are odd?

313. The three medians of the triangle ABC partition it into six triangles. Given that three of these triangles have equal perimeters, prove that the triangle ABC is equilateral.

314. For the real numbers a, b and c, it is known that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1 \ ,$$

and

$$a + b + c = 1 \ .$$

Find the value of the expression

$$M = \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \ .$$

315. The natural numbers 3945, 4686 and 5598 have the same remainder when divided by a natural number x. What is the sum of the number x and this remainder?

316. Solve the equation

$$|x^2 - 3x + 2| + |x^2 + 2x - 3| = 11 \ .$$

317. Let $P(x)$ be the polynomial

$$P(x) = x^{15} - 2004x^{14} + 2204x^{13} - \cdots - 2004x^2 + 2004x \ ,$$

Calculate $P(2003)$.

318. Solve for integers x, y, z the system

$$1 = x + y + z = x^3 + y^3 + z^2 \ .$$

[Note that the exponent of z on the right is 2, not 3.]

319. Suppose that a, b, c, x are real numbers for which $abc \neq 0$ and

$$\frac{xb + (1-x)c}{a} = \frac{xc + (1-x)a}{b} = \frac{xa + (1-x)b}{c} \ .$$

Prove that $a = b = c$.

320. Let L and M be the respective intersections of the internal and external angle bisectors of the triangle ABC at C and the side AB produced. Suppose that $CL = CM$ and that R is the circumradius of triangle ABC. Prove that

$$|AC|^2 + |BC|^2 = 4R^2 \ .$$

321. Determine all positive integers k for which $k^{1/(k-7)}$ is an integer.

322. The real numbers u and v satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 \ .$$

Determine $u + v$.

323. Alfred, Bertha and Cedric are going from their home to the country fair, a distance of 62 km. They have a motorcycle with sidecar that together accommodates at most 2 people and that can travel at a maximum speed of 50 km/hr. Each can walk at a maximum speed of 5 km/hr. Is it possible for all three to cover the 62 km distance within 3 hours?

324. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB| = a$, $|BC| = b$ and $|VA| = |VB| = |VC| = |VD| = c$. Determine the area of the intersection of the pyramid and the plane parallel to the edge VA that contains the diagonal BD.

325. Solve for positive real values of x, y, t:

$$(x^2 + y^2)^2 + 2tx(x^2 + y^2) = t^2y^2 \ .$$

Are there infinitely many solutions for which the values of x, y, t are all positive integers?

Optional rider: What is the smallest value of t for a positive integer solution?

326. In the triangle ABC with semiperimeter $s = \frac{1}{2}(a+b+c)$, points U, V, W lie on the respective sides BC, CA, AB. Prove that

$$s < |AU| + |BV| + |CW| < 3s \ .$$

Give an example for which the sum in the middle is equal to $2s$.

327. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

328. Let $\mathfrak{C}$ be a circle with diameter AC and centre D. Suppose that B is a point on the circle for which $BD \perp AC$. Let E be the midpoint of DC and let Z be a point on the radius AD for which $EZ = EB$.

Prove that

(a) The length c of BZ is the length of the side of a regular pentagon inscribed in $\mathfrak{C}$.

(b) The length b of DZ is the length of the side of a regular decagon (10-gon) inscribed in $\mathfrak{C}$.

(c) $c^2 = a^2 + b^2$ where a is the length of a regular hexagon inscribed in $\mathfrak{C}$.

(d) $(a + b) : a = a : b$.

329. Let x, y, z be positive real numbers. Prove that

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2} \ .$$

330. At an international conference, there are four official languages. Any two participants can communicate in at least one of these languages. Show that at least one of the languages is spoken by at least 60% of the participants.

331. Some checkers are placed on various squares of a $2m \times 2n$ chessboard, where m and n are odd. Any number (including zero) of checkers are placed on each square. There are an odd number of checkers in each row and in each column. Suppose that the chessboard squares are coloured alternately black and white (as usual). Prove that there are an even number of checkers on the black squares.

332. What is the minimum number of points that can be found (a) in the plane, (b) in space, such that each point in, respectively, (a) the plane, (b) space, must be at an irrational distance from at least one of them?

333. Suppose that a, b, c are the sides of triangle ABC and that a^2, b^2, c^2 are in arithmetic progression.

(a) Prove that $\cot A$, $\cot B$, $\cot C$ are also in arithmetic progression.

(b) Find an example of such a triangle where a, b, c are integers.

334. The vertices of a tetrahedron lie on the surface of a sphere of radius 2. The length of five of the edges of the tetrahedron is 3. Determine the length of the sixth edge.

335. Does the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a+b+c}$$

have infinitely many solutions in positive integers a, b, c?

336. Let $ABCD$ be a parallelogram with centre O. Points M and N are the respective midpoints of BO and CD. Prove that the triangles ABC and AMN are similar if and only if $ABCD$ is a square.

337. Let a, b, c be three real numbers for which $0 \leq c \leq b \leq a \leq 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \leq 1$?

338. A *triangular triple* (a, b, c) is a set of three positive integers for which $T(a) + T(b) = T(c)$. Determine the smallest triangular number of the form $a + b + c$ where (a, b, c) is a triangular triple. (*Optional investigations:* Are there infinitely many such triangular numbers $a + b + c$? Is it possible for the three numbers of a triangular triple to each be triangular?)

339. Let a, b, c be integers with $abc \neq 0$, and u, v, w be integers, not all zero, for which

$$au^2 + bv^2 + cw^2 = 0 .$$

Let r be any rational number. Prove that the equation

$$ax^2 + by^2 + cz^2 = r$$

is solvable.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

341. Let s, r, R respectively specify the semiperimeter, inradius and circumradius of a triangle ABC.

(a) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in geometric progression.

342. Prove that there are infinitely many solutions in positive integers of the system

$$a + b + c = x + y$$
$$a^3 + b^3 + c^3 = x^3 + y^3 .$$

343. A sequence $\{a_n\}$ of integers is defined by

$$a_0 = 0 , \quad a_1 = 1 , \quad a_n = 2a_{n-1} + a_{n-2}$$

for $n > 1$. Prove that, for each nonnegative integer k, 2^k divides a_n if and only if 2^k divides n.

344. A function f defined on the positive integers is given by

$$f(1) = 1 , \quad f(3) = 3 , \quad f(2n) = f(n) ,$$

$$f(4n + 1) = 2f(2n + 1) - f(n)$$
$$f(4n + 3) = 3f(2n + 1) - 2f(n) ,$$

for each positive integer n. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

345. Let $\mathfrak{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathfrak{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

346. Let n be a positive integer. Determine the set of all integers that can be written in the form

$$\sum_{k=1}^{n} \frac{k}{a_k}$$

where $a_1, a_2, \cdots, a_n$ are all positive integers.

347. Let n be a positive integer and $\{a_1, a_2, \cdots, a_n\}$ a finite sequence of real numbers which contains at least one positive term. Let S be the set of indices k for which at least one of the numbers

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \cdots, a_k + a_{k+1} + \cdots + a_n$$

is positive. Prove that

$$\sum\{a_k : k \in S\} > 0 \ .$$

348. (a) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that $f(x) - x^3$ is an increasing function. Must $f(x) - x - x^2$ also be increasing?

(b) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

349. Let s be the semiperimeter of triangle ABC. Suppose that L and N are points on AB and CB produced (*i.e.*, B lies on segments AL and CN) with $|AL| = |CN| = s$. Let K be the point symmetric to B with respect to the centre of the circumcircle of triangle ABC. Prove that the perpendicular from K to the line NL passes through the incentre of triangle ABC.

350. Let $ABCDE$ be a pentagon inscribed in a circle with centre O. Suppose that its angles are given by $\angle B = \angle C = 120°$, $\angle D = 130°$, $\angle E = 100°$. Prove that BD, CE and AO are concurrent.

351. Let $\{a_n\}$ be a sequence of real numbers for which $a_1 = 1/2$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1} \ .$$

Prove that, for all n, $a_1 + a_2 + \cdots + a_n < 1$.

352. Let $ABCD$ be a unit square with points M and N in its interior. Suppose, further, that MN produced does not pass through any vertex of the square. Find the smallest value of k for which, given any position of M and N, at least one of the twenty triangles with vertices chosen from the set $\{A, B, C, D, M, N\}$ has area not exceeding k.

Solutions

283. (a) Determine all quadruples (a, b, c, d) of positive integers for which the greatest common divisor of its elements is 1,

$$\frac{a}{b} = \frac{c}{d}$$

and $a + b + c = d$.

(b) Of those quadruples found in (a), which also satisfy

$$\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{1}{a} \ ?$$

(c) For quadruples (a, b, c, d) of positive integers, do the conditions $a+b+c = d$ and $(1/b)+(1/c)+(1/d) = (1/a)$ together imply that $a/b = c/d$?

Solution 1. (a) Suppose that the conditions on a, b, c, d are satisfied. Note that b and c have symmetric roles. Since $ad = bc$, if b and c were both even, then either a or d would be even, whence both would be even (since $a + b + c = d$), contradicting the fact that the greatest common divisor of a, b, c, d is equal to 1. Hence, at most one of b and c is even.

Suppose, if possible, b and c were both odd. Then a and d would be odd as well. If $b \equiv c \pmod 4$, then $bc \equiv 1$ and $b + c \equiv 2 \pmod 4$, whence $ad \equiv a(a+2) \equiv 3 \not\equiv bc \pmod 4$. If $b \equiv c + 2 \pmod 4$, it can similarly be shown that $ad \not\equiv bc \pmod 4$, In either case, we get an untenable conclusion. Hence, exactly one of b and c is even and the other is odd.

Without loss of generality, we may suppose that a and b have opposite parity. Let g be the greatest common divisor of a and b, so that $a = gu$ and $b = gv$ for some coprime pair (u, v) of positive integers with opposite parity. Since $d > c$, it follows that $b > a$ and $v > u$. Let $w = v - u$.

Since

$$\frac{b}{a} = \frac{a+b+c}{c} = \frac{a+b}{c} + 1 ,$$

it follows that

$$\frac{b-a}{a(b+a)} = \frac{1}{c}$$

whence

$$c = \frac{gu(u+v)}{w} \qquad \text{and} \qquad d = \frac{gv(u+v)}{w} .$$

Since the greatest common divisor of u and v is 1, w has no positive divisor in common with either u or v, save 1. Any common divisor of w and $u + v$ must divide $2u = (u + v) - (v - u)$ and $2v = (u + v) + (v - u)$; such a common divisor equals 1. Since u and v have opposite parity and so w is odd, w must divide g. Since the greatest common divisor of a, b, c, d is equal to 1, we must have that $g = w$. Hence

$$(a, b, c, d) = (u(v-u), v(v-u), u(v+u), v(v+u))$$

where u and v are coprime with opposite parity. Interchanging, the roles of b and c leads also to

$$(a, b, c, d) = (u(v-u), u(v+u), v(v-u), v(v+u))$$

with u, v coprime of opposite parity. On the other hand, any quadruples of this type satisfy the condition.

(b)

$$\begin{aligned} \frac{1}{b} + \frac{1}{c} + \frac{1}{d} &= \frac{1}{v(v-u)} + \frac{1}{u(v+u)} + \frac{1}{v(v+u)} \\ &= \frac{1}{v(v-u)} + \frac{1}{uv} = \frac{u + (v-u)}{uv(v-u)} = \frac{1}{v-u} = \frac{1}{a} . \end{aligned}$$

(c) Note that the conditions imply that $d - a$ and $b + c$ are nonzero. The conditions yield that $d - a = b + c$ and $(1/a) - (1/d) = (1/b) + (1/c)$. The second of these can be rewritten

$$\frac{ad}{d-a} = \frac{bc}{b+c}$$

so that $ad = bc$. Thus, all quadruples imply the required condition.

Solution 2. (a) [M. Lipnowski] Let $a/b = c/d = r/s$ where the greatest common divisor of r and s is equal to 1. Then $a = hr$, $b = hs$, $c = kr$, $d = ks$. Since the greatest common divisor of a, b, c, d equals 1, the greatest common divisor of h and k is 1. From $a + b + c = d$, we have that $(h + k)r = (k - h)s$. Observe that $\gcd(h + k, k - h) = 1$ when h and k have opposite parity and $\gcd(h + k, k - h) = 2$ when h and k are both odd. (Why?)

Thus, when h and k have oppposite parity, $r = k - h$, $s = k + h$ and

$$(a, b, c, d) = (h(k-h), h(k+h), k(k-h), k(k+h))$$

and, when h and k are both odd, then $r = \frac{1}{2}(k - h)$, $s = \frac{1}{2}(k + h)$ and

$$(a, b, c, d) = ((1/2)h(k-h), (1/2)h(k+h), (1/2)k(k-h), (1/2)k(k+h)) .$$

It can be checked that these always work. (Collate these with the result given in Solution 1.)

(b) Since $a/b = c/(a+b+c)$, $c = a(a+b)/(b-a)$ and $d = (a+b)+[a(a+b)/(b-a)] = b(a+b)/(b-a)$. Hence

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{d} = \frac{1}{b}+\frac{b-a}{a+b}\left(\frac{1}{a}+\frac{1}{b}\right)$$
$$= \frac{1}{b}+\frac{b-a}{ab} = \frac{1}{a} \ .$$

(c) [M. Lipnowski]

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{a+b+c} = \frac{1}{a}$$

is equivalent to

$$0 = bc(a+b+c) - a(b+c)(a+b+c) - abc$$
$$= (b+c)(bc - a^2 - ab - ac) \ ,$$

which in turn is equivalent to

$$0 = bc - a^2 - ab - ac \Longleftrightarrow bc = a(a+b+c) = ad \ .$$

284. Suppose that $ABCDEF$ is a convex hexagon for which $\angle A + \angle C + \angle E = 360°$ and

$$\frac{AB}{BC}\cdot\frac{CD}{DE}\cdot\frac{EF}{FA} = 1 \ .$$

Prove that

$$\frac{AB}{BF}\cdot\frac{FD}{DE}\cdot\frac{EC}{CA} = 1 \ .$$

Solution 1. [A. Zhang] Since the hexagon is convex, all its angles are less than 180°. A dilation of factor $|CD|/|DE|$ followed by a rotation, both with centre D, takes E to C and F to a point G so that $\Delta DCG \sim \Delta DEF$, $\angle DEF = \angle DCG$ and $DE : EF : FD = DC : CG : GD$. Since $DE : DC = FD : GD$ and $\angle EDC = \angle FDG$, $\Delta EDC \sim \Delta FDC$ and $DE : DC : CE = FD : DG : GF$. Now

$$\angle DCG + \angle BCD = \angle DEF + \angle BCD = 360° - \angle FAB > 180°$$

so that C lies within the triangle BDG and $\angle BCG = 360° - (\angle DCG + \angle BCD) = \angle FAB$.

Also,

$$\frac{CG}{CD} = \frac{EF}{DE} = \frac{AF}{AB}\cdot\frac{BC}{CD}$$

so that $CG : BC = AF : AB$, with the result that $\Delta BCG \sim \Delta BAF$, $AB : BF : FA = CB : BG : GC$ and $\angle FBG = \angle ABC$. From the equality of these angles and $AB : CB = BF : BG$, we have that $\Delta ABC \sim \Delta FBG$ and $AB : BC : CA = FB : BG : GF$. Hence

$$\frac{AB}{BF}\cdot\frac{FD}{DE}\cdot\frac{EC}{CA} = \frac{CA}{GF}\cdot\frac{GF}{CE}\cdot\frac{CE}{CA} = 1$$

as desired.

Solution 2. [T. Yin] *Lemma.* Let $ABCD$ be a convex quadrilateral with a, b, c, d, p, q the respective lengths of AB, BC, CD, DA, AC and BD. Then

$$p^2q^2 = (ac+bd)^2 - 4abcd\cos^2\theta$$

where $2\theta = \angle A + \angle C$.

Proof of Lemma. Locate E within the quadrilateral so that $\angle EDC = \angle ADB$ and $\angle ECD = \angle ABD$. Then $\Delta ABD \sim \Delta ECD$ whence $ac = qx$ where x is the length of EC. Now $\angle ADE = \angle BDC$ and $AD : DE = BD : CD$ whence $\Delta ADE \sim \Delta BDC$ and $bd = qy$ with y the length of AE.

Hence $abcd = q^2xy$ and $ac + bd = q(x + y)$. Therefore,

$$a^2c^2 + b^2d^2 + 2abcd = q^2(x^2 + 2xy + y^2) = q^2(x^2 + y^2) + 2abcd$$

which reduces to $a^2c^2 + b^2d^2 = q^2(x^2 + y^2)$.

Since $\angle DEC = \angle BAD$ and $\angle AED = \angle BCD$,

$$\angle AEC = \angle AED + \angle DEC = \angle C + \angle A = 2\theta \ .$$

By the law of cosines,

$$p^2 = x^2 + y^2 - 2xy\cos 2\theta = x^2 + y^2 - 2xy(2cos^2\theta - 1) \Longrightarrow$$

$$\begin{aligned} a^2c^2 + b^2d^2 &= p^2q^2 + 4q^2xy\cos^2\theta - 2q^2xy \\ &= p^2q^2 + 4abcd\cos^2\theta - 2abcd \end{aligned}$$

so that the desired result follows. ♠

Note that, when $\angle A + \angle C = 180°$, then we get Ptolemy's Theorem. Consider the hexagon of the problem with $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$, $|EF| = e$, $|FA| = f$, $|BF| = g$, $|CA| = h$, $|CF| = m$, $|DF| = u$ and $|CE| = v$. We are given that $ace = bdf$ and need to prove that $auv = dgh$.

From the lemma applied to $ABDF$, we obtain that

$$g^2h^2 = a^2m^2 + 2abfm + b^2f^2 - 4abfm\cos^2\alpha$$

where $2\alpha = \angle BAC + \angle BCF$. Applying the lemma to $CDEF$ yields that

$$u^2v^2 = d^2m^2 + 2cdem + c^2e^2 - 4cdem\cos^2\beta$$

where $2\beta = \angle FCD + \angle DEF$. Since $\angle A + \angle C + \angle E = 360°$, $\alpha + \beta = 180°$ and $\cos^2\alpha = \cos^2\beta$. Finally,

$$\begin{aligned} d^2g^2h^2 - a^2u^2v^2 &= (a^2d^2m^2 + 2abd^2fm + b^2d^2f^2 - 4abd^2fm\cos^2\alpha) \\ &\quad - (a^2d^2m^2 + 2a^2cdem + a^2c^2e^2 - 4a^2cdem\cos^2\beta) \\ &= 2adm(bdf - ace) + (b^2d^2f^2 - a^2c^2e^2) - 4adm(bdf - ace)\cos^2\alpha = 0 \ , \end{aligned}$$

whence $auv = dgh$ as required.

Solution 3. [Y. Zhao] The proof uses inversion in a circle and directed angles. Recall that, if O is the centre of a circle of radius r, then inversion is that involution $X \leftrightarrow X'$ for which X' is on the ray from O through X and $OX \cdot OX' = r^2$. It is not too hard to check using similar triangles that $\angle OPQ = \angle OQ'P'$ and using the law of cosines that $P'Q' = PQ \cdot (r^2/(OP \cdot OQ))$. For this problem, we make F the centre of the inversion. Then

$$\begin{aligned} 360° &= \angle FAB + \angle BCD + \angle DEF = \angle FAB + \angle BCF + \angle FCD + \angle DEF \\ &= \angle A'B'F + \angle FB'C' + \angle C'D'F + \angle FD'E' = \angle A'B'C' + \angle C'D'E' \end{aligned}$$

whence $\angle C'B'A' = \angle C'D'E'$.

In the following, we suppress the factor r^2. We obtain that

$$\begin{aligned} \frac{A'B'}{B'C'} \cdot \frac{C'D'}{D'E'} &= \left(\frac{AB}{FA \cdot FB} \cdot \frac{FB \cdot FC}{BC}\right) \cdot \left(\frac{CD}{FC \cdot FD} \cdot \frac{FD \cdot FE}{DE}\right) \\ &= \frac{AB}{FA} \cdot \frac{CD}{BC} \cdot \frac{EF}{DE} = 1 \end{aligned}$$

so that $A'B' : B'C' = D'E' : C'D'$. This, along with $\angle C'B'A' = \angle C'D'E'$ implies that $\Delta C'B'A' \sim \Delta C'D'E'$, so that $A'B' : A'C' = D'E' : E'C'$ or $A'B' \cdot E'C' = A'C' \cdot E'D'$.

Therefore

$$\begin{aligned}\frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} &= \left(\frac{A'B'}{FA' \cdot FB'} \cdot B'F \right) \cdot \left(\frac{1}{F'D'} \cdot \frac{FD' \cdot FE'}{D'E'} \right) \cdot \left(\frac{E'C'}{FE' \cdot FC'} \cdot \frac{FC' \cdot FA'}{C'A'} \right) \\ &= \frac{A'B'}{A'C'} \cdot \frac{E'C'}{E'D'} = 1 ,\end{aligned}$$

as desired.

Solution 4. [M. Abdeh-Kolahchi] Let A, B, C, D, E, F be points in the complex plane with

$$B - A = a = |a|(\cos\alpha + i\sin\alpha)$$

$$C - B = b = |b|(\cos\beta + i\sin\beta)$$

$$D - C = c = |c|(\cos\gamma + i\sin\gamma)$$

$$E - D = d = |d|(\cos\delta + i\sin\delta)$$

$$F - E = e = |e|(\cos\epsilon + i\sin\epsilon)$$

$$A - F = f = |f|(\cos\phi + i\sin\phi) .$$

Modulo $360°$, we have that

$$\angle A = \angle FAB \equiv 180° - (\phi - \alpha)$$

$$\angle C = \angle BCD \equiv 180° - (\delta - \beta)$$

$$\angle E = \angle DEF \equiv 180° - (\epsilon - \gamma) .$$

Also $a + b + c + d + e + f = 0$ and

$$\begin{aligned}\frac{ace}{bdf} &= \frac{|a||c||e|(\cos\alpha + i\sin\alpha)(\cos\gamma + i\sin\gamma)(\cos\epsilon + i\sin\epsilon)}{|b||d||f|(\cos\beta + i\sin\beta)(\cos\delta + i\sin\delta)(\cos\phi + i\sin\phi)} \\ &= 1(\cos(\alpha - \phi + \delta - \beta + \epsilon - \gamma)) \\ &= \cos(\angle A - 180° + \angle C - 180° + \angle E - 180°) = \cos(-180°) = -1 ,\end{aligned}$$

whence $ace + bdf = 0$. Therefore,

$$0 = ad(a + b + c + d + e + f) + (ace + bdf) = a(d + e)(c + d) + d(a + f)(a + b) ,$$

whence

$$\begin{aligned}\frac{a(d+e)(c+d)}{d(a+f)(a+b)} = -1 &\Longrightarrow \frac{|a|}{|a+f|} \cdot \frac{|d+e|}{|d|} \cdot \frac{|c+d|}{|a+b|} = 1 \\ &\Longrightarrow \frac{AB}{BF} \cdot \frac{FD}{DE} \cdot \frac{EC}{CA} = 1 .\end{aligned}$$

285. (a) Solve the following system of equations:

$$(1 + 4^{2x-y})(5^{1-2x+y}) = 1 + 2^{2x-y+1} ;$$

$$y^2 + 4x = \log_2(y^2 + 2x + 1) .$$

(b) Solve for real values of x:

$$3^x \cdot 8^{x/(x+2)} = 6 .$$

Express your answers in a simple form.

Solution. Let $u = 2x - y$. Then

$$(1 + 4^u)(5^{1-u}) = 1 + 2^{u+1}$$

so that

$$5^{u-1} = \frac{1 + 2^{2u}}{1 + 2^{u+1}} = 2^{u-1} + \frac{1 - 2^{u-1}}{1 + 2^{u+1}} \ .$$

Thus,

$$5^{u-1} - 2^{u-1} = \frac{1 - 2^{u-1}}{1 + 2^{u+1}} \ .$$

When $u > 1$, the left side of this equation is positive while the right is negative; when $u < 1$, the reverse is true. Hence, the only possible solution is $u = 1$, which checks out.

Substituting for x leads to

$$y^2 + 2y + 2 = \log_2(y^2 + y + 2) \ .$$

Since $y^2 + y + 2 = (y + \frac{1}{2})^2 + \frac{7}{4} > 0$, the right side is defined and is in fact positive. Let

$$\phi(y) = y^2 + 2y + 2 - \log_2(y^2 + y + 2) \ .$$

Then

$$\phi'(y) = \frac{2y(y+1)^2 + 4(y+1) - (\log_2 e)(2y+1)}{y^2 + y + 2} \ .$$

$$\phi'(y) = 0 \Longleftrightarrow (y+1)^2 = -\left((2 - \log_2 e) + \frac{4 - \log_2 e}{2y}\right) \ .$$

From the graphs of the two sides of the equation, we see that the left side and the right side have opposite signs when $y > 0$ and become equal for exactly one value of y. It follows that $\phi'(y)$ changes sign exactly once so that $\phi(y)$ decreases and then increases. Thus, $\phi(y)$ vanishes at most twice. Indeed, $\phi(-2) = \phi(-1) = 0$, and so $(x, y) = (0, -1), (-\frac{1}{2}, -2)$ are the only solutions of the equation.

(b) The equation can be rewritten

$$1 = 3^{1-x} 2^{2(1-x)/(x+2)}$$

whence

$$0 = (1 - x)(\log 3 + (2/(x+2)) \log 2) \ .$$

Thus, either $x = 1$ or $0 = \log_2 3 + 2/(x+2)$. The latter leads to

$$x = -2(1 + \log_3 2) = -2(\log_3 6) = -\log_3 36 \ .$$

286. Construct inside a triangle ABC a point P such that, if X, Y, Z are the respective feet of the perpendiculars from P to BC, CA, AB, then P is the centroid (intersection of the medians) of triangle XYZ.

Solution 1. Let AU, BV, CW be the medians of triangle ABC and let AL, BM, CN be their respective images in the bisectors of angles A, B, C. Since AU, BV, CW intersect in a common point (the centroid of ΔABC). AL, BM, CN must intersect in a common point P. This follows from the sine version of Ceva's theorem and its converse. Let X, Y, Z be the respective feet of the perpendiculars from P to sides BC, AC, AB.

Let I, J, K be the respective feet of the perpendiculars from the centroid G to the sides BC, AC and AB. The quadrilateral $PYAZ$ is the image of the quadrilateral $GJAK$ under a reflection in the angle bisector of A followed by a dilation with centre A and factor AP/AG. Hence $PY : PZ = GK : GI$. Since triangles AGB and AGC have the same area,

$$AB \cdot GK = AC \cdot GJ \Longrightarrow PY : PZ = AC : AB = b : c \ .$$

Applying a similar argument involving PX, we find that

$$PX : PY : PZ = a : b : c \ .$$

Let $PX = ae$, $PY = be$, $PZ = ce$. Then, since $\angle XPY + \angle ACB = 180°$,

$$[PXY] = \frac{1}{2}abe^2 \sin \angle XPY = e^2\left(\frac{1}{2}ab\sin C\right) = e^2[ABC] \ .$$

Similarly, $[PYZ] = [PZX] = e^2[ABC] = [PXY]$, whence P must be the centroid of triangle XYZ.

Solution 2. [M. Lipnowski] Erect squares $ARSB$, $BTUC$, $CVWA$ externally on the edges of the triangle. Suppose that RS and VW intersect at A', RS and TU at B' and TU and UW at C'.

We establish that AA', BB' and CC' are concurrent. They are cevians in the triangle $A'B'C'$. We have that

$$\begin{aligned}
&\frac{\sin \angle RA'A}{\sin \angle WA'A} \cdot \frac{\sin \angle VC'C}{\sin \angle UC'C} \cdot \frac{\sin \angle TB'B}{\sin \angle SB'B} \\
&\qquad = \frac{(AR/AA')}{(AW/AA')} \cdot \frac{(VC/CC')}{(UC/CC')} \cdot \frac{(TB/BB')}{(BS/BB')} \\
&\qquad = \frac{AR}{AW} \cdot \frac{VC}{UC} \cdot \frac{TB}{BC} = \frac{c}{b} \cdot \frac{b}{a} \cdot \frac{a}{c} = 1 \ .
\end{aligned}$$

Hence AA', BB', CC' intersect in a point P by the converse to Ceva's Theorem. P is the desired point.

To prove that this works, we first show that $PX : PY : PZ = a : b : c$, and then that $[XPY] = [YPZ] = [ZPX]$. Observe that, since $\Delta PZA \sim \Delta ARA'$ and $\Delta PYA \sim \Delta AWA'$,

$$\frac{PY}{PZ} = \frac{PY(AA'/PA)}{PZ(AA'/PA)} = \frac{AW}{AR} = \frac{b}{c} \ ,$$

and similarly that $PX : PZ = a : c$. Now

$$\angle XPY = 360° - \angle PXC - \angle PYC - \angle XCY = 180° - \angle XCY = 180° - \angle ACB \ ,$$

so that $[XPY] = \frac{1}{2}PX \cdot PY \sin \angle XPY = \frac{1}{2}PX \cdot PY \sin \angle ACB$. We find that

$$\begin{aligned}
[XPY] : [YPZ] : [ZPX] &= \frac{1}{2}PX \cdot PY \sin \angle ACB : \frac{1}{2}PY \cdot PZ \sin \angle ACB : \frac{1}{2}PZ \cdot PX \sin \angle ABC \\
&= \frac{1}{2}ab\sin C : \frac{1}{2}bc\sin A : \frac{1}{2}ca\sin B = [ABC] : [ABC] : [ABC] = 1 : 1 : 1 \ .
\end{aligned}$$

Hence $[XPY] = [YPZ] = [ZPX] = \frac{1}{3}[XYZ]$, so that the altitudes of these triangle from P to the sides of triangle XYZ are each one-third of the corresponding altitudes for triangle XYZ. Hence P must be the centroid of triangle XYZ.

Comment. A. Zhang and Y. Zhao gave the same construction. Zhang first gave an argument that P, being the centroid of triangle XYZ is characterized by $PX : PY : PZ = a : b : c$. This is a result of the characterization $[XPY] = [YPZ] = [ZPX]$ and the law of sines, with the argument similar to Lipnowski's. Zhao used the fact that $PX : PY : PZ = BC : CA : AB$ and that the vectors $\overrightarrow{PX}$, $\overrightarrow{PY}$, $\overrightarrow{PZ}$ were dilated versions of $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{AB}$ after a 90° rotation, so that $\overrightarrow{PX} + \overrightarrow{PY} + \overrightarrow{PZ} = \overrightarrow{O}$.

287. Let M and N be the respective midpoints of the sides BC and AC of the triangle ABC. Prove that the centroid of the triangle ABC lies on the circumscribed circle of the triangle CMN if and only if

$$4 \cdot |AM| \cdot |BN| = 3 \cdot |AC| \cdot |BC| \ .$$

Solution 1.

$$4|AM||BN| = 3|AC||BC| \Longleftrightarrow 12|AM||GN| = 12|AN||MC| \Longleftrightarrow |AM| : |MC| = |AN| : |GN|$$

$$\Longleftrightarrow \Delta AMC \sim \Delta ANG \Longleftrightarrow \angle AMC = \angle ANG$$

$\Longleftrightarrow GMGN$ is concyclic.

Solution 2. [A. Zhang] Since M and N are respective midpoints of BC and AC, $[ABC] = 4[NMC]$, so that

$$[ABMN] = \frac{3}{4}[ABC] = \frac{3}{8}|AC||BC| \sin \angle ACB \ .$$

However, $[ABMN] = \frac{1}{2}|AM||BN| \sin \angle NGM$ (why?). Hence

$$4|AM||BN| \sin \angle NGM = 3|AC||BC| \sin \angle ACB \ .$$

Observe that G lies inside triangle ABC, and so lies within the circumcircle of this triangle. Hence $\angle NGM = \angle AGB > \angle ACB$. We deduce that

$$4|AM||BN| = 3|AC||BC| \Longleftrightarrow \sin \angle NGM = \sin \angle ACB \Longleftrightarrow \angle NGM + \angle ACB = 180^\circ$$

$\Longleftrightarrow CMGN$ is concyclic.

288. Suppose that $a_1 < a_2 < \cdots < a_n$. Prove that

$$a_1 a_2^4 + a_2 a_3^4 + \cdots + a_n a_1^4 \geq a_2 a_1^4 + a_3 a_2^4 + \cdots + a_1 a_n^4 \ .$$

Solution. The result is trivial for $n = 2$. To deal with the $n = 3$ case, observe that, when $x < y < z$,

$$(xy^4 + yz^4 + zx^4) - (yx^4 + zy^4 + xz^4) = (1/2)(z-x)(y-x)(z-y)[(x+y)^2 + (x+z)^2 + (y+z)^2] \geq 0 \ .$$

As an induction hypothesis, assume that the result holds for the index $n \geq 3$. Then

$$\begin{aligned}(a_1 a_2^4 + a_2 a_3^4 + \cdots + a_n a_{n+1}^4 + a_{n+1} a_1^4) - (a_2 a_1^4 + a_3 a_2^4 + \cdots + a_{n+1} a_n^4 + a_1 a_{n+1}^4)& \\ = (a_1 a_2^4 + a_2 a_3^4 + \cdots + a_n a_1^4) - (a_2 a_1^4 + a_3 a_2^4 + \cdots + a_1 a_n^4)& \\ + (a_1 a_n^4 + a_n a_{n+1}^4 + a_{n+1} a_1^4) - (a_n a_1^4 + a_{n+1} a_n^4 + a_1 a_{n+1}^4) \geq 0 \ ,&\end{aligned}$$

as desired.

289. Let $n(r)$ be the number of points with integer coordinates on the circumference of a circle of radius $r > 1$ in the cartesian plane. Prove that

$$n(r) < 6\sqrt[3]{\pi r^2} \ .$$

Solution. Let $A = \pi r^2$ be the area of the circle, so that the right side of the inequality is $6A^{1/3}$. We observe that $A > 3$, $\pi^2 < (22/7)^2 < 10 < (2.2)^3$.

$$\begin{aligned}6A^{1/3} - 2\pi^{2/3}A^{1/3} &= (6 - 2\pi^{2/3})A^{1/3} > (6 - 2 \times 10^{1/3})A^{1/3} \\ &> (6 - 4.4) \times 3^{1/3} > 1.6 \times 1.25 = 2 \ ,\end{aligned}$$

so that there is an even integer k for which

$$6 = 2 \times 3^{2/3} \times 3^{1/3} < 2\pi^{2/3}A^{1/3} < k < 6A^{1/3} .$$

In particular, $8\pi^2 A < k^3$.

Let $P_1P_2 \cdots P_k$ be a regular $k-$gon inscribed in the circle. Locate the vertices so that none have integer coordinates. (How?) Identify $P_{k+1} = P_1$ and $P_{k+2} = P_2$, and let $\mathbf{v_i} = \overrightarrow{P_iP_{i+1}}$ for $1 \leq i \leq k$. Observe that $\mathbf{v_i}$ has length less than $2\pi r/k = (2/k)(\pi A)^{1/2}$. Then, for each i, the area of triangle $P_iP_{i+1}P_{i+2}$ is equal to

$$\frac{1}{2}|\mathbf{v_i} \times \mathbf{v_{i+1}}| = \frac{1}{2}|\mathbf{v_i}||\mathbf{v_{i+1}}|\sin(2\pi/k) < \frac{1}{2} \times \frac{4}{k^2} \times \pi A \times \frac{2\pi}{k} = \frac{1}{2} \times \frac{8\pi^2}{k^3} \times A < \frac{1}{2} .$$

Suppose, if possible, that the arc joining P_i and P_{i+2} (through P_{i+1}) contains points U, V, W, each with integer coordinates. Then, if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are the corresponding vectors for these points, then $|(\mathbf{v} - \mathbf{u}) \times (\mathbf{w} - \mathbf{u})|$ must be a positive integer, and so the area of triangle UVW must be at least 1/2. But each of the sides of triangle UVW has length less than the length of P_iP_{i+2} and the shortest altitude of triangle UVW is less than the altitude of triangle $P_iP_{i+1}P_{i+2}$ from P_{i+1} to side P_iP_{i+2}. Thus,

$$\frac{1}{2} \leq [UVW] \leq [P_iP_{i+1}P_{i+2}] < \frac{1}{2} ,$$

a contradiction. Hence, each arc P_iP_{i+2} has at most two points with integer coordinates. The whole circumference of the circle is the union of $k/2$ nonoverlapping such arcs, so that there Must be at most k points with integer coordinates. The result follows.

290. The School of Architecture in the *Olymon* University proposed two projects for the new Housing Campus of the University. In each project, the campus is designed to have several identical dormitory buildings, with the same number of one-bedroom apartments in each building. In the first project, there are 12096 apartments in total. There are eight more buildings in the second project than in the first, and each building has more apartments, which raises the total of apartments in the project to 23625. How many buildings does the second project require?

Solution. Let the number of buildings in the first project be n. Then there must be $12096/n$ apartments in each of them. The number of buildings in the second project is $n + 8$ with $23625/(n + 8)$ apartments in each of them. Since the number of apartments is an integer, $n + 8$, and so n, is odd. Furthermore, $12096 = 2^6 \cdot 3^3 \cdot 7$ and $23625 = 3^3 \cdot 5^3 \cdot 7$. Since n is an odd factor of 12096, n must take one of the values 1, 3, 7, 9, 21, 27, 63 or 189. Since $n + 8$ must be a factor of 23625, the only possible values for n are 1, 7 or 27. Taking into account that the number of apartments in each building of the second project is more than the number of apartments in each building of the first project, n must satisfy the inequality

$$\frac{12096}{n} < \frac{23625}{n+8} ,$$

which is equivalent to $n > 512/61$. Thus, $n \geq 9$. Therefore, $n = 27$ and $n + 8 = 35$. The second project requires 35 buildings.

291. The n-sided polygon $A_1, A_2, \cdots, A_n$ $(n \geq 4)$ has the following property: The diagonals from each of its vertices divide the respective angle of the polygon into $n - 2$ equal angles. Find all natural numbers n for which this implies that the polygon $A_1A_2 \cdots A_n$ is regular.

Solution. Let the measures of the angles of the polygon at each vertex A_i be a_i for $1 \leq i \leq n$. When $n = 4$, the polygon need not be regular. Any nonsquare rhombus has the property.

Let n exceed 4. Consider triangle $A_1A_2A_n$. We have that

$$a_1 + \frac{a_2}{n-2} + \frac{a_n}{n-2} = 180^\circ .$$

Since the sum of the exterior angles of an $n-$gon is $a_1 + a_2 + \cdots + a_n = (n-2)180°$, we find that

$$(n-3)a_1 = a_3 + a_4 + \cdots + a_{n-1} \ .$$

Suppose, wolog, that a_1 is a largest angle in the polygon. Then

$$(n-3)a_1 = a_3 + a_4 + \cdots + a_{n-1} \leq (n-3)a_1$$

with equality if and only if $a_1 = a_3 = \cdots = a_{n-1}$. Suppose, if possible that one of a_2 and a_n is strictly less than a_1. We have an inequality for a_4 analogous to the one given for a_1, and find that

$$(n-3)a_4 = a_1 + a_2 + a_6 + \cdots + a_n < (n-3)a_1 = (n-3)a_4 \ ,$$

a contradiction. Hence, all the angles a_i must be equal and the polygon regular.

292. 1200 different points are randomly chosen on the circumference of a circle with centre O. Prove that it is possible to find two points on the circumference, M and N, so that:

- M and N are different from the chosen 1200 points;
- $\angle MON = 30°$;
- there are *exactly* 100 of the 1200 points inside the angle MON.

Solution. The existence of the points M and N will be evident when we prove that there is a central angle of 30° which contains exactly 100 of the given points. Construct six diameters of the circle for which none of their endpoints coincide with any of the given points and they divide its circumference into twelve equal arcs of 30°; such a construction is always possible. If one of the angles contains exactly 100 points, then we have accomplished our task. Assume none of the angles contains 100 points. Then some contain more and others, less. Wolog, we can choose adjacent angles for which the first contains $d_1 > 100$ points and the second $d_2 < 100$ points. Define a function d which represents the number of points inside a rotating angle with respect to its position, and imagine this rotating angle moves from the position of the first of the adjacent angles to the second. As the angle rotates, one of the following occurs: (i) a new point enters the rotating angle; (ii) a point leaves the rotating angle; (iii) no point leaves or enters; (iv) one point leaves while another enters. Thus, the value of d changes by 1 at a time from d_1 to d_2, as so at some point must take the value 100. The desired result follows.

293. Two players, Amanda and Brenda, play the following game: Given a number n, Amanda writes n different natural numbers. Then, Brenda is allowed to erase several (including none, but not all) of them, and to write either $+$ or $-$ in front of each of the remaining numbers, making them positive or negative, respectively, Then they calculate their sum. Brenda wins the game is the sum is a multiple of 2004. Otherwise the winner is Amanda. Determine which one of them has a winning strategy, for the different choices of n. Indicate your reasoning and describe the strategy.

Solution. Amanda has a winning strategy, if $n \leq 10$. She writes the numbers $1, 2, 2^2, 2^3, \cdots, 2^{n-1}$. Recall that, for any natural number k, $2^k > 1 + 2 + \cdots + 2^{k-1}$. Since $2^{10} = 1024$, it is clear that, regardless of Brenda's choice, the result sum but lies between -1023 and 1023, inclusive, and it is not 0, since its sign coincides with the sign of the largest participating number. Hence, the sum cannot be a multiple of 2004.

Let $n \geq 11$. Then it is Brenda that has a winning strategy. The set C of numbers chosen by Amanda has $2^n - 1 > 2003$ different non-empty subsets. By the Pigeonhole Principle, two of these sums must leave the same remainder upon division by 2004. Let A and B be two sets with the same remainder. Brenda assigns a positive sign to all numbers that lie in A but not in B; she assigns a negative sign to all numbers that lie in B but not in A; she erases all the remaining numbers of C. The sum of the numbers remaining is equal to the difference of the sum of the numbers in A and B, which is divisible by 2004. Thus, Brenda wins.

Therefore, Amanda has a winning strategy when $n \leq 10$ and Brenda has a winning strategy when $n \geq 11$.

294. The number $N = 10101 \cdots 0101$ is written using $n+1$ ones and n zeros. What is the least possible value of n for which the number N is a multiple of 9999?

Solution. Observe that $N = 1 + 10^2 + \cdots + 10^{2n}$, $9999 = 3^2 \cdot 11 \cdot 101$ and $10^2 \equiv 1$ modulo 9 and modulo 11. Modulo 9 or modulo 11, $N \equiv n+1$, so that N is a multiple of 99 if and only if $n \equiv 98 \pmod{99}$. N is a multiple of 101 if and only if n is odd. Hence the smallest value of n for which N is a multiple of 9999 is 197.

295. In a triangle ABC, the angle bisectors AM and CK (with M and K on BC and AB respectively) intersect at the point O. It is known that

$$|AO| \div |OM| = \frac{\sqrt{6}+\sqrt{3}+1}{2}$$

and

$$|CO| \div |OK| = \frac{\sqrt{2}}{\sqrt{3}-1} \ .$$

Find the measures of the angles in triangle ABC.

Solution. Let $AB = c$, $BC = a$, $AC = b$, $CM = x$, $AK = y$, $\angle ABC = \beta$, $\angle ACB = \gamma$ and $\angle CAB = \alpha$. In triangle AMC, CO is an angle bisector, whence

$$AC : CM = AO : OM \Longleftrightarrow \frac{b}{x} = \frac{\sqrt{6}+\sqrt{3}+1}{2} \ . \tag{1}$$

In triangle ABC, AM is an angle bisector, whence

$$AB : AC = BM : CM \Longleftrightarrow \frac{c}{b} = \frac{a-x}{x} \Longleftrightarrow \frac{x}{a} = \frac{b}{c+b} \ . \tag{2}$$

Multiplying (1) and (2), we get

$$\frac{b}{a} = \frac{(\sqrt{6}+\sqrt{3}+1)b}{2(c+b)} \Longleftrightarrow a = \frac{2(b+c)}{\sqrt{6}+\sqrt{3}+1} \ . \tag{3}$$

Similarly, in triangle AKC, AO is an angle bisector. Hence

$$CO : OK = AC : AK \Longleftrightarrow \frac{\sqrt{3}-1}{\sqrt{2}} = \frac{y}{b} \ . \tag{4}$$

In triangle ABC, CK is an angle bisector. Hence

$$BC : AC = BK : AK \Longleftrightarrow \frac{a}{b} = \frac{c-y}{y} \Longleftrightarrow \frac{a+b}{b} = \frac{c}{y} \ . \tag{5}$$

Multiplying (4) and (5), we get

$$\frac{c}{b} = \frac{(a+b)(\sqrt{3}-1)}{b\sqrt{2}} \Longleftrightarrow c = \frac{(a+b)(\sqrt{3}-1)}{\sqrt{2}} \ . \tag{6}$$

Solve (3) and (6) to get b and c in terms of a. We find that

$$(b,c) = \left(\left(\frac{\sqrt{3}+1}{2} \right) a, \left(\frac{\sqrt{6}}{2} \right) a \right) \ .$$

From the Law of Cosines for triangle ABC,

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab} = \frac{1}{2} \Longleftrightarrow \gamma = 60^\circ \ .$$

From the Law of Sines, we have that

$$\frac{\sin\alpha}{\sin\gamma} = \frac{a}{c} \Longleftrightarrow \sin\alpha = \frac{1}{\sqrt{2}} \Longleftrightarrow \alpha = 45^\circ \ .$$

The remaining angle $\beta = 75^\circ$.

296. Solve the equation

$$5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x} \ .$$

Solution 1. [G. Siu; T. Liu] Let $u = \sin x$. For the equation to be meaningful, we require that $u \neq 0 \pmod{\pi}$. The equation is equivalent to

$$\begin{aligned} 0 &= 4u^4 - 10u^3 + 10u^2 - 5u + 1 = (u-1)(4u^3 - 6u^2 + 4u - 1) \\ &= (u-1)[u^4 - (1-u)^4] = (u-1)[u^2 - (u-1)^2][u^2 + (u-1)]^2 \\ &= u(u-1)(2u-1)[u^2 + (u-1)^2] \ . \end{aligned}$$

We must have that $u = 1$ or $u = 1/2$, whence $x \equiv \pi/6, \pi/2, 5\pi/6 \pmod{2\pi}$.

Solution 2. We exclude $x \equiv 0 \pmod{\pi}$. Let $y = \sin x + (2\sin x)^{-1}$. The given equation is equivalent to

$$0 = 2y^2 - 5y + 3 = (2y-3)(y-1) \ .$$

Thus

$$\sin x + \frac{1}{2\sin x} = 1$$

or

$$\sin x + \frac{1}{2\sin x} = \frac{3}{2} \ .$$

The first equation leads to

$$0 = \sin^2 x + (\sin x - 1)^2$$

with no real solutions, while the second leads to

$$0 = 2\sin^2 x - 3\sin x + 1 = (2\sin x - 1)(\sin x - 1) \ ,$$

whence it follows that $x \equiv \pi/6, \pi/2, 5\pi/6 \pmod{2\pi}$.

297. The point P lies on the side BC of triangle ABC so that $PC = 2BP$, $\angle ABC = 45^\circ$ and $\angle APC = 60^\circ$. Determine $\angle ACB$.

Solution 1. Let D be the image of C under a reflection with axis AP. Then $\angle APC = \angle APD = \angle DPB = 60^\circ$, $PD = PC = 2BP$, so that $\angle DBP = 90^\circ$. Hence AB bisects the angle DBP, and AP bisects the angle DPC, whence A is equidistant from BD, PC and PD.

Thus, AD bisects $\angle EDP$, where E lies on BD produced. Thus

$$\begin{aligned} \angle ACB = \angle ADP &= \frac{1}{2}\angle EDP \\ &= \frac{1}{2}(180^\circ - \angle BDP) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ \ . \end{aligned}$$

Solution 2. [Y. Zhao] Let Q be the midpoint of PC and R the intersection of AP and the right bisector of PQ, so that $PR = QR$ and $BR = CR$. Then $\angle RPQ = \angle RQP = 60^\circ$ and triangle PQR is equilateral. Hence $PB = PQ = PR = RQ = QC$ and $\angle PBR = \angle PRB = \angle QRC = \angle QCR = 30^\circ$.

Also, $\angle RBA = 15^\circ = \angle PAB = \angle RAB$, so $AR = BR = CR$. Thus, $\angle RAC = \angle RCA$. Now $\angle ARC = 180^\circ - \angle PRQ - \angle QRC = 90^\circ$, so that $\angle RCA = 45^\circ$ and $\angle ACB = 75^\circ$.

Solution 3. [R. Shapiro] Let H be the foot of the perpendicular from C to AP. Then CPH is a $30-60-90$ triangle, so that $BP = \frac{1}{2}PC = PH$ and $\angle PBH = \angle PHB = 30^\circ = \angle PCH$. Hence, $BH = HC$. As

$$\angle HAB = \angle PAB = 180^\circ - 120^\circ - 45^\circ = 15^\circ = \angle ABP - \angle HBP = \angle ABH \ ,$$

$AH = BH = HC$. Therefore, $\angle HAC = \angle HCA = 45^\circ$. Thus, $\angle ACB = \angle HCA + \angle PCH = 75^\circ$.

Solution 4. From the equation expressing $\tan 30^\circ$ in terms of $\tan 15^\circ$, we find that $\tan 15^\circ = 2 - \sqrt{3}$ and $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$. Let $\angle ACP = \theta$, so that

$$\angle PAC = 180^\circ - 60^\circ - \theta = 120^\circ - \theta \ .$$

Suppose, wolog, we set $|BP| = 1$, so that $|PC| = 2$. Then by the Law of Sines in triangle ABP,

$$|AP| = \frac{\sin 45^\circ}{\sin 15^\circ} = \sqrt{3} + 1.$$

By the Law of Sines in triangle APC,

$$\frac{\sin\theta}{\sqrt{3}+1} = \frac{\sin(120^\circ - \theta)}{2} = \frac{\sqrt{3}\cos\theta}{4} + \frac{\sin\theta}{4}$$

whence $(3 - \sqrt{3})\sin\theta = (3 + \sqrt{3})\cos\theta$. Hence

$$\tan\theta = 2 + \sqrt{3} = (2 - \sqrt{3})^{-1} = (\tan 15^\circ)^{-1} \ ,$$

so that $\theta = 75^\circ$.

298. Let O be a point in the interior of a quadrilateral of area S, and suppose that

$$2S = |OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \ .$$

Prove that $ABCD$ is a square with centre O.

Solution.

$$\begin{aligned} &|OA|^2 + |OB|^2 + |OC|^2 + |OD|^2 \\ &\quad = \frac{1}{2}(|OA|^2 + |OB|^2) + \frac{1}{2}(|OB|^2 + |OC|^2) + \frac{1}{2}(|OC|^2 + |OD|^2) + \frac{1}{2}(|OD|^2 + |OA|^2) \\ &\quad \geq |OA||OB| + |OB||OC| + |OC||OD| + |OD||OA| \\ &\quad \geq 2[AOB] + 2[BOC] + 2[COD] + 2[DOA] = 2S \end{aligned}$$

with equality if and only if $OA = OB = OC = OD$ and all the angles AOB, BOC, COD and DOA are right. The result follows.

299. Let $\sigma(r)$ denote the sum of all the divisors of r, including r and 1. Prove that there are infinitely many natural numbers n for which

$$\frac{\sigma(n)}{n} > \frac{\sigma(k)}{k}$$

whenever $1 \leq k < n$.

Solution 1. Let $u_m = \sigma(m)/m$ for each positive integer m. Since $d \leftrightarrow 2d$ is a one-one correspondence between the divisors of m and some even divisor of $2m$, $\sigma(2m) \geq 2\sigma(m) + 1$, so that

$$u_{2m} = \frac{\sigma(2m)}{2m} \geq \frac{2\sigma(m)+1}{2m} > \frac{\sigma(m)}{m} = u_m$$

for each positive integer m.

Let r be a given positive integer, and select $s \leq 2^r$ such that $u_s \geq u_k$ for $1 \leq k \leq 2^r$ (*i.e.*, u_s is the largest value of u_k for k up to and including 2^r). Then, as $u_{2s} > u_s$, it must happen that $2^r \leq 2s \leq 2^{r+1}$ and $u_{2s} \geq u_k$ for $1 \leq k \leq 2^r$.

Suppose that n is the smallest positive integer t for which $2^r \leq t$ and $u_k \leq u_t$ for $1 \leq k \leq 2^r$. Then $2^r \leq n \leq 2s \leq 2^{r+1}$. Suppose that $1 \leq k \leq n$. If $1 \leq k \leq 2^r$, then $u_k \leq u_n$ from the definition of n. If $2^r < k < n$, then there must be some number k' not exceeding 2^r for which $u_k < u_{k'} \leq u_n$. Thus, n has the desired property and $2^r \leq n \leq 2^{r+1}$. Since such n can be found for each positive exponent r, the result follows.

Comment. The sequence selected in this way starts off: $\{1, 2, 4, 6, 12, \cdots\}$.

Solution 2. [P. Shi] Define u_m as in Solution 1. Suppose, if possible, that there are only finitely many numbers n satisfying the condition of the problem. Let N be the largest of these, and let u_s be the largest value of u_m for $1 \leq m \leq N$. We prove by induction that $u_n \leq u_s$ for every positive integer n. This holds for $n \leq N$. Suppose that $n > N$. Then, there exists an integer $r < n$ for which $u_r > u_n$. By the induction hypothesis, $u_r \leq u_s$, so that $u_n < u_s$. But this contradicts the fact (as established in Solution 1) that $u_{2s} > u_s$.

300. Suppose that ABC is a right triangle with $\angle B < \angle C < \angle A = 90°$, and let $\mathfrak{K}$ be its circumcircle. Suppose that the tangent to $\mathfrak{K}$ at A meets BC produced at D and that E is the reflection of A in the axis BC. Let X be the foot of the perpendicular from A to BE and Y the midpoint of AX. Suppose that BY meets $\mathfrak{K}$ again in Z. Prove that BD is tangent to the circumcircle of triangle ADZ.

Solution 1. Let AZ and BD intersect at M, and AE and BC intersect at P. Since PY joints the midpoints of two sides of triangle AEX, $PY \| EX$. Since $\angle APY = \angle AEB = \angle AZB = \angle AZY$, the quadrilateral $AZPY$ is concyclic. Since $\angle AYP = \angle AXE = 90°$, AP is a diameter of the circumcircle of $AZPY$ and BD is a tangent to this circle. Hence $MP^2 = MZ \cdot MA$. Since

$$\angle PAD = \angle EAD = \angle EBA = \angle XBA,$$

triangles PAD and XBA are similar. Since

$$\angle MAD = \angle ZAD = \angle ZBA = \angle YBA,$$

it follows that

$$\angle PAM = \angle PAD - \angle MAD = \angle XBA - \angle YBA = \angle XBY$$

so that triangles PAM and XBY are similar. Thus

$$\frac{PM}{AP} = \frac{XY}{XB} = \frac{XA}{2XB} = \frac{PD}{2PA} \Longrightarrow PD = 2PM \Longrightarrow MD = PM \ .$$

Hence $MD^2 = MP^2 = MZ \cdot MA$ and the desired result follows.

Solution 2. [Y. Zhao] As in Solution 1, we see that there is a circle through the vertices of $AZPY$ and that BD is tangent to this circle. Let O be the centre of the circle $\mathfrak{K}$. The triangles OPA and OAD are similar, whereupon $OP \cdot OD = OA^2$. The inversion in the circle $\mathfrak{K}$ interchanges P and D, carries the line BD

to itself and takes the circumcircle of triangle AZP to the circumcircle of triangle AZD. As the inversion preserves tangency of circles and lines, the desired result follows.

301. Let $d = 1, 2, 3$. Suppose that M_d consists of the positive integers that *cannot* be expressed as the sum of two or more consecutive terms of an arithmetic progression consisting of positive integers with common difference d. Prove that, if $c \in M_3$, then there exist integers $a \in M_1$ and $b \in M_2$ for which $c = ab$.

Solution. M_1 consists of all the powers of 2, and M_2 consists of 1 and all the primes. We prove these assertions.

Since $k + (k+1) = 2k+1$, every odd integer exceeding 1 is the sum of two consecutive terms. Indeed, for each positive integers m and r,

$$(m-r) + (m-r+1) + \cdots + (m-1) + m + (m+1) + \cdots + (m+r-1) + (m+r) = (2r+1)m \ ,$$

and,

$$m + (m+1) + \cdots + (m+2r-1) = r[2(m+r)-1] \ ,$$

so that it can be deduced that every positive integer with at least one odd positive divisor exceeding 1 is the sum of consecutives, and no power of 2 can be so expressed. (If $m < r$ in the first sum, the negative terms in the sum are cancelled by positive ones.) Thus, M_1 consists solely of all the powers of 2.

Since $2n = (n+1) + (n-1)$, M_2 excludes all even numbers exceeding 2. Let $k \geq 2$ and $m \geq 1$. Then

$$m + (m+2) + \cdots + (m + 2(k-1)) = km + k(k-1) = k(m+k-1)$$

so that M_2 excludes all multiples of k from k^2 on. Since all such numbers are composite, M_2 must include all primes. Since each composite number is at least as large as the square of its smallest nontrivial divisor, each composite number must be excluded from M_2.

We now examine M_3. The result will be established if we show that M_3 does not contain any number of the form $2^r uv$ where r is a nonnegative integer and u, v are odd integers with $u \geq v > 1$. Suppose first that $r \geq 1$ and let $a = 2^r u - \frac{3}{2}(v-1)$. Then

$$a \geq 2u - \frac{3}{2}(v-1) \geq \frac{v}{2} + 1 > 1$$

and

$$a + (a+3) + \cdots + [a + 3(v-1)] = v[a + (3/2)(v-1)] = 2^r uv \ .$$

Since $m + (m+3) = 2m+3$, we see that M_3 excludes all odd numbers exceeding 3, and hence all odd composite numbers. Hence, every number in M_3 must be the product of a power of 2 and an odd prime or 1.

Comment. The solution provides more than necessary. It suffices to show only that M_1 contains all powers of 2, M_2 contains all primes and M_3 excludes all numbers with a composite odd divisor.

302. In the following, $ABCD$ is an arbitrary convex quadrilateral. The notation $[\cdots]$ refers to the area.

(a) Prove that $ABCD$ is a trapezoid if and only if

$$[ABC] \cdot [ACD] = [ABD] \cdot [BCD] \ .$$

(b) Suppose that F is an interior point of the quadrilateral $ABCD$ such that $ABCF$ is a parallelogram. Prove that

$$[ABC] \cdot [ACD] + [AFD] \cdot [FCD] = [ABD] \cdot [BCD] \ .$$

Solution 1. (a) Suppose that AB is not parallel to CD. Wolog, let these lines meet at E with A between E and B, and D between E and C. Let P, Q, R, S be the respective feet of the perpendiculars from A to CD, B to CD, C to AB, D to AB produced. Then

$$[ABC] \cdot [ACD] = [ABD][BCD] \Leftrightarrow |AB||CR||CD||AP| = |AB||DS||CD||BQ| \Leftrightarrow CR : DS = BQ : AP \ .$$

By similar triangles, we find that $CE : DE = CR : DS = BQ : AP = BE : AE$. The dilation with centre E and factor $|AE|/|BE|$ takes B to A, C to D and so the segment BC to the parallel segment AD. Thus $ABCD$ is a trapezoid.

(b) Let the quadrilateral be in the horizontal plane of three-dimensional space and let F be at the origin of vectors. Suppose that $\mathbf{u} = \overrightarrow{FA}$, $\mathbf{v} = \overrightarrow{FC}$, and $-p\mathbf{u} - q\mathbf{v} = \overrightarrow{FD}$, where p and q are nonnegative scalars. We have that $\overrightarrow{FB} = \mathbf{u} + \mathbf{v}$. Then

$$2[ABC] = |\mathbf{u} \times \mathbf{v}| \ ;$$

$$\begin{aligned} 2[ACD] =& 2([FAC] + [FAD] + [FCD]) \\ &= |\mathbf{u} \times \mathbf{v}| + |\mathbf{u} \times (p\mathbf{u} + q\mathbf{v})| + |\mathbf{v} \times (p\mathbf{u} + q\mathbf{v})| \\ &= (1 + q + p)|\mathbf{u} \times \mathbf{v}| \ ; \end{aligned}$$

$$2[FCD] = p|\mathbf{u} \times \mathbf{v}| \ ;$$

$$2[AFD] = q|\mathbf{u} \times \mathbf{v}| \ ;$$

$$2[ABD] = |(p\mathbf{u} + q\mathbf{v} + \mathbf{u}) \times \mathbf{v}| = (1 + p)|\mathbf{u} \times \mathbf{v}| \ ;$$

$$2[BCD] = |(p\mathbf{u} + q\mathbf{v} + \mathbf{v}) \times \mathbf{u}| = (1 + q)|\mathbf{u} \times \mathbf{v}|| \ .$$

The result follows.

Solution 2. [Y. Zhao] Observe that, since $(A + C) + (B + D) = 360°$,

$$\begin{aligned} \sin A \sin C - \sin B \sin D &= \frac{1}{2}[\cos(A - C) - \cos(A + C) - \cos(B - D) + \cos(B + D)] \\ &= \frac{1}{2}[\cos(A - C) - \cos(B - D)] = \frac{1}{2}[\cos(B + A - B - C) - \cos(B + A + B + C)] \\ &= \sin(B + A) \sin(B + C) \ . \end{aligned}$$

(a) Hence

$$\begin{aligned} 4[ABD][BCD] - 4[ABC][ACD] &= (AB \cdot DA \sin A)(BC \cdot CD \sin C) - (AB \cdot BC \sin B)(CD \cdot DA \sin D) \\ &= (AB \cdot BC \cdot CD \cdot DA)(\sin A \sin C - \sin B \sin D) \\ &= (AB \cdot BC \cdot CD \cdot DA) \sin(B + A) \sin(B + C) \ . \end{aligned}$$

The left side vanishes if and only if $A + B = C + D = 180°$ or $B + C = A + D = 180°$, *i.e.*, $AD \| BC$ or $AB \| CD$.

(b) From (a), we have that

$$\begin{aligned} 4[ABD][BCD] - 4[ABC][ACD] &= (AB \cdot BC \cdot CD \cdot DA) \sin(A + B) \sin(B + C) \\ &= (AB \cdot BC \cdot CD \cdot DA) \sin(A + B - 180°) \sin(B + C - 180°) \\ &= (FC \cdot AF \cdot CD \cdot DA)(\sin(\angle BAD - \angle BAF) \sin(\angle BCD - \angle BCF)) \\ &= [(DA \cdot AF) \sin \angle DAF][(DC \cdot CF) \sin \angle DCF] \\ &= 4[AFD][FCD] \ , \end{aligned}$$

as desired.

303. Solve the equation
$$\tan^2 2x = 2\tan 2x \tan 3x + 1 \ .$$

Solution 1. Let $u = \tan x$ and $v = \tan 2x$. Then

$$v^2 - 2v\left(\frac{u+v}{1-uv}\right) - 1 = 0$$

$$\Longleftrightarrow v^2 - uv^3 - 2uv - 2v^2 - 1 + uv = 0$$

$$\Longleftrightarrow 0 = uv + 1 + v^2 + uv^3 = (uv+1)(1+v^2)$$

$$\Longleftrightarrow uv = -1 \ .$$

Now $v = 2u(1-u^2)^{-1}$, so that $2u = v - u^2v = u + v$ and $u = v$. But then $u^2 = -1$ which is impossible. Hence the equation has no solution.

Solution 2.
$$\begin{aligned} 0 &= \tan^2 2x - 2\tan 3x \tan 2x - 1 \\ &= \tan^2 2x - 2\tan 3x \tan 2x + \tan^2 3x - \sec^2 3x \\ &= (\tan 2x - \tan 3x)^2 - \sec^2 3x \\ &= (\tan 2x - \tan 3x - \sec 3x)(\tan 2x - \tan 3x + \sec 3x) \ . \end{aligned}$$

Hence, either $\tan 2x = \tan 3x + \sec 3x$ or $\tan 2x = \tan 3x - \sec 3x$. Suppose that the former holds. Multiplying the equation by $\cos 2x \cos 3x$ yields $\sin 2x \cos 3x = \sin 3x \cos 2x + \cos 2x$. Hence

$$\begin{aligned} 0 &= \cos 2x + (\sin 3x \cos 2x - \sin 2x \cos 3x) \\ &= 1 - 2\sin^2 x + \sin x = (1 - \sin x)(1 + 2\sin x) \ , \end{aligned}$$

whence
$$x \equiv \frac{\pi}{2}, -\frac{\pi}{6}, \frac{7\pi}{6}$$
modulo 2π. But $\tan 3x$ is not defined at any of these angles, so the equation fails. Similarly, in the second case, we obtain $0 = (2\sin x - 1)(\sin x + 1)$ so that

$$x \equiv \frac{-\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

modulo 2π, and the equation again fails. Thus, there are no solutions.

Solution 3. Let $t = \tan x$, so that $\tan 2x = 2t(1-t^2)^{-1}$ and $\tan 3x = (3t - t^3)(1 - 3t^2)^{-1}$. Substituting for t in the equation and clearing fractions leads to

$$4t^2(1 - 3t^2) = 4t(3t - t^3)(1 - t^2) + (1 - t^2)^2(1 - 3t^2)$$

$$\Leftrightarrow 4t^2 - 12t^4 = (12t^2 - 16t^4 + 4t^6) + (1 - 5t^2 + 7t^4 - 3t^6)$$

$$\Leftrightarrow 0 = t^6 + 3t^4 + 3t^2 + 1 = (t^2 + 1)^3 \ .$$

There are no real solutions to the equation.

Solution 4. The equation is undefined if $2x$ or $3x$ is an odd multiple of $\pi/2$. We exclude this case. Then the equation is equivalent to
$$\frac{\sin^2 2x - \cos^2 2x}{\cos^2 2x} = \frac{2\sin 2x \sin 3x}{\cos 2x \cos 3x}$$

or

$$\begin{aligned}
0 &= \frac{2\sin 2x \sin 3x}{\cos 2x \cos 3x} + \frac{\cos 4x}{\cos^2 2x} \\
&= \frac{\sin 4x \sin 3x + \cos 4x \cos 3x}{\cos^2 2x \cos 3x} \\
&= \frac{\cos x}{\cos^2 2x \cos 3x} \ .
\end{aligned}$$

Since $\cos x$ vanishes only if x is an odd multiple of π, we see that the equation has no solution.

Solution 5. [Y. Zhao] Observe that, when $\tan(A-B) \neq 0$,

$$1 + \tan A \tan B = \frac{\tan A - \tan B}{\tan(A-B)} \ .$$

In particular,

$$1 + \tan x \tan 2x = \frac{\tan 2x - \tan x}{\tan x} \quad \text{and} \quad 1 + \tan 2x \tan 3x = \frac{\tan 3x - \tan 2x}{\tan x} \ .$$

There is no solution when $x \equiv 0 \pmod{\pi}$, so we exclude this possibility. Thus

$$\begin{aligned}
0 &= (1 + \tan 2x \tan 3x) + (\tan 2x \tan 3x - \tan^2 2x) \\
&= (\tan 3x - \tan 2x)(\cot x + \tan 2x) = \cot x(\tan 3x - \tan 2x)(1 + \tan x \tan 2x) \\
&= \cot^2 x(\tan 3x - \tan 2x)(\tan 2x - \tan x) \\
&= \cot^2 x \left(\frac{\sin x}{\cos 2x \cos 3x} \right) \left(\frac{\sin x}{\cos x \cos 2x} \right) \ .
\end{aligned}$$

This has no solution.

Solution 6. For a solution, neither $2x$ nor $3x$ can be a multiple of $\pi/2$, so we exclude these cases. Since

$$\tan 4x = \frac{2\tan 2x}{1 - \tan^2 2x} \ ,$$

we find that

$$\cot 4x = \frac{1 - \tan^2 2x}{2\tan 2x} = -\tan 3x \ ,$$

whence $1 + \tan 3x \tan 4x = 0$. Now

$$\tan 4x - \tan 3x = (1 + \tan 3x \tan 4x)\tan x = 0 \ ,$$

so that $4x \equiv 3x \pmod{\pi}$. But we have excluded this. Hence there is no solution to the equation.

304. Prove that, for any complex numbers z and w,

$$(|z| + |w|)\left| \frac{z}{|z|} + \frac{w}{|w|} \right| \leq 2|z + w| \ .$$

Solution 1.

$$\begin{aligned}
&(|z| + |w|)\left| \frac{z}{|z|} + \frac{w}{|w|} \right| \\
&= \left| z + w + \frac{|z|w}{|w|} + \frac{|w|z}{|z|} \right| \\
&\leq |z + w| + \frac{1}{|z||w|}|\bar{z}zw + \bar{w}zw| \\
&= |z + w| + \frac{|zw|}{|z||w|}|\bar{z} + \bar{w}| = 2|z + w| \ .
\end{aligned}$$

Solution 2. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, with a and b real and positive. Then the left side is equal to

$$\begin{aligned}|(a+b)(e^{i\alpha}+e^{i\beta})| &= |ae^{i\alpha}+ae^{i\beta}+be^{i\alpha}+be^{i\beta}| \\ &\leq |ae^{i\alpha}+be^{i\beta}|+|ae^{i\beta}+be^{i\alpha}| \ .\end{aligned}$$

Observe that

$$\begin{aligned}|z+w|^2 &= |(ae^{i\alpha}+be^{i\beta})(ae^{-i\alpha}+be^{-i\beta})| \\ &= a^2+b^2+ab[e^{i(\alpha-\beta)}+e^{i(\beta-\alpha)}] \\ &= |(ae^{i\beta}+be^{i\alpha})(ae^{-i\beta}+be^{-i\alpha})|\end{aligned}$$

from which we find that the left side does not exceed

$$|ae^{i\alpha}+be^{i\beta}|+|ae^{i\beta}+be^{i\alpha}| = 2|ae^{i\alpha}+be^{i\beta}| = 2|z+w| \ .$$

Solution 3. Let $z = ae^{i\alpha}$ and $w = be^{i\beta}$, where a and b are positive reals. Then the inequality is equivalent to

$$\left|\frac{1}{2}(e^{i\alpha}+e^{i\beta})\right| \leq |\lambda e^{i\alpha}+(1-\lambda)e^{i\beta}|$$

where $\lambda = a/(a+b)$. But this simply says that the midpoint of the segment joining $e^{i\alpha}$ and $e^{i\beta}$ on the unit circle in the Argand diagram is at least as close to the origin as another point on the segment.

Solution 4. [G. Goldstein] Observe that, for each $\mu \in \mathbf{C}$,

$$\left|\frac{\mu z}{|\mu z|}+\frac{\mu w}{|\mu w|}\right| = \left|\frac{z}{|z|}+\frac{w}{|w|}\right| \ ,$$

$$|\mu|[|z|+|w|] = |\mu z+\mu w| \ ,$$

and

$$|\mu||z+w| = |\mu z+\mu w| \ .$$

So the inequality is equivalent to

$$(|t|+1)\left|\frac{t}{|t|}+1\right| \leq 2|t+1|$$

for $t \in \mathbf{C}$. (Take $\mu = 1/w$ and $t = z/w$.)

Let $t = r(\cos\theta + i\sin\theta)$. Then the inequality becomes

$$(r+1)\sqrt{(\cos\theta+1)^2+\sin^2\theta} \leq 2\sqrt{(r\cos\theta+1)^2+r^2\sin^2\theta} = 2\sqrt{r^2+2r\cos\theta+1} \ .$$

Now,

$$\begin{aligned}4(r^2&+2r\cos\theta+1)-(r+1)^2(2+2\cos\theta) \\ &= 2r^2(1-\cos\theta)+4r(\cos\theta-1)+2(1-\cos\theta) \\ &= 2(r-1)^2(1-\cos\theta) \geq 0 \ ,\end{aligned}$$

from which the inequality follows.

Solution 5. [R. Mong] Consider complex numbers as vectors in the plane. $q = (|z|/|w|)w$ is a vector of magnitude z in the direction w and $p = (|w|/|z|)z$ is a vector of magnitude w in the direction z. A reflection about the angle bisector of vectors z and w interchanges p and w, q and z. Hence $|p+q| = |w+z|$. Therefore

$$\begin{aligned}(|z|&+|w|)\left|\frac{z}{|z|}+\frac{w}{|w|}\right| \\ &= |z+q+p+w| \leq |z+w|+|p+q| \\ &= 2|z+w| \ .\end{aligned}$$

305. Suppose that u and v are positive integer divisors of the positive integer n and that $uv < n$. Is it necessarily so that the greatest common divisor of n/u and n/v exceeds 1?

Solution 1. Let $n = ur = vs$. Then $uv < n \Rightarrow v < r, u < s$, so that $n^2 = uvrs \Rightarrow rs > n$. Let the greatest common divisor of r and s be g and the least common multiple of r and s be m. Then $m \leq n < rs = gm$, so that $g > 1$.

Solution 2. Let $g = \gcd(u, v)$, $u = gs$ and $v = gt$. Then $gst \leq g^2 st < n$ so that $st < n/g$. Now s and t are a coprime pair of integers, each of which divides n/g. Therefore, $n/g = dst$ for some $d > 1$. Therefore $n/u = n/(gs) = dt$ and $n/v = n/(gt) = ds$, so that n/u and n/v are divisible by d, and so their greatest common divisor exceeds 1.

Solution 3. $uv < n \Longrightarrow nuv < n^2 \Longrightarrow n < (n/u)(n/v)$. Suppose, if possible, that n/u and n/v have greatest common divisor 1. Then the least common multiple of n/u and n/v must equal $(n/u)(n/v)$. But n is a common multiple of n/u and n/v, so that $(n/u)(n/v) \leq n$, a contradiction. Hence the greatest common divisor of n/u and n/v exceeds 1.

Solution 4. Let P be the set of prime divisors of n, and for each $p \in P$. Let $\alpha(p)$ be the largest integer k for which p^k divides n. Since u and v are divisors of n, the only prime divisors of either u or v must belong to P. Suppose that $\beta(p)$ is the largest value of the integer k for which p^k divides uv.

If $\beta(p) \geq \alpha(p)$ for each $p \in P$, then n would divide uv, contradicting $uv < n$. (Note that $\beta(p) > \alpha(p)$ may occur for *some* p.) Hence there is a prime $q \in P$ for which $\beta(q) < \alpha(q)$. Then $q^{\alpha(q)}$ is not a divisor of either u or v, so that q divides both n/u and n/v. Thus, the greatest common divisor of n/u and n/v exceeds 1.

Solution 5. [D. Shirokoff] If n/u and n/v be coprime, then there are integers x and y for which $(n/u)x + (n/v)y = 1$, whence $n(xv + yu) = uv$. Since n and uv are positive, then so is the integer $xv + yu$. But $uv < n \Longrightarrow 0 < xv + yu < 1$, an impossibility. Hence the greatest common divisor of n/u and n/v exceeds 1.

306. The circumferences of three circles of radius r meet in a common point O. They meet also, pairwise, in the points P, Q and R. Determine the maximum and minimum values of the circumradius of triangle PQR.

Answer. The circumradius always has the value r.

Solution 1. [M. Lipnowski] $\angle QPO = \angle QRO$, since OQ is a common chord of two congruent circles, and so subtends equal angles at the respective circumferences. (Why are angle QPO and QRO not supplementary?) Similarly, $\angle OPR = \angle OQR$. Let P' be the reflected image of P in the line QR so that triangle $P'QR$ and PQR are congruent. Then

$$\angle QP'R + \angle QOR = \angle QPR + \angle QOR = \angle QPO + \angle RPO + \angle QOR$$
$$= \angle QRO + \angle OQR + \angle QOR = 180^\circ \; .$$

Hence P' lies on the circle through OQR, and this circle has radius r. Hence the circumradius of PQR equals the circumradius of $P'QR$, namely r.

Solution 2. [P. Shi; A. Wice] Let U, V, W be the centres of the circle. Then $OVPW$ is a rhombus, so that OP and VW intersect at right angles. Let H, J, K be the respective intersections of the pairs (OP, VW), (OQ, UW), (OP, UV). Then H (respectively J, K) is the midpoint of OP and VW (respectively OQ and UW, OP and UV). Triangle PQR is carried by a dilation with centre O and factor $\frac{1}{2}$ onto HJK. Also, HJK is similar with factor $\frac{1}{2}$ to triangle UVW (determined by the midlines of the latter triangle). Hence triangles PQR and UVW are congruent. But the circumcircle of triangle UVW has centre O and radius r, so the circumradius of triangle PQR is also r.

Solution 3. [G. Zheng] Let U, V, W be the respective centres of the circumcircles of OQR, ORP, OPQ. Place O at the centre of coordinates so that

$$U \sim (r\cos\alpha, r\sin\alpha)$$

$$V \sim (r\cos\beta, r\sin\beta)$$

$$W \sim (r\cos\gamma, r\sin\gamma)$$

for some α, β, γ. Since $OVPW$ is a rhombus,

$$P \sim (r(\cos\beta + \cos\gamma), r(\sin\beta + \sin\gamma)) \ .$$

Similarly, $Q \sim (r(\cos\alpha + \cos\gamma), r(\sin\alpha + \sin\gamma)$, so that

$$|PQ| = r\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2} = |UV| \ .$$

Similarly, $|PR| = |UW|$ and $|QR| = |VW|$. Thus, triangles PQR and UVW are congruent. Since O is the circumcentre of triangle UVW, the circumradius of triangle PQR equals the circumradius of triangle UVW which equals r.

Solution 4. Let U, V, W be the respective centres of the circles QOR, ROP, POQ. Suppose that $\angle OVR = 2\beta$; then $\angle OPR = \beta$. Suppose that $\angle OWQ = 2\gamma$; then $\angle OPQ = \gamma$. Hence $\angle QPR = \beta + \gamma$. Let ρ be the circumradius of triangle PQR. Then $|QR| = 2\rho\sin(\beta + \gamma)$.

Consider triangle QUR. The reflection in the axis OQ takes W to U so that $\angle QUO = \angle QWO = 2\gamma$. Similarly, $\angle RUO = 2\gamma$, whence $\angle QUR = 2(\beta + \gamma)$. Thus triangle QUR is isosceles with $|QU| = |QR| = r$ and apex angle QUR equal to $2(\beta + \gamma)$. Hence $|QR| = 2r\sin(\beta + \gamma)$. It follows that $\rho = r$.

Comment. This problem was the basis of the logo for the 40th International Mathematical Olympiad held in 1999 in Romania.

307. Let p be a prime and m a positive integer for which $m < p$ and the greatest common divisor of m and p is equal to 1. Suppose that the decimal expansion of m/p has period $2k$ for some positive integer k, so that

$$\frac{m}{p} = .ABABABAB\ldots = (10^k A + B)(10^{-2k} + 10^{-4k} + \cdots)$$

where A and B are two distinct blocks of k digits. Prove that

$$A + B = 10^k - 1 \ .$$

(For example, $3/7 = 0.428571\ldots$ and $428 + 571 = 999$.)

Solution. We have that

$$\frac{m}{p} = \frac{10^k A + B}{10^{2k} - 1} = \frac{10^k A + B}{(10^k - 1)(10^k + 1)}$$

whence

$$m(10^k - 1)(10^k + 1) = p(10^k A + B) = p(10^k - 1)A + p(A + B) \ .$$

Since the period of m/p is $2k$, $A \neq B$ and p does not divide $10^k - 1$. Hence $10^k - 1$ and p are coprime and so $10^k - 1$ must divide $A + B$. However, $A \leq 10^k - 1$ and $B \leq 10^k - 1$ (since both A and B have k digits), and equality can occur at most once. Hence $A + B < 2 \times 10^k - 2 = 2(10^k - 1)$. It follows that $A + B = 10^k - 1$ as desired.

Comment. This problem appeared in the *College Mathematics Journal* 35 (2004), 26-30. In writing up the solution, it is clearer to set up the equation and clear fractions, so that you can argue in terms of factors of products.

308. Let a be a parameter. Define the sequence $\{f_n(x) : n = 0, 1, 2, \cdots\}$ of polynomials by

$$f_0(x) \equiv 1$$

$$f_{n+1}(x) = xf_n(x) + f_n(ax)$$

for $n \geq 0$.

(a) Prove that, for all n, x,

$$f_n(x) = x^n f_n(1/x) \ .$$

(b) Determine a formula for the coefficient of x^k $(0 \leq k \leq n)$ in $f_n(x)$.

Solution 1. The polynomial $f_n(x)$ has degree n for each n, and we will write

$$f_n(x) = \sum_{k=0}^{n} b(n,k)x^k \ .$$

Then

$$x^n f_n(1/x) = \sum_{k=0}^{n} b(n,k)x^{n-k} = \sum_{k=0}^{n} b(n, n-k)x^k \ .$$

Thus, (a) is equivalent to $b(n,k) = b(n, n-k)$ for $0 \leq k \leq n$.

When $a = 1$, it can be established by induction that $f_n(x) = (x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^n$. Also, when $a = 0$, $f_n(x) = x^n + x^{n-1} + \cdots + x + 1 = (x^{n+1} - 1)(x-1)^{-1}$. Thus, (a) holds in these cases and $b(n,k)$ is respectively equal to $\binom{n}{k}$ and 1.

Suppose, henceforth, that $a \neq 1$. For $n \geq 0$,

$$\begin{aligned} f_{n+1}(k) &= \sum_{k=0}^{n} b(n,k)x^{k+1} + \sum_{k=0}^{n} a^k b(n,k)x^k \\ &= \sum_{k=1}^{n} b(n,k-1)x^k + b(n,n)x^{n+1} + b(n,0) + \sum_{k=1}^{n} a^k b(n,k)x^k \\ &= b(n,0) + \sum_{k=1}^{n} [b(n,k-1) + a^k b(n,k)]x^k + b(n,n)x^{n+1} \ , \end{aligned}$$

whence $b(n+1, 0) = b(n, 0) = b(1, 0)$ and $b(n+1, n+1) = b(n, n) = b(1, 1)$ for all $n \geq 1$. Since $f_1(x) = x+1$, $b(n, 0) = b(n, n) = 1$ for each n. Also

$$b(n+1, k) = b(n, k-1) + a^k b(n, k) \tag{1}$$

for $1 \leq k \leq n$.

We conjecture what the coefficients $b(n,k)$ are from an examination of the first few terms of the sequence:

$$f_0(x) = 1; \qquad f_1(x) = 1 + x; \qquad f_2(x) = 1 + (a+1)x + x^2;$$

$$f_3(x) = 1 + (a^2 + a + 1)x + (a^2 + a + 1)x^2 + x^3;$$

$$f_4(x) = 1 + (a^3 + a^2 + a + 1)x + (a^4 + a^3 + 2a^2 + a + 1)x^2 + (a^3 + a^2 + a + 1)x^3 + x^4;$$

$$f_5(x) = (1 + x^5) + (a^4 + a^3 + a^2 + a + 1)(x + x^4) + (a^6 + a^5 + 2a^4 + 2a^3 + 2a^2 + a + 1)(x^2 + x^3) \ .$$

We make the empirical observation that

$$b(n+1, k) = a^{n+1-k} b(n, k-1) + b(n, k) \tag{2}$$

which, with (1), yields

$$(a^{n+1-k}-1)b(n,k-1)=(a^k-1)b(n,k)$$

so that

$$b(n+1,k)=\left[\frac{a^k-1}{a^{n+1-k}-1}+a^k\right]b(n,k)=\left[\frac{a^{n+1}-1}{a^{n+1-k}-1}\right]b(n,k)$$

for $n\geq k$. This leads to the conjecture that

$$b(n,k)=\left(\frac{(a^n-1)(a^{n-1}-1)\cdots(a^{k+1}-1)}{(a^{n-k}-1)(a^{n-k-1}-1)\cdots(a-1)}\right)b(k,k) \tag{3}$$

where $b(k,k)=1$.

We establish this conjecture. Let $c(n,k)$ be the right side of (3) for $1\leq k\leq n-1$ and $c(n,n)=1$. Then $c(n,0)=b(n,0)=c(n,n)=b(n,n)=1$ for each n. In particular, $c(n,k)=b(n,k)$ when $n=1$.

We show that

$$c(n+1,k)=c(n,k-1)+a^kc(n,k)$$

for $1\leq k\leq n$, which will, through an induction argument, imply that $b(n,k)=c(n,k)$ for $0\leq k\leq n$. The right side is equal to

$$\left(\frac{a^n-1}{a^{n-k}-1}\right)\cdots\left(\frac{a^{k+1}-1}{a-1}\right)\left[\frac{a^k-1}{a^{n-k+1}-1}+a^k\right]=\frac{(a^{n+1}-1)(a^n-1)\cdots(a^{k+1}-1)}{(a^{n+1-k}-1)(a^{n-k}-1)\cdots(a-1)}=c(n+1,k)$$

as desired. Thus, we now have a formula for $b(n,k)$ as required in (b).

Finally, (a) can be established in a straightforward way, either from the formula (3) or using the pair of recursions (1) and (2).

Solution 2. (a) Observe that $f_0(x)=1$, $f_1(x)=x+1$ and $f_1(x)-f_0(x)=x=a^0xf_0(x/a)$. Assume as an induction hypothesis that $f_k(x)=x^kf(1/x)$ and

$$f_k(x)-f_{k-1}(x)=a^{k-1}xf_{k-1}(x/a)$$

for $0\leq k\leq n$. This holds for $k=1$.

Then

$$\begin{aligned}f_{n+1}(x)-f_n(x)&=x[f_n(x)-f_{(n-1)}(x)]+[f_n(ax)-f_{n-1}(ax)]\\&=a^{n-1}x^2f_{n-1}(x/a)+a^{n-1}axf_{n-1}(x)\\&=a^nx[f_{n-1}(x)+(x/a)f_{n-1}(x/a)=a^nxf_n(x/a)\ ,\end{aligned}$$

whence

$$\begin{aligned}f_{n+1}(x)&=f_n(x)+a^nxf_n(x/a)=f_n(x)+a^nx(x/a)^nf_n(a/x)\\&=x^nf_n(1/x)+x^{n+1}f_n(a/x)=x^{n+1}[(1/x)f_n(1/x)+f_n(a/x)]=x^{n+1}f_{n+1}(1/x)\ .\end{aligned}$$

The desired result follows.

Comment. Because of the appearance of the factor $a-1$ in denominators, you should dispose of the case $a=1$ separately. Failure to do so on a competition would likely cost a mark.

309. Let $ABCD$ be a convex quadrilateral for which all sides and diagonals have rational length and AC and BD intersect at P. Prove that AP, BP, CP, DP all have rational length.

Solution 1. Because of the symmetry, it is enough to show that the length of AP is rational. The rationality of the lengths of the remaining segments can be shown similarly. Coordinatize the situation by taking $A \sim (0,0)$, $B \sim (p,q)$, $C \sim (c,0)$, $D \sim (r,s)$ and $P \sim (u,0)$. Then, equating slopes, we find that

$$\frac{s}{r-u} = \frac{s-q}{r-p}$$

so that

$$\frac{sr-ps}{s-q} = r-u$$

whence $u = r - \frac{sr-ps}{s-q} = \frac{ps-qr}{s-q}$.

Note that $|AB|^2 = p^2+q^2$, $|AC|^2 = c^2$, $|BC|^2 = (p^2-2pc+c^2)+q^2$, $|CD|^2 = (c^2-2cr+r^2)+s^2$ and $|AD|^2 = r^2+s^2$, we have that

$$2rc = AC^2 + AD^2 - CD^2$$

so that, since c is rational, r is rational. Hence s^2 is rational.

Similarly

$$2pc = AC^2 + AB^2 - BC^2 \ .$$

Thus, p is rational, so that q^2 is rational.

$$2qs = q^2+s^2-(q-s)^2 = q^2+s^2-[(p-r)^2+(q-s)^2]+p^2-2pr+r^2$$

is rational, so that both qs and $q/s = (qs)/s^2$ are rational. Hence

$$u = \frac{p-r(q/s)}{1-(q/s)}$$

is rational.

Solution 2. By the cosine law, the cosines of all of the angles of the triangle ACD, BCD, ABC and ABD are rational. Now

$$\frac{AP}{AB} = \frac{\sin\angle ABP}{\sin\angle APB}$$

and

$$\frac{CP}{BC} = \frac{\sin\angle PBC}{\sin\angle BPC} \ .$$

Since $\angle APB + \angle BPC = 180°$, therefore $\sin\angle APB = \sin\angle BPC$ and

$$\begin{aligned}\frac{AP}{CP} &= \frac{AB\sin\angle ABP}{BC\sin\angle PBC} = \frac{AB\sin\angle ABP\sin\angle PBC}{BC\sin^2\angle PBC} \\ &= \frac{AB(\cos\angle ABP\cos\angle PBC - \cos(\angle ABP+\angle PBC))}{BC(1-\cos^2\angle PBC)} \\ &= \frac{AB(\cos\angle ABD\cos\angle DBC - \cos\angle ABC)}{BC(1-\cos^2\angle DBC)}\end{aligned}$$

is rational. Also $AP+CP$ is rational, so that $(AP/CP)(AP+CP) = ((AP/CP)+1)AP$ is rational. Hence AP is rational.

310. (a) Suppose that n is a positive integer. Prove that

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x(x+k)^{k-1}(y-k)^{n-k} \ .$$

(b) Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k} \ .$$

Comments. (a) and (b) are equivalent. To obtain (b) from (a), replace x by $-x/z$ and y by $-y/z$. On the other hand, the substitution $z=-1$ yields (a) from (b).

The establishment of the identities involves the recognition of a certain sum which arise in the theory of finite differences. Let $f(x)$ be a function of x and define the following operators that take functions to functions:

$$If(x) = f(x)$$
$$Ef(x) = f(x+1) = (I+\Delta)f(x)$$
$$\Delta f(x) = f(x+1) - f(x) = (E-I)f(x) \ .$$

For any operator P, $P^n f(x)$ is defined recursively by $P^0 f(x) = f(x)$ and $P^{k+1}f(x) = P(P^{k-1})f(x))$, for $k \geq 1$. Thus $E^k f(x) = f(x+k)$ and

$$\Delta^2 f(x) = \Delta f(x+1) - \Delta f(x) = f(x+2) - 2f(x+1) + f(x) = (E^2 - 2E + I)f(x) = (E-I)^2 f(x) \ .$$

We have an *operational calculus* in which we can treat polynomials in I, E and Δ as satisfying the regular rules of algebra. In particular

$$E^n f(x) = (I+\Delta)^n f(x) = \sum \binom{n}{k} \Delta^k f(x)$$

and

$$\Delta^n f(x) = (E-I)^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^k f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) \ ,$$

for each positive integer n, facts than can be verified directly by unpacking the operational notation.

Now let $f(x)$ be a polynomial of degree $d \geq 0$. If $f(x)$ is constant ($d=0$), then $\Delta f(x) = 0$. If $d \geq 1$, then $\Delta f(x)$ is a polynomial of degree $d-1$. It follows that $\Delta^d f(x)$ is constant, and $\Delta^n f(x) = 0$ whenever $n > d$. This yields the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+k) = 0$$

for all x whenever $f(x)$ is a polynomial of degree strictly less than n.

Solution 1. [G. Zheng]

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1}(y-k)^{n-k} &= \sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1}[(x+y)-(x+k)]^{n-k} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n}{k} x(x+k)^{k-1} \binom{n-k}{j} (x+y)^j (-1)^{n-k-j} (x+k)^{n-k-j} \\
&= \sum_{0 \leq k \leq n-j \leq n} (-1)^{n-k-j} \binom{n}{k} \binom{n-k}{j} x(x+k)^{n-j-1}(x+y)^j \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} (-1)^{n-k-j} \binom{n}{j} \binom{n-j}{k} x(x+k)^{n-j-1}(x+y)^j \\
&= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} (x+y)^j \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} x(x+k)^{n-j-1} \\
&= (x+y)^n x(x+0)^{-1} + x \sum_{j=1}^{n-1} (-1)^{n-j} \binom{n}{j} (x+y)^j \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} (x+k)^{n-j-1} \ .
\end{aligned}$$

Let $m = n - j$ so that $1 \le m \le n$. Then

$$\begin{aligned}\sum_{k=0}^{n-j}(-1)^k\binom{n-j}{k}(x+k)^{n-j-1} &= \sum_{k=0}^{m}(-1)^k\binom{m}{k}(x+k)^{m-1} \\ &= \sum_{k=0}^{m}(-1)^k\binom{m}{k}\sum_{l=0}^{m-1}\binom{m-1}{l}x^{m-l}k^l \\ &= \sum_{l=0}^{m-1}\binom{m-1}{l}x^{m-l}\sum_{k=0}^{m}(-1)^k\binom{m}{k}k^l = 0 \ .\end{aligned}$$

The desired result now follows.

Solution 2. [M. Lipnowski] We prove that

$$\sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k} = (x+y)^n$$

by induction. When $n = 1$, this becomes

$$1 \cdot x(x)^{-1}y + 1 \cdot x(x-z)^0(y+z)^0 = y + x = x + y \ .$$

Assume that for $n \ge 2$,

$$\sum_{k=0}^{n-1}\binom{n-1}{k}x(x-kz)^{k-1}(y+zk)^{n-k-1} = (x+y)^{n-1} \ .$$

Let $f(y) = (x+y)^n$ and $g(y) = \sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k}$. We can establish that $f(y) = g(y)$ for all y by showing that $f'(y) = g'(y)$ for all y (equality of the derivatives with respect to y) and $f(-x) = g(-x)$ (equality when y is replaced by $-x$).

That $f'(y) = g'(y)$ is a consequence of the induction hypothesis and the identity $\binom{n}{k}(n-k) = n\binom{n-1}{k}$. Also

$$\begin{aligned}g(-x) &= \sum_{k=0}^{n}\binom{n}{k}x(x-kz)^{k-1}(-x+kz)^{n-k} \\ &= x\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(x-kz)^{n-1} = 0 \ ,\end{aligned}$$

by appealing to the finite differences result. The desired result now follows.

311. Given a square with a side length 1, let P be a point in the plane such that the sum of the distances from P to the sides of the square (or their extensions) is equal to 4. Determine the set of all such points P.

Solution. If the square is bounded by the lines $x = 0$, $x = 1$, $y = 0$ and $y = 1$ in the Cartesian plane, then the required locus is equal the octagon whose vertices are $(0,2)$, $(1,2)$, $(2,1)$, $(2,0)$, $(1,-1)$, $(0,-1)$, $(-1,0)$, $(-1,1)$. Any point on the locus must lie outside of the square, as within the square the sum of the distances is equal to 2. If, for example, a point on the locus lies between $x = 0$ and $x = 1$, the sum of the distances to the vertical sides is 1, and it must be 1 unit from the nearer horizontal side and 2 units from the farther horizontal side. If, for example, a point on the locus lies to the left of $x = 0$ and above $y = 1$ and has coordinates (u, v), then

$$|u| + (1 + |u|) + v + (v - 1) = 4$$

or

$$-u+1-u+v+v-1=4 \quad \text{or} \quad v-u=2\ .$$

Thus it can be shown that every point on the locus lies on the octagon, and conversely, it is straightforward to verify that each point on the octagon lies on the locus.

312. Given ten arbitrary natural numbers. Consider the sum, the product, and the absolute value of the difference calculated for any two of these numbers. At most how many of all these calculated numbers are odd?

Solution. Suppose that there are k odd numbers and $10-k$ even numbers, where $0 \leq k \leq 10$. There are $k(10-k)$ odd sums, $k(10-k)$ odd differences and $\frac{1}{2}k(k-1)$ odd products (on the presumption that the numbers chosen are distinct), giving a total of

$$\frac{1}{2}(39k-3k^2) = \frac{3}{2}\left[\left(\frac{13}{2}^2\right) - \left(\frac{13}{2}-k\right)^2\right]$$

odd results. This quantity achieves its maximum when $k=13/2$, so the maximum number 63 of calculated numbers occurs when $k=6$ or $k=7$.

Comment. If we allow a number to be operated with itself, then the maximum occurs when $k=7$.

313. The three medians of the triangle ABC partition it into six triangles. Given that three of these triangles have equal perimeters, prove that the triangle ABC is equilateral.

Solution. [P. Shi] Let a, b, c be the respective lengths of the sides BC, CA, AB, and u, v, w the respective lengths of the medians AP, BQ, CR (P, Q, R the respective midpoints of BC, CA, AB). If G is the centroid of the triangle ABC, then

$$AG : GP = BG : GQ = CG : GR = 2 : 1\ .$$

We need three preliminary lemmata.

Lemma 1. $4u^2 = 2b^2+2c^2-a^2; 4v^2 = 2c^2+2a^2-b^2; 4w^2 = 2a^2+2b^2-c^2$.

Proof. This can be established, either by representing the medians vertorially in terms of the sides and applying the cosine law to the whole triangle, or by applying the cosine law to pairs of inner triangles along an edge. ♠

Lemma 2. $u < v; u = v; u > v$ according as $a > b; a = b; a < b$, with analogous results for other pairs of medians and sides.

Proof. $4(u^2-v^2) = 3(b^2-a^2)$. ♠

Lemma 3. If triangle BCR and BCQ have the same perimeter, then $b=c$.

Proof. Equality of the perimeters is equivalent to $BR+RC = BQ+QC$, so that Q and R are points on an ellipse with foci B and C. Since RQ is parallel to the major axis containing BC, R and Q are reflections of each other in the minor axis, so that $RB = QC$. Hence $b=c$. ♠

We now establish conditions under which the triangle must be isosceles.

Lemma 4. Suppose that two adjacent inner triangles along the same side of triangle ABC have the same perimeter. Then triangle ABC is isosceles.

Solution. For example, the equality of the perimeters of BPG and CPG is equivalent to

$$\frac{1}{2}a+\frac{1}{3}u+\frac{2}{3}v = \frac{1}{2}a+\frac{1}{3}u+\frac{2}{3}w$$

$$\Leftrightarrow v=w \Leftrightarrow b=c\ .♠$$

Lemma 5. Suppose that two adjacent inner triangles sharing a vertex of triangle ABC have the same perimeter. Then triangle ABC is isosceles.

Proof. For example, suppose that triangle BRG and PGB have the same perimeter. Produce CR to point T so that $GR = RT$. Thus, BR is a median of triangle BGT. Produce AP to point S so that $GP = PS$. Thus, CP is a median of triangle CPS.

Since GP joins midpoints of two sides of triangle CTB, $TB \| GP$ and $TB = 2GP = GS$. Since triangle PGB and PSC are congruent (SAS), $BG = SC$. Also, $TG = 2RG = GC$. Hence, triangles TBG and GSC are congruent (SSS).

Let X be the midpoint of GC. A translation that takes T to G takes triangle TBG to triangle GSC and median BR to median SX. We have that

$$\text{Perimeter}(SXC) = \text{Perimeter}(BRG) = \text{Perimeter}(PGB) = \text{Perimeter}(PSC) \; .$$

Applying Lemma 3, we deduce that $GS = GC$, whence $u = w$ and $a = c$. ♠

Lemma 6. If two opposite triangles (say, BGP and AQG) have equal perimeters, then triangle ABC is isosceles.

Proof. The equality of the perimeters of BGP and AQG implies that

$$\frac{1}{2}a + \frac{1}{3}u + \frac{2}{3}v = \frac{1}{2}b + \frac{1}{3}v + \frac{2}{3}u$$

$$\Leftrightarrow 3(a - b) = 2(u - v) \; .$$

By Lemma 2, the latter equation holds if and only if $a = b$. ♠

Let us return to the problem. There are essentially four different cases for the three inner triangles with equal perimeters.

Case 1. The three are adjacent (say BRG, BPG, CPG). Then by Lemmata 4 and 5, $a = b = c$.

Case 2. Two are adjacent along a side and the third is opposite (say BPG, CPG, AQG). Then, by Lemmata 4 and 6, $a = b = c$.

Case 3. Two are adjacent at a vertex and the third is opposite (say BPG, CQG, AQG.) Then, by Lemmata 5 and 6, $a = b = c$.

Case 4. No two are adjacent (say BPG, CQG, ARG). Then we have

$$\frac{1}{2}a + \frac{1}{3}u + \frac{2}{3}v = \frac{1}{2}b + \frac{1}{3}v + \frac{2}{3}w = \frac{1}{2}c + \frac{1}{3}w + \frac{2}{3}v \; .$$

Thus

$$3(a - b) = 2(w - u) + 2(w - v) \; . \tag{1}$$

Similarly,

$$3(b - c) = 2(u - v) + 2(u - w) \; ; \tag{2}$$

$$3(c - a) = 2(v - u) + 2(v - w) \; . \tag{3}$$

Suppose, wolog, that $a \geq b \geq c$. Then $u \leq v \leq w$, so that, by (2), $3(b - c) \leq 0 \Rightarrow b = c \Rightarrow v = w$. But then, by (3), $3(c - a) = 2(v - u) \geq 0 \Rightarrow c = a$.

The result follows.

314. For the real numbers a, b and c, it is known that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = 1 \; ,$$

and

$$a + b + c = 1 \ .$$

Find the value of the expression

$$M = \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \ .$$

Solution 1. Putting the first equation over a common denominator and using the second equation yields that

$$a + b + c = abc = 1$$

whence

$$\begin{aligned} M &= \frac{1}{1+a+(1/c)} + \frac{1}{1+b+(1/a)} + \frac{1}{1+c+(1/b)} \\ &= \frac{c}{1+c+ac} + \frac{a}{1+a+ab} + \frac{b}{1+b+bc} \\ &= \frac{c}{1+c+(1/b)} + \frac{a}{1+a+(1/c)} + \frac{b}{1+b+(1/a)} \\ &= \frac{bc}{1+b+bc} + \frac{ac}{1+c+ac} + \frac{ab}{1+a+ab} \ . \end{aligned}$$

This yields three different expressions for M over denominators of the form $1 + a + ab$, which when added together yield $3M = 3$ or $M = 1$.

Solution 2. [V. Krakovna] It is clear that $abc \neq 0$ for the expressions to be defined. As before, $abc = 1$, and

$$\frac{1}{1+a+ab} = \frac{1}{1+a+(1/c)} = \frac{c}{c+ca+1} \ .$$

Hence

$$\begin{aligned} M &= \frac{1}{1+a+ab} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \\ &= \frac{c}{c+ca+1} + \frac{1}{1+b+bc} + \frac{1}{1+c+ca} \\ &= \frac{c+1}{1+c+ca} + \frac{1}{1+b+bc} \\ &= \frac{b(c+1)}{b+bc+1} + \frac{1}{1+b+bc} = \frac{bc+b+1}{b+bc+1} = 1 \ . \end{aligned}$$

315. The natural numbers 3945, 4686 and 5598 have the same remainder when divided by a natural number x. What is the sum of the number x and this remainder?

Solution. Observe that $5598 - 4686 = 912 = 16 \times 57$ and $4686 - 3945 = 741 = 13 \times 57$, so that if a divisor leaves equal remainders for the three numbers, the divisor must also divide evenly into 57. Since $5598 = 98 \times 57 + 12$, $4686 = 82 \times 57 + 12$ and $3945 = 69 \times 57 + 12$. the number x must be 1, 3, 19 or 57. The sums of the number and the remainder are respectively 1, 3, 31 and 69.

316. Solve the equation

$$|x^2 - 3x + 2| + |x^2 + 2x - 3| = 11 \ .$$

Solution. The equation can be rewritten

$$|x - 1|[|x - 2| + |x + 3|] = 11 \ .$$

When $x \leq -3$, the equation is equivalent to

$$2x^2 - x - 12 = 0$$

neither of whose solutions satisfies $x \leq -3$. When $-3 \leq x \leq 1$, the equation is equivalent to $-5x + 5 = 11$ and we get the solution $x = -6/5$. When $1 < x < 2$, the equation is equivalent to $5x - 5 = 11$ which has no solution with $1 < x < 2$. Finally, when $2 \leq x$, the equation is equivalent to $2x^2 - x - 12 = 0$ and we obtain the solution $x = \frac{1}{4}(1 + \sqrt{97})$.

Thus the solutions are $x = -6/5, (1 + \sqrt{97})/4$.

317. Let $P(x)$ be the polynomial

$$P(x) = x^{15} - 2004x^{14} + 2004x^{13} - \cdots - 2004x^2 + 2004x \ ,$$

Calculate $P(2003)$.

Solution 1. For each nonnegative integer n, we have that

$$x^{n+2} - 2004x^{n+1} + 2003x^n = x^n(x-1)(x-2003) \ .$$

Therefore,

$$\begin{aligned} P(x) &= (x^{15} - 2004x^{14} + 2003x^{13}) + (x^{13} - 2004x^{12} + x^{11}) + \cdots + (x^3 - 2004x^2 + 2003x) + x \\ &= (x^{13} + x^{11} + \cdots + x)(x-1)(x-2003) + x \ \ , \end{aligned}$$

whereupon $P(2003) = 2003$.

Solution 2. [R. Tseng]

$$P(x) = (x^{15} - 2003x^{14}) - (x^{14} - 2003x^{13}) + (x^{13} - 2003x^{12}) - \cdots - (x^2 - 2003x) + x$$

whence $P(2003) = 0 - 0 + 0 - \cdots - 0 + 2003 = 2003$.

318. Solve for integers x, y, z the system

$$1 = x + y + z = x^3 + y^3 + z^2 \ .$$

[Note that the exponent of z on the right is 2, not 3.]

Solution 1. Substituting the first equation into the second yields that

$$x^3 + y^3 + [1 - (x+y)]^2 = 1$$

which holds if and only if

$$\begin{aligned} 0 &= (x+y)(x^2 - xy + y^2) + (x+y)^2 - 2(x+y) \\ &= (x+y)(x^2 - xy + y^2 + x + y - 2) \\ &= (1/2)(x+y)[(x-y)^2 + (x+1)^2 + (y+1)^2 - 6] \ . \end{aligned}$$

It is straightforward to check that the only possibilities are that either $y = -x$ or $(x, y) = (0, -2), (-2, 0)$ or $(x, y) = (-3, -2), (-2, -3)$ or $(x, y) = (1, 0), (0, 1)$. Hence

$$(x, y, z) = (t, -t, 1), (1, 0, 0), (0, 1, 0), (-2, -3, 6), (-3, -2, 6), (-2, 0, 3), (0, -2, 3)$$

where t is an arbitrary integer. These all check out.

Solution 2. As in Solution 1, we find that either $x + y = 0, z = 1$ or $x^2 + (1 - y)x + (y^2 + y - 2) = 0$. The discriminant of the quadratic in x is

$$-3y^2 - 6y + 9 = -3(y + 1)^2 + 12 \ ,$$

which is nonnegative when $|y + 1| \leq 4$. Checking out the possibilities leads to the solution.

Solution 3.

$$\begin{aligned}(1 - z)(1 + z) &= 1 - z^2 = x^3 + y^3 \\ &= (x + y)[(x + y)^2 - 3xy] = (1 - z)[(1 - z)^2 - 3xy] \ ,\end{aligned}$$

whence either $z = 1$ or $3xy = (1 - 2z + z^2) - (1 + z) = z(z - 3)$. The former case yields $(x, y, z) = (x, -x, 1)$ while the latter yields

$$x + y = 1 - z \qquad xy = \frac{1}{3}z(z - 3) \ .$$

Thus, we must have that $z \equiv 0 \pmod 3$ and that x, y are roots of the quadratic equation

$$t^2 - (1 - z)t + \frac{z(z - 3)}{3} = 0 \ .$$

The discriminant of this equation is $[12 - (z - 3)^2]/3$. Thus, the only possibilities are that $z = 0, 3, 6$; checking these gives the solutions.

319. Suppose that a, b, c, x are real numbers for which $abc \neq 0$ and

$$\frac{xb + (1 - x)c}{a} = \frac{xc + (1 - x)a}{b} = \frac{xa + (1 - x)b}{c} \ .$$

Is it true that, necessarily, $a = b = c$?

Comment. There was an error in the original formulation of this problem, and it turns out that the three numbers a, b, c are not necessarily equal. Note that in the problem, a, b, c, x all have the same status. Some solvers, incorrectly, took the given conditions as an identity in x, so that they assumed that the equations held for some a, b, c and all x.

Solution 1. Suppose first that $a + b + c \neq 0$. Then the three equal fractions are equal to the sum of their numerators divided by the sum of the denominators [why?]:

$$\frac{x(a + b + c) + (1 - x)(a + b + c)}{a + b + c} = 1 \ .$$

Hence $a = xb + (1 - x)c$, $b = xc + (1 - x)a$, $c = xa + (1 - x)b$, from which $x(b - c) = (a - c)$, $x(c - a) = (b - a)$, $x(a - b) = (c - b)$. Multiplying these three equations together yields that $x^3(b - c)(c - a)(a - b) = (a - c)(b - a)(c - b)$. Therefore, either $x = -1$ or at least two of a, b, c are equal.

If $x = -1$, then $a + b = 2c$, $b + c = 2a$ and $c + a = 2b$. This implies for example that $a - c = 2(c - a)$, whence $a = c$. Similarly, $a = b$ and $b = c$. Suppose on the other hand that, say, $a = b$; then $b = c$ and $c = a$.

The remaining case is that $a + b + c = 0$. Then each entry and sum of pairs of entries is nonzero, and

$$\frac{xa + (1 - x)b}{-(a + b)} = \frac{x(-a - b) + (1 - x)a}{b}$$

$$\Longrightarrow xab + (1 - x)b^2 = x(a + b)^2 - (1 - x)(a^2 + ab)$$

$$\Longrightarrow (1 - x)(a^2 + ab + b^2) = x(a^2 + ab + b^2) \ .$$

Since $2(a^2+ab+b^2) = (a+b)^2+a^2+b^2 > 0$, $1-x = x$ and $x = 1/2$. But in this case, the equations become

$$\frac{b+c}{2a} = \frac{c+a}{2b} = \frac{a+b}{2c}$$

each member of which takes the value $-1/2$ for all a, b, c for which $a+b+c = 0$.

Hence, the equations hold if and only if either $a = b = c$ and x is arbitrary, or $x = 1/2$ and $a+b+c = 0$.

Comment. On can get the first part another way. If d is the common value of the three fractions, then

$$xb + (1-x)c = da \ ; \quad xc + (1-x)a = db \ ; \quad xa + (1-x)b = dc \ .$$

Adding these yeilds that $a+b+c = d(a+b+c)$, whence $d = 1$ or $a+b+c = 0$.

Solution 2. The first inequality leads to

$$xb^2 + (1-x)bc = xac + (1-x)a^2$$

or

$$x(a^2+b^2) - x(a+b)c = a^2 - bc \ .$$

Similarly

$$x(c^2+a^2) - x(c+a)b = b^2 - ca \ ;$$
$$x(b^2+c^2) - x(b+c)a = c^2 - ab \ .$$

Adding these three equations together leads to

$$2x[(a-b)^2 + (b-c)^2 + (c-a)^2] = (a-b)^2 + (b-c)^2 + (c-a)^2 \ .$$

Hence, either $a = b = c$ or $x = 1/2$.

If $x = 1/2$, then for some constant k,

$$\frac{b+c}{a} = \frac{c+a}{b} = \frac{a+b}{c} = k \ ,$$

whence

$$-ka + b + c = a - kb + c = a + b - kc = 0 \ .$$

Add the three left members to get

$$(2-k)(a+b+c) = 0 \ .$$

Therefore, $k = 2$ or $a+b+c = 0$. If $k = 2$, then $a = b = c$, as in Solution 1. If $a+b+c = 0$, then $k = -1$ for any relevant values of a, b, c. Hence, either $a = b = c$ or $x = 1/2$ and $a+b+c = 0$.

320. Let L and M be the respective intersections of the internal and external angle bisectors of the triangle ABC at C and the side AB produced. Suppose that $CL = CM$ and that R is the circumradius of triangle ABC. Prove that

$$|AC|^2 + |BC|^2 = 4R^2 \ .$$

Solution 1. Since $\angle LCM = 90^\circ$ and $CL = CM$, we have that $\angle CLM = \angle CML = 45^\circ$. Let $\angle ACB = 2\theta$. Then $\angle CAB = 45^\circ - \theta$ and $\angle CBA = 45^\circ + \theta$. It follows that

$$\begin{aligned}|BC|^2 + |AC|^2 &= (2R\sin\angle CAB)^2 + (2R\sin\angle CBA)^2 \\ &= 4R^2(\sin^2(45^\circ - \theta) + \sin^2(45^\circ + \theta)) \\ &= 4R^2(\sin^2(45^\circ - \theta) + \cos^2(45^\circ - \theta)) = 4R^2 \ .\end{aligned}$$

Solution 2. [B. Braverman] $\angle ABC$ is obtuse [why?]. Let AD be a diameter of the circumcircle of triangle ABC. Then $\angle ADC = \angle CBM = 45° + \angle LCB$ (since $ABCD$ is concyclic). Since $\angle ACD = 90°$, $\angle DAC = 45° - \angle LCB = \angle CAB$. Hence, chords DC and CB, subtending equal angles at the circumference of the circumcircle, are equal. Hence

$$4R^2 = |AC|^2 + |CD|^2 = |AC|^2 + |BC|^2 \ .$$

321. Determine all positive integers k for which $k^{1/(k-7)}$ is an integer.

Solution. When $k = 1$, the number is an integer. Suppose that $2 \leq k \leq 6$. Then $k - 7 < 0$ and so

$$0 < k^{1/(k-7)} = 1/(k^{1/7-k}) < 1$$

and the number is not an integer. When $k = 7$, the expression is undefined.

When $k = 8$, the number is equal to 8, while if $k = 9$, the number is equal to 3. When $k = 10$, the number is equal to $10^{1/3}$, which is not an integer [why?].

Suppose that $k \geq 11$. We establish by induction that $k < 2^{k-7}$. This is clearly true when $k = 11$. Suppose it holds for $k = m \geq 11$. Then

$$m + 1 < 2^{m-7} + 2^{m-7} = 2^{(m+1)-7} \ ;$$

the desired result follows by induction. Thus, when $k \geq 11$, $1 < k^{1/(k-7)} < 2$ and the number is not an integer.

Thus, the number is an integer if and only if $k = 1, 8, 9$.

322. The real numbers u and v satisfy

$$u^3 - 3u^2 + 5u - 17 = 0$$

and

$$v^3 - 3v^2 + 5v + 11 = 0 \ .$$

Determine $u + v$.

Solution 1. The equations can be rewritten

$$u^3 - 3u^2 + 5u - 3 = 14 \ ,$$
$$v^3 - 3v^2 + 5v - 3 = -14 \ .$$

These can be rewritten as

$$(u - 1)^3 + 2(u - 1) = 14 \ ,$$
$$(v - 1)^3 + 2(v - 1) = -14 \ .$$

Adding these equations yields that

$$\begin{aligned} 0 &= (u - 1)^3 + (v - 1)^3 + 2(u + v - 2) \\ &= (u + v - 2)[(u - 1)^2 - (u - 1)(v - 1) + (v - 1)^2 + 2] \ . \end{aligned}$$

Since the quadratic $t^2 - st + s^2$ is always positive [why?], we must have that $u + v = 2$.

Solution 2. Adding the two equations yields

$$\begin{aligned} 0 &= (u^3 + v^3) - 3(u^2 + v^2) + 5(u + v) - 6 \\ &= (u + v)[(u + v)^2 - 3uv] - 3[(u + v)^2 - 2uv] + 5(u + v) - 6 \\ &= [(u + v)^3 - 3(u + v)^2 + 5(u + v) - 6] - 3uv(u + v - 2) \\ &= \frac{1}{2}(u + v - 2)[(u - v)^2 + (u - 1)^2 + (v - 1)^2 + 4] \ . \end{aligned}$$

Since the second factor is positive, we must have that $u + v = 2$.

Solution 3. [N. Horeczky] Since $x^3 - 3x^2 + 5x = (x-1)^3 + 2(x-1) + 3$ is an increasing function of x (since $x - 1$ is increasing), the equation $x^3 - 3x^2 + 5x - 17 = 0$ has exactly one real solution, namely $x = u$. But

$$\begin{aligned} 0 &= v^3 - 3v^2 + 5v + 11 \\ &= (v-2)^3 + 3(v-2)^2 + 5(v-2) + 17 \\ &= -[(2-v)^3 - 3(2-v)^2 + 5(2-v) - 17] \ . \end{aligned}$$

Thus $x = 2 - v$ satisfies $x^3 - 3x^2 + 5x - 17 = 0$, so that $2 - v = u$ and $u + v = 2$.

Comment. One can see also that each of the two given equations has a unique real root by noting that the sum of the squares of the roots, given by the cofficients, is equal to $3^2 - 2 \times 5 = -1$.

Solution 4. [P. Shi] Let m and n be determined by $u + v = 2m$ and $u - v = 2n$. Then $u = m + n$, $v = m - n$, $u^2 + v^2 = 2m^2 + 2n^2$, $u^2 - v^2 = 4mn$, $u^2 + uv + v^2 = 3m^2 + n^2$, $u^2 - uv + v^2 = m^2 + 3n^2$, $u^3 + v^3 = 2m(m^2 + 3n^2)$ and $u^3 - v^3 = 2n(3m^2 + n^2)$. Adding the equations yields that

$$\begin{aligned} 0 &= (u^3 + v^3) - 3(u^2 + v^2) + 5(u + v) - 6 \\ &= 2m^3 + 6mn^2 - 6m^2 - 6n^2 + 10m - 6 \\ &= 6(m-1)n^2 + 2(m^3 - 3m^2 + 5m - 3) \\ &= 6(m-1)n^2 + 2(m-1)(m^2 - 2m + 3) \\ &= 2(m-1)[3n^2 + (m-1)^2 + 2] \ . \end{aligned}$$

Hence $m = 1$.

323. Alfred, Bertha and Cedric are going from their home to the country fair, a distance of 62 km. They have a motorcycle with sidecar that together accommodates at most 2 people and that can travel at a maximum speed of 50 km/hr. Each can walk at a maximum speed of 5 km/hr. Is it possible for all three to cover the 62 km distance within 3 hours?

Solution 1. We consider the following regime. A begins by walking while B and C set off on the motorcycle for a time of t_1 hours. Then C dismounts from the motorcycle and continues walking, while B drives back to pick up A for a time of t_2 hours. Finally, B and A drive ahead until they catch up with C, taking a time of t_3 hours. Suppose that all of this takes $t = t_1 + t_2 + t_3$ hours.

The distance from the starting point to the point where B picks up A is given by

$$5(t_1 + t_2) = 50(t_1 - t_2)$$

km, and the distance from the point where B drops off C until the point where they all meet again is given by

$$5(t_2 + t_3) = 50(t_3 - t_2) \ .$$

Hence $45t_3 = 45t_1 = 55t_2$, so that $t_1 = t_3 = (11/9)t_2$ and so $t = (31/9)t_2$ and

$$t_1 = \frac{11}{31}t \ , \quad t_2 = \frac{9}{31}t \ , \quad t_3 = \frac{11}{31}t \ .$$

The total distance travelled in the t hours is equal to

$$50t_1 + 5(t_2 + t_3) = \frac{650}{31}$$

kilometers. In three hours, they can travel $1950/31 = 60 + (90/31) > 62$ kilometers in this way, so that all will reach the fair before the three hours are up.

Solution 2. Follow the same regime as in Solution 1. Let d be the distance from the start to the point where B drops C in kilometers. The total time for for C to go from start to finish, namely

$$\frac{d}{50} + \frac{62-d}{5}$$

hours, and we wish this to be no greater than 3. The condition is that $d \geq 470/9$.

The time for B to return to pick up A after dropping C is $9d/550$ hours in which he covers a distance of $9d/11$ km. The total distance travelled by the motorcycle is

$$d + \frac{9d}{11} + (62 - \frac{2d}{11}) = \frac{18d+682}{11}$$

km, and this is covered in

$$\frac{18d+682}{550}$$

hours. To get A and B to their destinations on time, we wish this to not exceed 3; the condition for this is that $d \leq 484/9$. Thus, we can get everyone to the fair on time if

$$\frac{470}{9} \leq d \leq \frac{484}{9} \ .$$

Thus, if $d = 53$, for example, we can achieve the desired journey.

Solution 3. [D. Dziabenko] Suppose that B and C take the motorcycle for exactly $47/45$ hours while A walks after them. After $47/45$ hours, B leaves C to walk the rest of the way, while B drives back to pick up A. C reaches the destination in exactly

$$\frac{62-(47/45)50}{5} + \frac{47}{45} = 3$$

hours. Since B and A start and finish at the same time, it suffices to check that that B reaches the fair on time. When B drops C off, B and A are 47 km apart. It takes B $47/55$ hours to return to pick up A. At this point, they are now

$$62 - 5\left(\frac{47}{45} + \frac{47}{55}\right) = 62 - 47\left(\frac{20}{99}\right) = \frac{5198}{99}$$

km from the fair, which they will reach in a further

$$\frac{5198}{99 \times 50} = \frac{2599}{2475}$$

hours. The total travel time for A and B is

$$\begin{aligned}\frac{47}{45} + \frac{47}{55} + \frac{1}{50}\left[62 - 5\left(\frac{47}{45} + \frac{47}{55}\right)\right] \\ = \frac{9 \times 47}{10 \times 5}\left[\frac{1}{9} + \frac{1}{11}\right] + \frac{31}{25} = \frac{517+423+682}{550} = \frac{811}{275}\end{aligned}$$

hours. This is less than three hours.

324. The base of a pyramid $ABCDV$ is a rectangle $ABCD$ with $|AB| = a$, $|BC| = b$ and $|VA| = |VB| = |VC| = |VD| = c$. Determine the area of the intersection of the pyramid and the plane parallel to the edge VA that contains the diagonal BD.

Solution 1. A dilation with centre C and factor 1/2 takes A to S, the centre of the square and V to M, the midpoint of VC. The plane of intersection is the plane that contains triangle BMD. Since BM is a median of triangle BVC with sides c, c, b, its length is equal to $\frac{1}{2}\sqrt{2b^2+c^2}$ [why?]; similarly, $|DM| = \frac{1}{2}\sqrt{2a^2+c^2}$. Also, $|BD| = \sqrt{a^2+b^2}$. Let $\theta = \angle BMD$. Then, by the law of Cosines,

$$\cos\theta = \frac{c^2-a^2-b^2}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}} \ ,$$

whence

$$\sin\theta = \frac{\sqrt{4c^2(a^2+b^2)-(a^2-b^2)^2}}{\sqrt{2b^2+c^2}\sqrt{2a^2+c^2}} \ .$$

The required area is

$$\frac{1}{2}|BM||DM|\sin\theta = \frac{1}{8}\sqrt{4c^2(a^2+b^2)-(a^2-b^2)^2} \ .$$

Comment. One can also use Heron's formula to get the area of the triangle, but this is more labourious. Another method is to calculate $(1/2)|BD||MN|$, where N is the foot of the perpendicular from M to BD, Note that, when $a \neq b$, N is not the same as S [do you see why?]. If $d = |BD|$ and $x = |SN|$ and, say $|MB| \leq |MD|$, then

$$|MN|^2 = |MB|^2 - \left(\frac{d}{2} - x\right)^2 = |MD|^2 - \left(\frac{d}{2} + x\right)^2$$

whence

$$x = \frac{|MD|^2 - |MB|^2}{2d} \ .$$

If follows that

$$|MN|^2 = \frac{2a^2b^2 - a^4 - b^4 + 4a^2c^2 + 4b^2c^2}{16(a^2+b^2)} \ .$$

325. Solve for positive real values of x, y, t:

$$(x^2+y^2)^2 + 2tx(x^2+y^2) = t^2y^2 \ .$$

Are there infinitely many solutions for which the values of x, y, t are all positive integers? What is the smallest value of t for a positive integer solution?

Solution. Considering the equation as a quadratic in t, we find that the solution is given by

$$t = \frac{(x^2+y^2)[x+\sqrt{x^2+y^2}]}{y^2} = \frac{x^2+y^2}{\sqrt{x^2+y^2}-x} \ .$$

where x and y are arbitrary real numbers. The choice of sign before the radical is governed by the condition that $t > 0$. Integer solutions are those obtained by selecting $(x, y) = (k(m^2-n^2), 2kmn)$ for integers m, n, k where m and n are coprime and k is a multiple of $2n^2$. Then

$$t = \frac{k(m^2+n^2)^2}{2n^2} \ .$$

The smallest solution is found by taking $n = 1$, $m = k = 2$ to yield $(x, y, t) = (6, 8, 25)$.

Comment. L. Fei gives the solution set

$$(x, y, t) = (2n^2+2n, 2n+1, (2n^2+2n+1)^2) \ .$$

This set of solutions satisfies in particular $y^2 = 2x+1$. If, in the above solution, one takes $(x, y) = (2mn, m^2-n^2)$, then $t = (m^2+n^2)^2/(m-n)^2$; in particular, this gives the solution $(x, y, t) = (4, 3, 25)$.

326. In the triangle ABC with semiperimeter $s = \frac{1}{2}(a+b+c)$, points U, V, W lie on the respective sides BC, CA, AB. Prove that

$$s < |AU| + |BV| + |CW| < 3s \ .$$

Give an example for which the sum in the middle is equal to $2s$.

Solution. The triangle inequality yields that $|AB| + |BU| > |AU|$ and $|AC| + |CU| > |AU|$. Adding these and dividing by 2 gives $s > |AU|$. Applying the same inequality to BV and CW yields that

$$3s > |AU| + |BV| + |CW| \ .$$

Again, by the triangle inequality, $|AU|+|BU| > |AB|$ and $|AU|+|CU| > |AC|$. Adding these inequalities gives $2|AU| > |AB| + |AC| - |BC|$. Adding this to analogous inequalities for BV and CW and dividing by 2 yields that $|AU| + |BV| + |CW| > s$.

Comment. For the last part, most students gave a degenerate example in which the points U, V, W coincided with certain vertices of the triangle. A few gave more interesting examples. However, it was necessary to make clear that the stated lengths assigned to AU, BV and CW were indeed possible, *i.e.* they were at least as great as the altitudes. The nicest example came from D. Dziabenko: $|AB| = 6$, $|BC| = 8$, $|CA| = 10$, $|AU| = 8$, $|BV| = 7$ $|CW| = 9$.

327. Let A be a point on a circle with centre O and let B be the midpoint of OA. Let C and D be points on the circle on the same side of OA produced for which $\angle CBO = \angle DBA$. Let E be the midpoint of CD and let F be the point on EB produced for which $BF = BE$.

(a) Prove that F lies on the circle.

(b) What is the range of angle EAO?

Solution 1. [Y. Zhao] When $\angle CBO = \angle DBA = 90°$, the result is obvious. Wolog, suppose that $\angle CBO = \angle DBA < 90°$. Suppose that the circumcircle of triangle OBD meets the given circle at G. Since $OBDG$ is concyclic and triangle OGD is isosceles,

$$\angle OBC = \angle ABD = 180° - \angle OBD = \angle OGD = \angle ODG = \angle OBG \ ,$$

so that $G = C$ and $OBDC$ is concyclic.

Let H lie on OA produced so that $OA = AH$. Since $OB \cdot OH = OA^2$, the inversion in the given circle with centre O interchanges B and H, fixes C and D, and carries the circle $OBDC$ (which passes through the centre O of inversion) to a straight line passing through H, D, C. Thus C, D, H are collinear.

This means that CD always passes through the point H on OA produced for which $OA = AH$. Since E is the midpoint of CD, a chord of the circle with centre O, $\angle OEH = \angle OED = 90°$. Hence E lies on the circle with centre A and radius OA.

Consider the reflection in the point B (the dilation with centre B and factor -1). This takes the circle with centre O and radius OA to the circle with centre A and the same radius, and also interchanges E and F. Since E is on the latter circle, F is on the given circle.

Ad (b), E lies on the arc of the circle with centre A and radius OA that joins O to the point R of intersection of this circle and the given circle. Since $RB \perp OA$, and $OA = OR = RA$, $\angle RAO = 60°$. It can be seen that $\angle EAO$ ranges from $0°$ (when CD is a diameter) to $60°$ (when $C = D = R$).

Solution 2. [A. Wice] We first establish a Lemma.

Lemma. Let UZ be an angle bisector of triangle UVW with Z on VW. Then

$$UZ^2 = UV \cdot UW - VZ \cdot WZ \ .$$

Proof. By the Cosine Law,

$$UV^2 = UZ^2 + VZ^2 - 2UZ \cdot VZ \cos \angle UZW$$

and

$$UW^2 = UZ^2 + WZ^2 + 2UZ \cdot WZ \cos \angle UZW \ .$$

Eliminating the cosine term yields that

$$UV^2 \cdot WZ + UW^2 \cdot VZ = (UZ^2 + VZ \cdot WZ)(WZ + VZ) \ .$$

Now,

$$UV : VZ = UW : WZ = (UV + UW) : (VZ + WZ) \ ,$$

so that

$$UV \cdot WZ = UW \cdot VZ$$

and

$$(UV + UW) \cdot WZ = UW \cdot (WZ + VZ) \ .$$

Thus

$$\begin{aligned} UV^2 \cdot WZ + UW^2 \cdot VZ &= (UV + UW) \cdot WZ \cdot UV \\ &= (WZ + VZ) \cdot UW \cdot UV \ . \end{aligned}$$

It follows that $UW \cdot UV = UZ^2 + VZ \cdot WZ$. ♠

Let R be a point on the circle with $BR \perp OA$, S be the intersection of CD in OA produced, and D' be the reflection of D in OA. (Wolog, $\angle CBO < 90°$.) Since SB is an angle bisector of triangle SCD', from the Lemma, we have that

$$BS^2 = SC \cdot SD' - CB \cdot D'B = SC \cdot SD - BR^2 = SC \cdot SD - (SR^2 - BS^2)$$

whence $SC \cdot SD = SR^2$. Using power of a point, we deduce that SR is tangent to the given circle and $OR \perp SR$.

Now

$$(OA + AS)^2 - OR^2 = RS^2 = BR^2 + (AB + AS)^2 = 3AB^2 + AB^2 + 2AB \cdot AS + AS^2$$

from which $4AB \cdot AS = 4AB^2 + 2AB \cdot AS$, whence $AS = 2AB = OA$. Since $OE \perp CD$, E lies on the circle with diameter OS.

Consider the reflection in the point B (dilation in B with factor -1). It interchanges E and F, interchanges O and A, and switches the circles $ADRC$ and $OERS$. Since E lies on the latter circle, F must lie on the former circle, and the desired result (a) follows.

Ad (b), the locus of E is that part of the circle with centre A that lies within the circle with centre O. Angle EAO is maximum when E coincides with R, and minimum when D coincides with A. Since triangle ORA is equilateral, the maximum angle is $60°$ and the minimum angle is $0°$.

Solution 3. [M. Elqars] Let the radius of the circle be r. Let $\angle CBO = \angle DBA = \alpha$, $\angle DOB = \beta$ and $\angle COF = \gamma$. By the Law of Sines, we have that

$$\sin \alpha : \sin(\gamma - \alpha) = OC : OB = OD : OB = \sin(180° - \alpha) : \sin(\alpha - \beta) = \sin \alpha : \sin(\alpha - \beta) \ ,$$

whence $\alpha - \beta = \gamma - \alpha$. Thus $2\alpha = \beta + \gamma$. Therefore,

$$\angle DOC = 180° - (\beta + \gamma) = 180° - 2\alpha = \angle CBD \ .$$

Thus,

$$\angle DOE = \frac{1}{2}\angle DOC = 90^\circ - \alpha \ ,$$

whence $\angle EDO = \alpha$ and $|OE| = r \sin \alpha$.

Observe that $\sin(\alpha - \beta) : \sin \alpha = OB : OD = 1 : 2$, so that $\sin(\alpha - \beta) = \frac{1}{2} \sin \alpha$.

$$\begin{aligned} |AE|^2 &= |OE|^2 + |OA|^2 - 2|OE||OA| \cos \angle EOA \\ &= r^2 \sin^2 \alpha + r^2 - 2r^2 \sin \alpha \cos(90^\circ - \alpha + \beta) \\ &= r^2[1 + \sin^2 \alpha - 2 \sin \alpha \sin(\alpha - \beta)] \\ &= r^2[1 + \sin^2 \alpha - \sin^2 \alpha] = r^2 \ . \end{aligned}$$

The segments EF and OA bisect each other, so they are diagonals of a parallelogram $OFAE$. Hence $|OF| = |AE| = r$, as desired. As before, we see that $\angle EAO$ ranges from 0° to 60°.

Solution 4. Assign coordinates: $O \sim (0, 0)$, $B \sim (\frac{1}{2}, 0)$, $C \sim (1, 0)$, and let the slope of the lines BC and BD be respectively $-m$ and m. Then $C \sim (\frac{1}{2} + s, -ms)$ and $D \sim (\frac{1}{2} + t, mt)$ for some s and t. Using the fact that the coordinates of C and D satisfy $x^2 + y^2 = 1$, we find that

$$C \sim \left(\frac{m^2 - \sqrt{3m^2 + 4}}{2(m^2 + 1)}, \frac{-m^3 + m\sqrt{3m^2 + 4}}{2(m^2 + 1)} \right)$$

$$D \sim \left(\frac{m^2 + \sqrt{3m^2 + 4}}{2(m^2 + 1)}, \frac{m^3 + m\sqrt{3m^2 + 4}}{2(m^2 + 1)} \right)$$

$$E \sim \left(\frac{m^2}{2(m^2 + 1)}, \frac{m\sqrt{3m^2 + 4}}{2(m^2 + 1)} \right)$$

$$F \sim \left(\frac{m^2 + 2}{2(m^2 + 1)}, \frac{-m\sqrt{3m^2 + 4}}{2(m^2 + 1)} \right) \ .$$

It can be checked that the coordinates of F satisfy the equation $x^2 + y^2 = 1$ and the result follows.

Solution 5. [P. Shi] Assign the coordinates $O \sim (0, -1)$, $B \sim (0, 0)$, $A \sim (0, 1)$. Taking the coordinates of C and D to be of the form $(x, y) = (r \cos \theta, r \sin \theta)$ and $(x, y) = (s \cos \theta, -s \sin \theta)$ and using the fact that both lie of the circle of equation $x^2 + (y + 1)^2 = 4$, we find that

$$C \sim ((\sqrt{\sin^2 \theta + 3} - \sin \theta) \cos \theta, (\sqrt{\sin^2 \theta + 3} - \sin \theta) \sin \theta)$$

$$D \sim ((\sqrt{sin^2 \theta + 3} + \sin \theta) \cos \theta, -(\sqrt{\sin^2 \theta + 3} + \sin \theta) \sin \theta)$$

$$E \sim ((\sqrt{\sin^2 \theta + 3}) \cos \theta, -\sin^2 \theta)$$

$$F \sim (-(\sqrt{\sin^2 \theta + 3}) \cos \theta, \sin^2 \theta) \ .$$

It is straightforward to verify that $|OF|^2 = 4$, from which the result follows.

328. Let $\mathfrak{C}$ be a circle with diameter AC and centre D. Suppose that B is a point on the circle for which $BD \perp AC$. Let E be the midpoint of DC and let Z be a point on the radius AD for which $EZ = EB$.

Prove that

(a) The length c of BZ is the length of the side of a regular pentagon inscribed in $\mathfrak{C}$.

(b) The length b of DZ is the length of the side of a regular decagon (10-gon) inscribed in $\mathfrak{C}$.

(c) $c^2 = a^2 + b^2$ where a is the length of a regular hexagon inscribed in $\mathfrak{C}$.

(d) $(a+b) : a = a : b$.

Comment. We begin by reviewing the trigonetric functions of certain angles. Since

$$\begin{aligned}\cos 72^\circ &= 2\cos^2 36^\circ - 1 = 2\cos^2 144^\circ - 1 \\ &= 2(2\cos^2 72^\circ - 1)^2 = 8\cos^4 72^\circ - 8\cos^2 72^\circ + 1 \ ,\end{aligned}$$

$t = \cos 72^\circ$ is a root of the equation

$$0 = 8t^4 - 8t^2 - t + 1 = (t-1)(2t+1)(4t^2 + 2t - 1) \ .$$

Since it must be the quadratic factor that vanishes when $t = \cos 72^\circ$, we find that

$$\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5}-1}{4} \ .$$

Hence $\cos 144^\circ = -(\sqrt{5}+1)/4$ and $\cos 36^\circ = (\sqrt{5}+1)/4$.

Solution. [J. Park] Wolog, suppose that the radius of the circle is 2.

(a) Select W on the arc of the circle joining A and B such that $BW = BZ$. We have that $|BE| = |EZ| = \sqrt{5}$, $b = |ZD| = \sqrt{5} - 1$, $c^2 = |BZ|^2 + |BW|^2 = 10 - 2\sqrt{5}$ and, by the Law of Cosines applied to triangle BDW,

$$\cos \angle BDW = \frac{1}{4}(\sqrt{5} - 1) \ .$$

Hence BW subtends an angle of 72° at the centre of the circle and so BW is a side of an inscribed regular pentagon.

(b) The angle at each vertex of a regular decagon is 144°. Thus, the triangle formed by the side of an inscribed regular pentagon and two adjacent sides of an inscribed regular decagon has angles $144^\circ, 18^\circ, 18^\circ$. Conversely, if a triangle with these angles has its longest side equal to that of an inscribed regular pentagon, then its two equal sides have lengths equal to those of the sides of a regular decagon inscribed in the same circle.

Now $b = \sqrt{5} - 1$, $c^2 = 10 - 2\sqrt{5}$, so $4b^2 - c^2 = 14 - 2\sqrt{45} > 0$. Thus, $c < 2b$, and we can construct an isosceles triangle with sides b, b, c. By the Law of Cosines, the cosine of the angle opposite c is equal to $(2b^2 - c^2)/(2b^2) = -\frac{1}{4}(\sqrt{5} - 1)$. This angle is equal to 144°, and so b is the side length of a regular inscribed decagon.

(c) The side length of a regular inscribed hexagon is equal to the radius of the circle. We have that

$$a^2 = |BD|^2 = |BZ|^2 - |ZD|^2 = c^2 - b^2 \ .$$

(d) Since $b^2 + 2b - 4 = (b+1)^2 - 5 = 0$, $a^2 = 4 = (2+b)b = (a+b)b$, whence $(a+b) : a = a : b$.

329. Let x, y, z be positive real numbers. Prove that

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2} \ .$$

Solution 1. Let ABC be a triangle for which $|AB| = x$, $|AC| = z$ and $\angle BAC = 120^\circ$. Let AD be a ray through A that bisects angle BAC and has length y. By the law of cosines applied respectively to triangle ABC, ABD and ACD, we find that

$$|BC| = \sqrt{x^2 + xz + z^2} \ ,$$

$$|BD| = \sqrt{x^2 - xy + y^2} \ ,$$

$$|CD| = \sqrt{y^2 - yz + z^2}\ .$$

Since $|BD| + |CD| \geq |BC|$, the desired result follows.

Solution 2. [B. Braverman; B.H. Deng] Note that $x^2 - xy + y^2$, $y^2 - yz + z^2$ and $z^2 + xz + z^2$ are always positive [why?]. By squaring, we see that the given inequality is equivalent to

$$x^2 + z^2 + 2y^2 - xy - yz + 2\sqrt{(x^2 - xy + y^2)(y^2 - yz + z^2)} \geq x^2 + xz + z^2$$

which reduces to

$$2\sqrt{(x^2 - xy + y^2)(y^2 - yz + z^2)} \geq xy + yz + zx - 2y^2\ .$$

If the right side is negative, then the inequality holds trivially. If the right side is positive, the inequality is equivalent (by squaring) to

$$4(x^2 - xy + y^2)(y^2 - yz + z^2) \geq (xy + yz + zx - 2y^2)^2\ .$$

Expanding and simplifying gives the equivalent inequality

$$x^2y^2 + x^2z^2 + y^2z^2 - 2x^2yz - 2xyz^2 + 2xy^2z \geq 0$$

or $(xy + yz - zx)^2 \geq 0$. Since the last always holds, the result follows.

Comment. The above write-up proceeds by a succession of equivalent inequalities to one that is trivial, a working-backwards from the result. The danger of this approach is that one may come to a step where the reasoning is not necessarily reversible, so that instead of a chain of equivalent statements, you get to a stage where the logical implication is in the wrong direction. The possibility that $xy + yz + zx < 2y^2$ complicates the argument a litle and needs to be dealt with. To be on the safe side, you could frame the solution by starting with the observation that $(xy + yz - zx)^2 \geq 0$ and deducing that

$$(xy + yz + zx - 2y^2)^2 \leq 4(x^2 - xy + y^2)(y^2 - yz + z^2)\ .$$

From this, we get that

$$xy + yz + zx - 2y^2 \leq |xy + yz + zx - 2y^2| \leq \sqrt{(x^2 - xy + y^2)(y^2 - yz + z^2)}\ ,$$

from which the required inequality follows by rearranging the terms between the outside members and taking the square root.

Solution 3. [L. Fei] Let $x = ay$ and $z = by$ for some positive reals a and b. Then the given inequality is equivalent to

$$\sqrt{a^2 - a + 1} + \sqrt{b^2 - b + 1} \geq \sqrt{a^2 + ab + b^2}$$

which in turn (by squaring) is equivalent to

$$2 - a - b + 2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1} \geq ab\ . \qquad (*)$$

We have that

$$\begin{aligned} 0 &\leq 3(ab - a - b)^2 = 3a^2b^2 - 6a^2b - 6ab^2 + 3a^2 + 3b^2 + 6ab \\ &= 4(a^2b^2 - a^2b + a^2 - ab^2 + ab - a + b^2 - b + 1) - (a^2b^2 + a^2 + b^2 + 4 + 2a^2b + 2ab^2 - 2ab - 4a - 4b) \\ &= [2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1}]^2 - (ab + a + b - 2)^2\ . \end{aligned}$$

Hence

$$2\sqrt{a^2 - a + 1}\sqrt{b^2 - b + 1} \geq |ab + a + b - 2| \geq ab + a + b - 2\ .$$

Taking the inequality of the outside members and rearranging the terms yields $(*)$.

Comment. The student who produced this solution worked backwards down to the obvious inequality $(ab-a-b)^2 \geq 0$, However, for a proper argument, you need to show how by logical steps you can go in the other direction, *i.e.* from the obvious inequality to the desired one. You will notice that this has been done in the write-up above; the price that you pay is that the evolution from $(ab-a-b)^2 \geq 0$ seems somewhat artificial. There is a place where care is needed. It is conceivable that $ab+a+b-2$ is negative, so that $2\sqrt{a^2-a+1}\sqrt{b^2-b+1} \geq ab+a+b-2$ is always true, even when $4(a^2-a+1)(b^2-b+1) \geq (ab+a+b-1)^2$ fails. The inequality $A^2 \geq B^2$ is equivalent to $A \geq B$ only if you know that A and B are both positive.

330. At an international conference, there are four official languages. Any two participants can communicate in at least one of these languages. Show that at least one of the languages is spoken by at least 60% of the participants.

Solution 1. Let the four languages be E, F, G, I. If anyone speaks only one language, then everyone else must speak that language, and the result holds. Suppose there is an individual who speaks exactly two languages, say E and F. Then everyone else must speak at least one of E and F. If 60% of the participants speaks a particular one of these languages, then the result holds. Otherwise, at least 40% of the participants, constituting set A, must speak E and not F, and 40%, constituting set B, must speak F and not E. Since each person in A must communicate with each person in B, each person in either of these sets must speak G or I. At least half the members of A must speak a particular one of these latter languages, say G. If any of them speaks only G (as well as E), then everyone in B must speak G and so at least 20% + 40% = 60% of the participants speak G. The remaining possibility is that everyone in A speaks both G and I. At least half the members of B speaks a particular one of these languages, say G, and so 40% + 20% = 60% of the participants speak G. Thus, if anyone speaks only two languages, the result holds.

Finally, suppose that every participant speaks at least three languages. Let p% speak E, F and G (and possibly, but not necessarily I), q% speak E, F, I but not G, r% speak E, G, I but not F, s% speak F, G, I but not E. Then $p+q+r+s = 100$ and so

$$(p+q+r)+(p+q+s)+(p+r+s)+(q+r+s) = 300 \ .$$

At least one of the four summands on the left is at least 75, Suppose it is $p+r+s$, say. Then at least 75% speaks G. The result holds once again.

Solution 2. [D. Rhee] As in Solution 1, we take the languages to be E, F, G, I, and can dispose of the case where someone speaks exactly one of these. Let $(A \cdots C)$ denote the set of people who speaks the languages $A \cdots C$ and no other language, and suppose that each person speaks at least two languages.

We first observe that in each of the pairs of sets, $\{(EF), (GI)\}$, $\{(EG), (FI)\}$, $\{(EI), (FG)\}$, at least one of the sets in each pair is empty. So either there is one language that is spoken by everyone speaking exactly two languages, or else there are only three languages spoken among those that speak exactly two languages. Thus, wolog, we can take the two language sets among the participants to be either $\{(EF), (EG), (EI)\}$ or $\{(EF), (FG), (EG)\}$.

Case 1. Everyone is in exactly one of the language groups

$$(EF), (EG), (EI), (EFG), (EFI), (EGI), (FGI), (EFGI) \ .$$

If no more than 40% of the participants are in (FGI), then at least 60% of the participants speak E. Otherwise, more than 40% of the participants are in (FGI). If no more that 40% of the participants are in $(EG) \cup (EI) \cup (EGI)$, then at least 60% of the participants speak F. The remaining case is that more that 40% of the participants are in each of (FGI) and $(EG) \cup (EI) \cup (EGI)$. It follows that, either at least 20% of the participants are in $(EG) \cup (EGI)$, in which case at least 60% speak G, or at least 20% of the participants are in $(EI) \cup (EGI)$, in which case at least 60% speak I.

Case 2. Everyone is in exactly one of the language groups

$$(EF), (EG), (FG), (EFG), (EFI), (EGI), (FGI), (EFGI) \ .$$

Since the three sets $(EF) \cup (EFI)$, $(EG) \cup (EGI)$ amd $(FG) \cup (FGI)$ are disjoint, one of them must include fewer than 40% of the participants. Suppose, say, it is $(EF) \cup (EFI)$. Then more than 60% must belong to its complement, and each of these must speak G. The result follows.

331. Some checkers are placed on various squares of a $2m \times 2n$ chessboard, where m and n are odd. Any number (including zero) of checkers are placed on each square. There are an odd number of checkers in each row and in each column. Suppose that the chessboard squares are coloured alternately black and white (as usual). Prove that there are an even number of checkers on the black squares.

Solution 1. Rearrange the rows so that the m odd-numbered rows move into the top m positions and the m even-numbered rows move into the bottom m positions, while all the columns remain intact except for order of entries. Now move all the n odd-numbered columns to the left n positions and all the n even-numbered columns to the right n positions, while the rows remain intact except for order of entries. The conditions of the problem continue to hold. Now the chessboard consists of two diagonally opposite $m \times n$ arrays of black squares and two diagonally opposite $m \times n$ arrays of white squares. Let a and b be the number of checkers in the top $m \times n$ arrays of black and white squares respectively, and c and d be the number of checkers in the arrays of white and black squares respectively. Since each row has an odd number of checkers, and m is odd, then $a + b$ is odd. By a similar argument, $b + d$ is odd. Hence

$$a + d = (a + b) + (b + d) - 2b$$

must be even. But $a + d$ is the total number of checkers on the black squares, and the result follows.

Solution 2. [F. Barekat] Suppose that the $(1, 1)$ square on the chessboard is black. The set of black squares is contained in the union U of the odd-numbered columns along with the union V of all the even-numbered rows. Note that U contains an odd number of columns and V an odd number of rows. Since each row and each column contains an odd number of checkers, U has an odd number u of checkers and V has an odd number v of checkers. Thus $u + v$ is even. Note that each white square belongs either to both of U and V, or to neither of them. Thus, the $u + v$ is equal to the number of black checkers plus twice the number of checkers on the white squares common to U and V. The result follows.

332. What is the minimum number of points that can be found (a) in the plane, (b) in space, such that each point in, respectively, (a) the plane, (b) space, must be at an irrational distance from at least one of them?

Solution 1. We solve the problem in space, as the planar problem is subsumed in the spatial problem. Two points will never do, as we can select on the right bisector of the segment joining them a point that is the same rational distance from both of them (why?).

However, we can find three points that will serve. Select three collinear points A, B, C such that $|AB| = |BC| = u$ where u^2 is not rational. Let P be any point in space. If P, A, B, C are collinear and $|PA| = a$, $|PC| = c$, then $|PB|$ is equal to either $a - u$ or $c - u$. If a and c are rational, then both $a - u$ and $c - u$ are nonrational. Hence at least one of the three distances is rational.

If P, A, C are not collinear, then PB is a median of triangle PAC. Let $b = |PB|$. Then $a^2 + c^2 = 2b^2 + 2u^2$ (why?). Since u^2 is non rational, at least one of a, b, c is nonrational.

Comment. You can check that $a^2 + c^2 = 2(b^2 + u^2)$ holds for the collinear case as well.

Solution 2. [F. Barekat] As in the foregoing, the number has to be at least three. Consider the points $(0, 0, 0)$, $(u, 0, 0)$ and $(v, 0, 0)$, where v is irrational and u and v^2 are rational. Let $P \sim (x, y, z)$. Then the distances from P to the three points are the respective square roots of $x^2 + y^2 + z^2$, $x^2 - 2ux + u^2 + y^2 + z^2$ and $x^2 - 2vx + v^2 + y^2 + z^2$. If the first of these is irrational, then we have one irrational distance. Suppose that $x^2 + y^2 + z^2$ is rational. If x is irrational, then

$$x^2 - 2ux + u^2 + y^2 + z^2 = (x^2 + y^2 + z^2 + u^2) - 2ux$$

is irrational. If x is rational, then

$$x^2 - 2vx + v^2 + y^2 + z^2 = (x^2 + y^2 + z^2 + v^2) - 2vx$$

is irrational. Hence not all the three distances can be rational.

333. Suppose that a, b, c are the sides of triangle ABC and that a^2, b^2, c^2 are in arithmetic progression.

(a) Prove that $\cot A$, $\cot B$, $\cot C$ are also in arithmetic progression.

(b) Find an example of such a triangle where a, b, c are integers.

Solution 1. [F. Barekat] (a) Suppose, without loss of generality that $a \le b \le c$. Let AH be an altitude of the triangle. Then

$$\cot B = \frac{|BH|}{|AH|} \qquad \text{and} \qquad \cot C = \frac{|CH|}{|AH|} \ .$$

Therefore,

$$2[ABC](\cot B + \cot C) = a|AH|\left(\frac{|BH| + |CH|}{|AH|}\right) = a^2 \ .$$

Similar equalities hold for b^2 and c^2. Therefore

$$2b^2 = a^2 + c^2 \Leftrightarrow 2(\cot A + \cot C) = (\cot B + \cot C) + (\cot B + \cot A) \Leftrightarrow \cot A + \cot C = 2 \cot B \ .$$

The result follows from this.

(b) Observe that $a^2 + c^2 = 2b^2$ if and only if $(c-a)^2 + (c+a)^2 = (2b)^2$. So, if (x, y, z) is a Pythagorean triple with z even and x and y of the same parity, then

$$(a, b, c) = \left(\frac{y-x}{2}, \frac{z}{2}, \frac{y+x}{2}\right) \ .$$

Let $(x, y, z) = (m^2 - n^2, 2mn, m^2 + n^2)$ where m and n have the same parity and $m > n$. Then

$$(a, b, c) = \left(\frac{n^2 + 2mn - m^2}{2}, \frac{m^2 + n^2}{2}, \frac{m^2 + 2mn - n^2}{2}\right) \ .$$

To ensure that these are sides of a triangle, we need to impose the additional conditions that

$$n^2 + 2mn - m^2 > 0 \Leftrightarrow 2n^2 > (m-n)^2 \Leftrightarrow m < (\sqrt{2} + 1)n$$

and

$$m^2 + 2mn - n^2 < (m^2 + n^2) + (n^2 + 2mn - m^2) \Leftrightarrow m < n\sqrt{3} \ .$$

Thus, we can achieve our goal as long as $n^2 < m^2 < 3n^2$ and $m \equiv n \pmod 2$.

For example, $(m, n) = (5, 3)$ yields $(a, b, c) = (7, 17, 23)$.

Comment. Take $(x, y, z) = (2mn, m^2 - n^2, m^2 + n^2)$ to give the solution

$$(a, b, c) = \left(\frac{m^2 - 2mn - n^2}{2}, \frac{m^2 + n^2}{2}, \frac{m^2 + 2mn - n^2}{2}\right) \ .$$

For example, $(m, n) = (5, 1)$ yields $(a, b, c) = (7, 13, 17)$. Here are some further numerical examples; note how they come in chains with the first of each triple equal to the last of the preceding one:

$$(a, b, c) = [(1, 5, 7),](7, 13, 17), (17, 25, 31), (31, 41, 49), \cdots$$

$$(a, b, c) = (7, 17, 23), (23, 37, 47), (47, 65, 79), \cdots$$

Solution 2. Since $c^2 = a^2 + b^2 - 2ab\cos C$, we have that

$$\cot C = \frac{a^2 + b^2 - c^2}{2ab\sin C} = \frac{a^2 + b^2 - c^2}{4[ABC]}$$

with similar equations for $\cot A$ and $\cot B$. Hence

$$\cot A + \cot C - 2\cot B = (2[ABC])^{-1}(2b^2 - a^2 - c^2) \ .$$

Thus, a^2, b^2, c^2 are in arithmetic progression if and only if $\cot A, \cot B, \cot C$ are in arithmetic progression.

(b) We need to solve the Diophantine equation $a^2 + c^2 = 2b^2$ subject to the condition that $c < a + b$. The inequality is equivalent to $2b^2 - a^2 < (a + b)^2$ which reduces to $(b - a)^2 < 3a^2$ or $b < (1 + \sqrt{3})a$. To ensure the inequality, let us try $b = 2a + k$, so that $b^2 = 4a^2 + 4ka + k^2$ and we have to solve $7a^2 + 8ka + 2k^2 = c^2$. Upon multiplication by 7 and shifting terms, the equation becomes

$$(7a + 4k)^2 - 7c^2 = 2k^2 \ .$$

(Note that $b/a = 2 + (k/a) < 1 + \sqrt{3}$ as long as $a > \frac{1}{2}(\sqrt{3} + 1)k$.)

We solve a *Pell's Equation* $x^2 - 7y^2 = 2k^2$ with the condition that $x \equiv 4k \pmod 7$. There is a standard technique for solving such equations. We find the fundamental solution of $x^2 - 7y^2 = 1$; this is the solution with the smallest positive values of x and y, and in this case is $(x, y) = (8, 3)$. We need a particular solution of $x^2 - 7y^2 = 2k^2$; the solution $(x, y) = (3k, k)$ will do. Then we get an infinite set of solutions (x_n, y_n) for $x^2 - 7y^2 = 2k^2$ by defining $(x_0, y_0) = (3k, k)$ and, for $n \geq 1$,

$$x_n + y_n\sqrt{7} = (8 + 3\sqrt{7})(x_{n-1} + y_{n-1}\sqrt{7}) \ .$$

(Note that the same equation holds when we replace the plus signs in the three terms by minus signs, so we can see that this works by multiplying this equation by its surd conjugate.) Separating out the terms, we get the recursion

$$x_n = 8x_{n-1} + 21y_{n-1}$$

$$y_n = 3x_{n-1} + 8y_{n-1}$$

for $n \geq 1$. Observe that $x_n \equiv x_{n-1} \pmod 7$. Thus, to get the solution we want, we need to select k such that $k \equiv 0 \pmod 7$.

An infinite family of solutions of $x^2 - 7y^2 = 2k^2$ starts with

$$(x, y) = (3k, k), (45k, 17k), (717k, 271k), \cdots \ .$$

All but the first of these will yield a triangle, since we will have $7a = x - 4k \geq 41k$ whence $a > 5k > \frac{1}{2}(\sqrt{3} + 1)k$. Let $k = 7$. Then, we get the triangles $(a, b, c) = (41, 89, 119), (713, 1433, 1897), \cdots$. We get only similar triangles to these from other multiples of 7 for k.

334. The vertices of a tetrahedron lie on the surface of a sphere of radius 2. The length of five of the edges of the tetrahedron is 3. Determine the length of the sixth edge.

Solution 1. Let $ABCD$ be the tetrahedron with the lengths of AB, AC, AD, BC and BD all equal to 3. The plane that contains the edge AB and passes through the centre of the sphere is a plane of symmetry for the tetrahedron and is thus orthogonal to CD. This plane meets CD in P, the midpoint of CD. Likewise, the plane orthogonal to AB passing through the midpoint M of AB is a plane of symmetry of the tetrahedron that passes through C, D and P, as well as the centre O of the sphere.

Consider the triangle OCM with altitude CP. Since OM is the altitude of the triangle OAB with $|OA| = |OB| = 2$ and $|AB| = 3$, $|OM| = (\sqrt{7})/2$. Since MC is an altitude of the equilateral triangle ABC, $|MC| = (3\sqrt{3})/2$. Since OC is a radius of the sphere, $|OC| = 2$.

Let $\theta = \angle OCM$. Then, by the Cosine Law, $\frac{7}{4} = 4 + \frac{27}{4} - 2\sqrt{27}\cos\theta$, so that $\cos\theta = (\sqrt{3})/2$ and $\sin\theta = 1/2$. Hence, the area $[OCM]$ of triangle OCM is equal to $\frac{1}{2}|OC||MC|\sin\theta = (3\sqrt{3})/4$. But this area is also equal to $\frac{1}{2}|CP||OM| = ((\sqrt{7})/4)|CP|$. Therefore,

$$|CD| = 2|CP| = 2(3\sqrt{3})/(\sqrt{7}) = (6\sqrt{3})/(\sqrt{7}) = (6\sqrt{21})/7 .$$

Comment. An alterntive way is to note that CMP is a right triangle with hypotenuse CM with O a point on PM. Let $u = |CP|$. We have that $|CM| = (3\sqrt{3})/2$, $|OM| = (\sqrt{7})/2$, $|CO| = 2$. so that $|OP|^2 = 4 - u^2$ and $|MP| = [(\sqrt{7})/2] + \sqrt{4 - u^2}$. Hence, by Pythagoras' Theorem,

$$\frac{27}{4} = \left[\frac{\sqrt{7}}{2} + \sqrt{4-u^2}\right]^2 + u^2$$

$$\Longrightarrow \frac{27}{4} = \frac{7}{4} + \sqrt{7}\sqrt{4-u^2} + 4$$

$$\Longrightarrow u^2 = 27/7 \Longrightarrow u = (3\sqrt{3})/\sqrt{7} .$$

Hence $|CD| = 2u = (6\sqrt{3})/\sqrt{7}$.

Solution 2. [F. Barekat] As in Solution 1, let CD be the odd side, and let O be the centre of the sphere. Let G be the centroid of the triangle ABC. Since the right bisecting plane of the three sides of triangle ABC each pass through the centroid G and the centre O, $OG \perp ABC$. Observe that $|CG| = \sqrt{3}$ and $|GM| = (\sqrt{3})/2$, where M is the midpoint of AB.

As CGO is a right triangle, $|GO|^2 = |CO|^2 - |CG|^2 = 4 - 3 = 1$, so that $|GO| = 1$. Hence, $|OM|^2 = |OG|^2 + |GM|^2 = 1 + \frac{3}{4} = \frac{7}{4}$, so that $|OM| = (\sqrt{7})/2$. Therefore,

$$\sin\angle OMC = \sin\angle OMG = |OG|/|OM| = 2/(\sqrt{7}) .$$

With P the midpoint of CD, the line MP (being the intersection of the planes right bisecting AB and CD) passes through O. Since MC and MD are altitudes of equilateral triangles, $|MC| = |MD| = 3(\sqrt{3})/2$, so that MCD is isosceles with MP bisecting the apex angle. Hence

$$|CD| = 2|MC|\sin(\frac{1}{2}\angle CMD) = 2|MC|\sin\angle OMC = (6\sqrt{3})/(\sqrt{7}) .$$

Solution 3. Let A be at $(-3/2, 0, 0)$ and B be at $(3/2, 0, 0)$ in space. The locus of points equidistant from A and B is the plane $x = 0$. Let the centre of the sphere be at the point $(0, u, 0)$, so that $u^2 + 9/4 = 4$ and $u = (\sqrt{7})/2$. Suppose that C is at the point $(0, y, z)$ where $\frac{9}{4} + y^2 + z^2 = 9$. We have that $(y - (\sqrt{7})/2))^2 + z^2 = 4$ whence

$$(\sqrt{7})y + (9/4) = y^2 + z^2 = 9 - (9/4)$$

and $y = 9/(2\sqrt{7})$. Therefore $z = (3\sqrt{3})/(\sqrt{7})$.

We find that $C \sim (0, 9/(2\sqrt{7}), (3\sqrt{3})/(\sqrt{7}))$ and $D \sim (0, 9/(2\sqrt{7}), (-3\sqrt{3})/(\sqrt{7}))$, whence $|CD| = (6\sqrt{3})/(\sqrt{7})$.

Solution 4. [Y. Zhao] Use the notation of Solution 1, and note that the right bisecting plane of AB and CD intersect in a diameter of the sphere. Let us first determine the volume of the tetrahedron $ABDO$. Consider the triangle ABD with centroid X. We have that $|AB| = |BD| = |AD| = 3$, $|AX| = \sqrt{3}$ and $[ABD] = (9\sqrt{3})/4$. Since O is equidistant from the vertices of triangle ABD, $OX \perp ABD$. By

Pythagoras' Theorem applied to triangle AXO, $|OX| = 1$, so that the volume of tetrahedron $ABDO$ is $(1/3)|OX|[ABD] = (3\sqrt{3})/4$.

Triangle ABO has sides of lengths 2, 2, 3, and (by Heron's formula) area $(3\sqrt{7})/4$. Since $CD \perp ABO$, CD is the production of an altitude of tetrahedron $ABDO$; the altitude has length $\frac{1}{2}|CO|$. Hence

$$|CD| = \frac{2[3\text{Volume}(ABDO)]}{[ABO]} = \frac{2(9\sqrt{3})/4}{(3\sqrt{7})/4} = \frac{6\sqrt{3}}{\sqrt{7}} \ .$$

Solution 5. [A. Wice] Let the tetrahedron $ABCD$ have its vertices on the surface of the sphere of equation $x^2 + y^2 + z^2 = 4$ with A at $(0, 0, 2)$. The remaining vertices have coordinates (u, v, w) and satisfy the brace of equations: $u^2 + v^2 + (w - 2)^2 = 9$ and $u^2 + v^2 + w^2 = 4$. Hence, $w = -1/4$. Thus, the points B, C, D lie on the circle of equations $z = -1/4$, $x^2 + y^2 = 63/16$. The radius R of this circle and the circumradius of triangle BCD is $3\sqrt{7}/4$.

Triangle BCD has sides of length $(b, c, d) = (b, 3, 3)$ and area $bcd/4R = \frac{1}{2}b\sqrt{9 - (b^2/4)}$. Hence

$$\frac{3}{\sqrt{7}} = \frac{1}{2}\sqrt{9 - (b^2/4)} \Leftrightarrow 36 = 7(9 - (b^2/4)) \Leftrightarrow 144 = 7(36 - b^2) \Leftrightarrow b = \frac{6\sqrt{3}}{\sqrt{7}} \ .$$

Thus, the length of the remaining side is $(6\sqrt{3})/\sqrt{7}$.

335. Does the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} = \frac{12}{a + b + c}$$

have infinitely many solutions in positive integers a, b, c?

Comment. The equation is equivalent to

$$(a + b + c)(bc + ca + ab + 1) = 12abc \ .$$

This is quadratic in each variable, and for any integer solution, has integer coefficients. The general idea of the solution is to start with a particular solution ($(a, b, c) = (1, 1, 1)$ is an obvious one), fix two of the variables at these values and regard the equation as a quadratic in the third. Since the sum and the product of the roots are integers and one root is known, one can find another root and so bootstrap one's way to other solutions. Thus, we have the quadratic for a:

$$(b + c)a^2 + [(b + c)^2 + (bc + 1) - 12bc]a + (b + c)(bc + 1) = 0 \ ,$$

so that if (a, b, c) satisfies the equation, then so also does (a', b, c) where

$$a + a' = \frac{9bc - b^2 - c^2 - 1}{b + c} \qquad \text{and} \qquad aa' = bc + 1 \ .$$

We can start constructing solutions using these relations:

$$(1, 1, 1), (1, 1, 2), (1, 2, 3), (2, 3, 7), (2, 5, 7), (3, 7, 11), (5, 7, 18), (5, 13, 18), \cdots \ .$$

However, some triples do not lead to a second solution in integers. For example, $(b, c) = (2, 5)$ leades to $(7, 2, 5)$ and $(11/7, 2, 5)$, and $(b, c) = (3, 11)$ leads to $(7, 3, 11)$ and $(34/7, 3, 11)$. So we have no guarantee that this process will not peter out.

Solution 1. [Y. Zhao] Yes, there are infinitely many solutions. Specialize to the case that $c = a + b$. Then the equation is equivalent to

$$2(a + b)[(a + b)^2 + ab + 1] = 12ab(a + b) \Longleftrightarrow a^2 - 3ab + b^2 + 1 = 0 \ .$$

This has at least one solution $(a, b) = (1, 1)$. Suppose that $(a, b) = (p, q)$ is a solution with $p \leq q$. Then $(a, b) = (q, 3q - p)$ is a solution. (To see this, note that $x^2 - 3qx + (q^2 + 1) = 0$ is a quadratic equation with one root $x = p$ and root sum $3q$.) Observe that $q < 3q - p$.

Define a sequence $\{x_n\}$ for $n \geq 0$ by $x_0 = x_1 = 1$ and $x_n = 3x_{n-1} - x_{n-2}$ for $n \geq 2$. Then $(a, b) = (x_0, x_1)$ satisfies the equation, and by induction, so does $(a, b) = (x_n, x_{n+1})$ for $n \geq 1$. Since $x_{n+1} - x_n = 2x_n - x_{n_1}$, one sees by induction that $\{x_n\}$ is strictly increasing for $n \geq 1$. Hence, an infinite set of solutions for the given equation is given by

$$(a, b, c) = (x_n, x_{n+1}, x_n + x_{n+1})$$

for $n \geq 0$. Some examples are

$$(a, b, c) = (1, 1, 2), (1, 2, 3), (2, 5, 7), (5, 13, 18), \cdots$$

Comment. If $\{f_n\}$ is the Fibonacci sequence defined by $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, then the solutions are $(a, b, c) = (f_{2k-1}, f_{2k+1}, f_{2k-1} + f_{2k+1})$.

Solution 2. Yes. Let $u_0 = u_1 = 1$, $v_0 = 1, v_1 = 2$ and

$$u_n = 4u_{n-1} - u_{n-2}$$

$$v_n = 4v_{n-1} - v_{n-2}$$

for $n \geq 2$, so that $\{u_n\} = \{1, 1, 3, 11, 41, \cdots\}$ and $\{v_n\} = \{1, 2, 7, 26, 97, \cdots\}$. It can be proven by induction that both sequences are strictly increasing for $n \geq 1$. We prove that the equation of the problem is satisfied by

$$(a, b, c) = (u_n, v_n, u_{n+1}), (v_n, u_{n+1}, v_{n+1})$$

for $n \geq 0$. In other words, the equation holds if (a, b, c) consists of three consecutive terms of the sequence $\{1, 1, 1, 2, 3, 7, 11, 26, 41, 97, \cdots\}$.

Observe that, for $n \geq 2$,

$$\begin{aligned} u_n v_n - u_{n+1} v_{n-1} &= u_n(4v_{n-1} - v_{n-2}) - (4u_n - u_{n-1})v_{n-1} \\ &= u_{n-1}v_{n-1} - u_n v_{n-2} \end{aligned}$$

so that, by induction, it can be established that $u_n v_v - u_{n+1} v_{n-1} = -1$. Similarly,

$$\begin{aligned} u_n v_{n+1} - u_{n+1} v_n &= u_n(4v_n - v_{n-1}) - (4u_n - u_{n-1})v_n \\ &= u_{n-1}v_n - u_n v_{n-1} = 1 \ . \end{aligned}$$

It can be checked that $(a, b, c) = (1, 1, 1)$ satisfies the equation. Suppose, as an induction hypothesis, that $(a, b, c) = (u_n, v_n, u_{n+1})$ satisfies the equation. Then u_n is a root of the quadratic

$$\begin{aligned} 0 &= [x + v_n + u_{n+1}][x(v_n + u_{n+1}) + v_n u_{n+1} + 1] - 12 v_n u_{n+1} x \\ &= (v_n + u_{n+1})x^2 + (v_n^2 + u_{n+1}^2 + 1 - 9v_n u_{n+1})x + (v_n + u_{n+1})(v_n u_{n+1} + 1) \ . \end{aligned}$$

The second root is

$$v_{n+1} = \frac{v_n u_{n+1} + 1}{u_n}$$

so $(a, b, c) = (v_n, u_{n+1}, v_{n+1})$ satisfies the equation. Similarly, v_n is a root of a quadratic, the product of whose roots is $u_{n+1} v_{n+1} + 1$. The other root of this quadratic is

$$u_{n+2} = \frac{u_{n+1} v_{n+1} + 1}{v_n}$$

and so, $(a, b, c) = (u_{n+1}, v_{n+1}, u_{n+2})$ satisfies the equation.

Solution 3. [P. Shi] Yes. Let $u_0 = v_0 = 1$, $v_0 = 1$, $v_1 = 2$, and

$$u_n = \frac{u_{n-1}v_{n-1} + 1}{v_{n-2}} \qquad \text{and} \qquad v_n = \frac{u_n v_{n-1} + 1}{u_{n-1}}$$

for $n \geq 2$. We establish that, for $n \geq 1$,

$$u_n = 2v_{n-1} - u_{n-1} \; ; \tag{1}$$

$$v_n = 3u_n - v_{n-1} \; ; \tag{2}$$

$$3u_n^2 + 2v_{n-1}^2 + 1 = 6u_n v_{n-1} \; ; \tag{3}$$

$$3u_n^2 + 2v_n^2 + 1 = 6u_n v_n \; . \tag{4}$$

Note that $(3u_n^2 + 2v_n^2 + 1 - 6u_nv_n) - (3u_n^2 + 2v_{n-1}^2 + 1 - 6u_nv_{n-1}) = 2(v_n - v_{n-1})(v_n + v_{n-1} - 3u_n)$, so that the truth of any two of (2), (3), (4) implies the truth of the third.

The proof is by induction. The result holds for $n = 1$. Suppose it holds for $1 \leq n \leq k - 1$. From (3),

$$3u_{k-1}^2 + 2v_{k-2}^2 + 1 = 6u_{k-1}v_{k-2} \Longrightarrow$$

$$(3u_{k-1} - v_{k-2})u_{k-1} + 1 = 2(3u_{k-1} - v_{k-2})v_{k-2} - u_{k-1}v_{k-2} \; .$$

Substituting in (2) gives

$$u_k = \frac{u_{k-1}v_{k-1} + 1}{v_{k-2}} = \frac{u_{k-1}(3u_{k-1} - v_{k-2}) + 1}{v_{k-2}} = 2v_{k-1} - u_{k-1}$$

which establishes (1) for $n = k$. From (4),

$$(2v_{k-1} - u_{k-1})v_{k-1} + 1 = [3(2v_{k-1} - u_{k-1}) - v_{k-1}]u_{k-1}$$

whence

$$v_k = \frac{u_k v_{k-1} + 1}{u_{k-1}} = \frac{(2v_{k-1} - u_{k-1})v_{k-1} + 1}{u_{k-1}} = 3u_k - v_{k-1}$$

which establishes (2) for $n = k$. From (4),

$$3(2v_{k-1} - u_{k-1})^2 + 2v_{k-1}^2 + 1 = 6(2v_{k-1} - u_{k-1})v_{k-1}$$

$$\Longrightarrow 3u_k^2 + 2v_{k-1}^2 + 1 = 6u_kv_k \; .\heartsuit$$

Let $w_{2n} = u_n$ and $w_{2n+1} = v_n$ for $n \geq 0$. Then

$$w_n = \frac{w_{n-1}w_{n-2} + 1}{w_{n-3}}$$

for $n \geq 3$. It can be checked that $(a, b, c) = (1, 1, 1) = (w_0, w_1, w_2)$ satisfies the equation. Suppose that the equation is satisfied by $(a, b, c) = (w_{n-1}, w_n, w_{n+1})$. Thus

$$(x + w_n + w_{w+1})(x(w_n + w_{n+1}) + w_nw_{n+1} + 1) = 12xw_nw_{n+1} = 0$$

is a quadratic equation in x one of whose roots is $x = w_{n-1}$. The quadratic can be rewritten

$$x^2 + [(w_n + w_{n+1}) - (11w_nw_{n+1} - 1)/(w_n + w_{n+1})]x + (w_nw_{n+1} + 1) = 0 \; .$$

Since the product of the roots is equal to $w_n w_{n+1} + 1$, the second root is equal to

$$\frac{w_n w_{n+1} + 1}{w_{n-1}} = w_{n+2} \ .$$

The result follows.

Solution 4. Let $c = a + b$. Then the equation becomes $2c(ab + c^2 + 1) = 12abc$, whence $ab = (c^2 + 1)/5$. Hence, a, b are roots of the quadratic equation

$$0 = t^2 - ct + \left(\frac{c^2 + 1}{5}\right) = \frac{1}{4}\left[(2t - c)^2 - \left(\frac{5c^2 - 20}{25}\right)\right] \ .$$

For this to have an integer solution, it is necessary that $5c^2 - 20 = s^2$, the square of an integer s. Now $s^2 - 5c^2 = -20$ is a Pell's equation, three of whose solutions are $(s, c) = (0, 2), (5, 3), (15, 7)$. Since $x^2 - 5y^2 = 1$ is satisfied by $(x, y) = (9, 4)$, solutions (s_n, c_n) of $s^2 - 5c^2 = -20$ are given by the recursion

$$s_{n+1} = 9s_n + 20c_n$$

$$c_{n+1} = 4s_n + 9c_n$$

for $n \geq 0$. where (s_0, c_0) is a starter solution. Taking $(s_0, c_0) = (0, 2)$, we get the solutions

$$(s; a, b, c) = (0; 1, 1, 2), (40; 5, 13, 18), (720; 89, 233, 322), \cdots \ .$$

Taking $(s_0, c_0) = (5, 3)$, we get the solutions

$$(s; a, b, c) = (5; 1, 2, 3), (105; 13, 34, 47), (1885; 233, 610, 843), \cdots \ .$$

Taking $(s_0, c_0) = (15, 7)$, we get the solutions

$$(s; a, b, c) = (15; 2, 5, 7), (275; 34, 89, 123), \cdots \ .$$

Comment. Note that the values of c seem to differ from a perfect square by 2; is this a general phenomenon?

Solution 5. [Z. Guo] Let $r_1 = r_2 = 1$, and, for $n \geq 1$,

$$r_{2n+1} = 2r_{2n} - r_{2n-1}$$

$$r_{2n+2} = 3r_{2n+1} - r_{2n} \ .$$

Thus, $\{r_n\} = \{1.1, 1, 2, 3, 7, 11, 26, 41, 87, 133, \cdots\}$. Observe that $(a, b, c) = (r_m, r_{m+1}, r_{m+2})$ satisfies the equation for $m = 1, 2$.

For each positive integer n, let $k_n = r_{2n} - r_{2n-1}$, so that

$$r_{2n-1} = r_{2n} - k_n \ ,$$

$$r_{2n+1} = r_{2n} + k_n \ ,$$

$$r_{2n+2} = 2r_{2n} + 3k_n \ .$$

Suppose that $(a, b, c) = (r_{2m-1}, r_{2m}, r_{2m+1})$ and $(a, b, c) = (r_{2m}, r_{2m+1}, r_{2m+2})$ satisfy the equation. Substituting $(a, b, c) = (r_{2m} - k_m, r_{2m}, r_{2m} + k_m)$ into the equation and simplifying yields that $r_{2m} = \sqrt{3k_m^2 + 1}$. (This can be verified by substituting $(a, b, c) = (r_{2m}, r_{2m} + k_m, 2r_{2m} + 3k_{2m})$ into the equation.) In fact, this condition is equivalent to these values of (a, b, c) satisfying the equation.

We have that

$$\begin{aligned} 3k_{m+1}^2 + 1 &= 3(r_{2m+2} - r_{2m+1})^2 + 1 \\ &= 3(r_{2m} + 2k_m)^2 + 1 \\ &= 3(r_{2m}^2 + 4k_m^2 + 4r_{2m}k_m) + 1 \\ &= 3(7k_m^2 + 1) + 12r_{2m}k_m + 1 = 21k_m^2 + 12r_{rm}k_m + 4 \ . \end{aligned}$$

Thus

$$\begin{aligned} r_{2m+2}^2 &= (2r_{2m} + 3k_m)^2 = 4r_{2m}^2 + 9k_m^2 + 12r_{2m}k_m \\ &= 12k_m^2 + 4 + 9k_m^2 + 12r_{2m}k_m = 3k_{m+1}^2 + 1 \ . \end{aligned}$$

This is the condition that

$$(a, b, c) = (r_{2m+2} - k_{m+1}, r_{2m+2}, r_{2m+2} + k_{m+1}) = (r_{2m+1}, r_{2m+2}, r_{2m+3})$$

and

$$(a, b, c) = (r_{2m+2}, r_{2m+2} + k_{m+1}, 2r_{2m+2} + 3k_{m+1}) = (r_{2m+2}, r_{2m+3}, r_{2m+4})$$

satisfy the equation. The result follows by induction.

Solution 6. [D. Rhee] Try for solutions of the form

$$(a, b, c) = \left(x, \frac{1}{2}(x + y), y\right)$$

where x and y are postive integers of the same parity. Plugging this into the equation and simplifying yields the equivalent equation

$$x^2 - 4xy + y^2 + 2 = 0 \Longleftrightarrow x^2 - 4yx + (y^2 + 2) = 0 \ .$$

Suppose that $z_1 = 1$, $z_2 = 3$ and $z_{n+1} = 4z_n - z_{n-1}$ for $n \geq 1$. Then, it can be shown by induction that $z_{n+1} > z_n$ and z_n is odd for $n \geq 1$. We prove by induction that $(x, y) = (z_n, z_{n+1})$ is a solution of the quadratic equation in x and y.

This is true for $n = 1$. Suppose that it holds for $n \geq 1$. Then z_n is a root of the quadratic equation $x^2 - 4z_{n+1}x + (z_{n+1}^2 + 2) = 0$. Since the sum of the roots is the integer $4z_{n+1}$, the second root is $4z_{n+1} - z_n = z_{n+2}$ and the desired result holds because of the symmetry of the equation in x and y.

Thus, we obtain solutions $(a, b, c) = (1, 2, 3), (3, 7, 11), (11, 26, 41), (41, 97, 153), \cdots$ of the given equation.

Solution 7. [J. Park; A. Wice] As before, we note that if $(a, b, c) = (u, v, w)$ satisfies the equation, then so also does $(a, b, c) = (v, w, (vw + 1)/u)$. Define a sequence $\{x_n\}$ by $x_1 = x_2 = x_3 = 1$ and

$$x_{n+3} = \frac{x_{n+2}x_{n+1} + 1}{x_n}$$

for $n \geq 1$. We prove by induction, that for each n, the following properties hold:

(a) $x_1, \cdots, x_{n+3}$ are integers; in particular, x_n divides $x_{n+2}x_{n+1} + 1$;
(b) x_{n+1} divides $x_n + x_{n+2}$;
(c) x_{n+2} divides $x_n x_{n+1} + 1$;
(d) $x_1 = x_2 = x_3 < x_4 < x_5 < \cdots < x_n < x_{n+1} < x_{n+2}$;
(e) $(a, b, c) = (x_n, x_{n+1}, x_{n+2})$ satisfies the equation.

These hold for $n = 1$. Suppose they hold for $n = k$. Since

$$x_{k+2}x_{k+3} + 1 = \frac{x_{k+2}(x_{k+1}x_{k+2} + 1)}{x_k} + 1 = \frac{x_{k+2}^2 x_{k+1} + (x_k + x_{k+2})}{x_k} \ ,$$

from (b) we find that x_{k+1} divides the numerator. Since, by (a), x_k and x_{k+1} are coprime, x_{k+1} must divide $x_{k+2}x_{k+3}+1$.

Now

$$x_{k+1}+x_{k+3} = \frac{(x_k x_{k+1}+1)+x_{k+1}x_{k+2}}{x_k} \ .$$

By (a), x_k and x_{k+2} are coprime, and, by (c), x_{k+2} divides the numerator. Hence x_{k+2} divides $x_{k+1}+x_{k+3}$.

Since $x_k = (x_{k+1}x_{k+2}+1)/x_{k+3}$ is an integer, x_{k+3} divides $x_{k+1}x_{k+2}+1$. Next,

$$x_{k+3} = \frac{x_{k+1}x_{k+2}+1}{x_k} > \left(\frac{x_{k+1}}{x_k}\right)x_{k+2} > x_{k+2} \ .$$

Finally, the theory of the quadratic delivers (e) for $n=k+1$. The result follows.

336. Let $ABCD$ be a parallelogram with centre O. Points M and N are the respective midpoints of BO and CD. Prove that the triangles ABC and AMN are similar if and only if $ABCD$ is a square.

Comment. Implicit in the problem, but what should have been stated, is that the similarity intended is in the order of the vertices as given, *i.e.*, $AB:BC:CA = AM:MN:NA$. In the first solution, other possible orderings of the vertices in the similarity are considered (in which case, the result becomes false); in the remaining solutions, the restricted sense of the similarity is discussed.

Solution 1. Let P be the midpoint of CN. Observe that

$$[AMD] = \frac{3}{4}[ABD] = \frac{3}{4}[ABC]$$

and

$$[AMC] = \frac{1}{2}[ABC] \ .$$

As N is the midpoint of CD, the area of triangle AMN is the average of these two (why?), so that

$$[AMN] = \frac{5}{8}[ABC] \ .$$

Thus, in any case, we have the ratio of the areas of the two triangles, so, if they are similar, we know exactly what the factor of similarity must be.

Let $|AB|=|CD|=2a$, $|AD|=|BC|=2b$, $|AC|=2c$ and $|BD|=2d$. Recall, that if UVW is a triangle with sides $2u$, $2v$, $2w$ opposite the respective vertices U,V,W and m is the length of the median from U, then

$$m^2 = 2v^2+2w^2-u^2 \ .$$

Since AN is a median of triangle ACD, with sides $2a$, $2b$, $2c$,

$$|AN|^2 = 2c^2+2b^2-a^2 \ .$$

Since AM is a median of triangle ABO, with sides $2a$, c, d,

$$|AM|^2 = 2a^2+(c^2/2)-(d^2/4) \ .$$

Since CM is a median of triangle CBO, with sides $2b$, c, d,

$$|CM|^2 = 2b^2+(c^2/2)-(d^2/4) \ .$$

Since MN is a median of triangle MCD with sides $|CM|$, $3d/2$, $2a$,

$$|MN|^2 = \frac{|CM|^2}{2} + \frac{1}{2}\left(\frac{3d}{2}\right)^2 - a^2$$
$$= b^2 + \frac{c^2}{4} - \frac{d^2}{8} + \frac{9d^2}{8} - a^2 = b^2 + \frac{c^2}{4} + d^2 - a^2 \ .$$

Suppose that triangle ABC and AMN are similar (with any ordering of the vertices). Then

$$8(|AM|^2 + |AN|^2 + |MN|^2) = 5(|AB|^2 + |AC|^2 + |BC|^2)$$
$$\Longleftrightarrow 24b^2 + 22c^2 + 6d^2 = 20a^2 + 20b^2 + 20c^2$$
$$\Longleftrightarrow 2b^2 + c^2 + 3d^2 = 10a^2 \quad . \tag{1}$$

Also, for the parallelogram with sides $2a$, $2b$ and diagonals $2c$, $2d$, we have that

$$c^2 + d^2 = 2(a^2 + b^2) \quad . \tag{2}$$

Equations (1) and (2) together yield $d^2 = 4a^2 - 2b^2$ and $c^2 = 4b^2 - 2a^2$. We now have that

$$\frac{|AN|^2}{|AC|^2} = \frac{2c^2 + 2b^2 - a^2}{4c^2} = \frac{4c^2 + c^2}{8c^2} = \frac{5}{8}$$

which is the desired ratio. Thus, when triangles ABC and AMN are similar, the sides AN and AC are in the correct ratio, and so must correspond in the similarity. (There is a little more to this than meets the eye. This is obvious when triangle AMN is scalene; if triangle AMN were isosceles or equilateral, then it is self-congruent and the similarity can be set up to make AN and BC correspond.)

We also have that

$$\frac{|MN|^2}{|AB|^2} = \frac{4b^2 + c^2 + 4d^2 - 4a^2}{16a^2} = \frac{4b^2 + 4b^2 - 2a^2 + 16a^2 - 8b^2 - 4a^2}{16a^2} = \frac{10a^2}{16a^2} = \frac{5}{8}$$

and

$$\frac{|AM|^2}{|BC|^2} = \frac{8a^2 + 2c^2 - d^2}{16b^2} = \frac{8a^2 + 8b^2 - 4a^2 - 4a^2 + 2b^2}{16b^2} = \frac{10b^2}{16b^2} = \frac{5}{8} \ .$$

Thus, if triangle ABC and AMN are similar, then

$$AB : BC : AC = MN : AM : AN$$

and $\cos\angle ABC = \cos\angle ADC = 3(a^2 - b^2)/(2ab)$. The condition that the cosine has absolute value not exceeding 1 yields that $(\sqrt{10} - 1)b \le 3a \le (\sqrt{10} + 1)b$.

Now we look at the converse. Suppose that $ABCD$ is a parallelogram with sides $2a$ and $2b$ as indicated above and that $\cos\angle ABC = \cos\angle ADC = 3(a^2 - b^2)/(2ab)$. Then, the lengths $2c$ and $2d$ of the diagonals are given by

$$4c^2 = |AC|^2 = 4a^2 + 4b^2 - 12a^2 + 12b^2 = 16b^2 - 8a^2 = 8(2b^2 - a^2)$$

and

$$4a^2 = |BD|^2 = 4a^2 + 4b^2 + 12a^2 - 12b^2 = 16a^2 - 8b^2 = 8(2a^2 - b^2)$$

so that $|AC| = 2\sqrt{4b^2 - 2a^2}$ and $|BD| = 2\sqrt{4a^2 - 2b^2}$. Using the formula for the lengths of the medians, we have that

$$|AN|^2 = 2(4b^2 - 2a^2) + 2b^2 - a^2 = 5(2b^2 - a^2)$$

$$|AM|^2 = 2a^2 + \frac{4b^2 - 2a^2}{2} - \frac{4a^2 - 2b^2}{4} = \frac{5b^2}{2}$$

$$|MN|^2 = b^2 + \left(\frac{4b^2 - 2a^2}{4}\right) + (4a^2 - 2b^2) - a^2 = \frac{5a^2}{2} \ .$$

Thus

$$\frac{|AN|^2}{|AC|^2} = \frac{|AM|^2}{|BC|^2} = \frac{|MN|^2}{|AB|^2} = \frac{5}{8}$$

and triangles ABC and AMN are similar with $AB : BC : AC = MN : AM : AN$.

If triangles ABC and AMN are similar with $AB : BC : AC = AM : MN : AN$, then we must have that $AB = BC$, *i.e.* $a = b$ and so $\angle ABC = \angle ADC = 90°$, *i.e.*, $ABCD$ is a square.

Comment. A direct geometric argument that triangles ABC and AMN are similar when $ABCD$ is a square can be executed as follows. By reflection about an axis through M parallel to BC, we see that $MN = MC$. By reflection about axis BD, we see that $AM = CM$ and $\angle BAM = \angle BCM$. Hence $AM = MN$ and $\angle BAM = \angle BCM = \angle CMP = \angle NMP$ where P is the midpoint of CN.

Consider a rotation with centre A through an angle BAM followed by a dilation of factor $|AM|/|AB|$. This transformation takes $A \longrightarrow A$, $B \longrightarrow M$ and the line BC to a line through M making an angle $\angle BAM$ with BC; the image line must be MN. Since $AB = BC$ and $AM = MN$, $C \longrightarrow N$. Thus, the image of triangle ABC is triangle AMN and the two triangles are similar.

An alternative argument uses the fact that triangles AOM and ADN are similar, so that $\angle AMO = \angle AND$ and $AMND$ is concyclic. From this follows $\angle AMN = 180° - \angle ADN = 90°$.

Solution 2. [S. Eastwood; Y. Zhao] If $ABCD$ is a square, we can take B at 1 and D at i, whereupon M is at $(3+i)/4$ and N at $(1+2i)/2$. The vector $\overrightarrow{MN}$ is represented by

$$\frac{1+2i}{2} - \frac{3+i}{4} = \frac{i(3+i)}{4} \ .$$

Since $\overrightarrow{AM}$ is represented by $(3+i)/4$, MN is obtained from AM by a $90°$ rotation about M so that $MN = AM$ and $\angle AMN = 90°$. Hence the triangles ABC and AMN are similar.

For the converse, let the parallelogram $ABCD$ be represented in the complex plane with A at 0, B at z and D at w, Then C is at $z+w$, M is at $(3z+w)/4$ and N is at $\frac{1}{2}(z+w) + \frac{1}{2}w = (z+2w)/2$. Suppose that triangles ABC and AMN are similar. Then, since $\angle AMN = \angle ABC$ and $AM : AN = AB : AC$, we must have that

$$\frac{\frac{1}{4}(3z+w)}{\frac{1}{2}(z+2w)} = \frac{z}{z+w} \Longleftrightarrow (3z+w)(z+w) = 2z(z+2w)$$

$$\Longleftrightarrow 3z^2 + 4zw + w^2 = 2z^2 + 4zw \Longleftrightarrow z^2 + w^2 = 0 \Longleftrightarrow z = \pm iw \ .$$

Thus AD is obtained from AB by a $90°$ rotation and $ABCD$ is a square.

Comment. Strictly speaking, the reasoning in the last paragraph is reversible, so we could use it for the proof in both directions. However, the particularization may aid in understanding what is going on.

Solution 3. [F. Barekat] Let $ABCD$ be a square. Then $\Delta ADC \sim \Delta AOB$ so that $AB : AC = OB : DC = MB : NC$. Since also $\angle ABM = \angle ACN = 45°$, $\Delta AMB \sim \Delta ANC$. Therefore $AM : AB = AN : AC$. Also

$$\angle MAB = \angle NAC \Longrightarrow \angle CAB = \angle CAM + \angle MAB = \angle CAM + \angle NAC = \angle NAM \ .$$

Therefore $\Delta AMN \sim \Delta ABC$.

On the other hand, suppose that $\Delta AMN \sim \Delta ABC$. Then $AM : AN = AB : AC$ and

$$\angle NAC = \angle NAM - \angle CAM = \angle CAB - \angle CAM = \angle MAB \ ,$$

whence $\Delta AMB = \Delta ANC$.

Therefore, $AB : AC = MB : NC = BO : DC$. Since also $\angle ABO = \angle ACD$, $\Delta ABO \sim \Delta ACD$, so that $\angle ABO = \angle ACD$ and $\angle AOB = \angle ADC$. Because $\angle ABO = \angle ACD$, $ABCD$ is a concyclic quadrilateral and $\angle DAB + \angle DCB = 180°$. Since $ABCD$ is a parallelogram, $\angle DAB = \angle DCB = 90°$ and $ABCD$ is a rectangle. Thus $\angle AOB = \angle ADC = 90°$, from which it can be deduced that $ABCD$ is a square.

Solution 4. [B. Deng] Let $ABCD$ be a square. Since $MN = MC = MA$, triangles MNC and AMN are isosceles. We have that

$$AM^2 + MN^2 = 2AM^2 = 2(AO^2 + OM^2) = (5/4)AB^2$$

and

$$AN^2 = AD^2 + DN^2 = (5/4)AB^2 = AM^2 + MN^2 ,$$

whence $\angle AMN = 90°$ and $\Delta AMN \sim \Delta ABC$.

Suppose on the other hand that $\Delta AMN \sim \Delta ABC$. Then $AN : AM = AC : AB$ and $\angle NAM = \angle CAB$ together imply that

$$\angle NAC = \angle MAB \Longrightarrow \Delta NAC \sim \Delta MAB \Longrightarrow \angle NCA = \angle ABM .$$

But $\angle NCA = \angle OAB \Rightarrow \angle OAB = \angle OBA \Rightarrow ABCD$ is a rectangle.

Let H be the foot of the perpendicular from M to CN. Then ON, MH and BC are all parallel and M is the midpoint of OB. Hence H is the midpoint of CN and $\Delta HMN \equiv \Delta HMC$. Therefore $MC = MN$. Now, from the median length formula,

$$AM^2 = \frac{1}{4}(2BA^2 + 2AO^2 - BO^2) = \frac{1}{4}(2BA^2 + AO^2)$$

and

$$CM^2 = \frac{1}{4}(2BC^2 + 2CO^2 - BO^2) = \frac{1}{4}(2BC^2 + CO^2)$$

whence

$$(2BA^2 + AO^2) : (2BC^2 + CO^2) = AM^2 : CM^2 = AM^2 : MN^2 = BA^2 : BC^2$$

so that

$$2BA^2 \cdot BC^2 + AO^2 \cdot BC^2 = 2BA^2 \cdot BC^2 + BA^2 \cdot CO^2 \Rightarrow BC^2 = BA^2 \Rightarrow BC = BA$$

and $ABCD$ is a square.

337. Let a, b, c be three real numbers for which $0 \le c \le b \le a \le 1$ and let w be a complex root of the polynomial $z^3 + az^2 + bz + c$. Must $|w| \le 1$?

Solution 1. [L. Fei] Let $w = u+iv$, $\overline{w} = u-iv$ and r be the three roots. Then $a = -2u-r$, $b = |w|^2+2ur$ and $c = -|w|^2r$. Substituting for b, ac and c, we find that

$$|w|^6 - b|w|^4 + ac|w|^2 - c^2 = 0$$

so that $|w|^2$ is a nonnegative real root of the cubic polynomial $q(t) = t^3 - bt^2 + act - c^2 = (t-b)t^2 + c(at-c)$. Suppose that $t > 1$, then $t - b$ and $at - c$ are both positive, so that $q(t) > 0$. Hence $|w| \le 1$.

Solution 2. [P. Shi; Y.Zhao]

$$\begin{aligned} 0 &= (1-w)(w^3 + aw^2 + bw + c) \\ &= -w^4 + (1-a)w^3 + (a-b)w^2 + (b-c)w + c \\ &\Longrightarrow w^4 = (1-a)w^3 + (a-b)w^2 + (b-c)w + c \\ &\Longrightarrow |w|^4 \le (1-a)|w|^3 + (a-b)|w|^2 + (b-c)|w| + c . \end{aligned}$$

Suppose, if possible, that $|w| > 1$. Then

$$|w|^4 \leq |w|^3[(1-a)+(a-b)+(b-c)+c] = |w|^3$$

which implies that $|w| \leq 1$ and yields a contradiction. Hence $|w| \leq 1$.

Solution 3. There must be one real solution v. If $v = 0$, then the remaining roots w and $\overline{w}$, the complex conjugate of w, must satisfy the quadratic equation $z^2 + az + b = 0$. Therefore $|w|^2 = w\overline{w} = b \leq 1$ and the result follows. Henceforth, let $v \neq 0$.

Observe that

$$f(-1) = -1 + a - b + c = -(1-a) - (b-c) \leq 0$$

and that

$$f(-c) = -c^3 + ac^2 - bc + c \geq -c^3 + c^3 - bc + c = c(1-b) \geq 0 \ ,$$

so that $-1 \leq v \leq -c$. The polynomial can be factored as

$$(z-v)(z^2+pz+q)$$

where $c = -qv$ so that $q = c/(-v) \leq 1$. But $q = w\overline{w}$, and the result again follows.

338. A *triangular triple* (a,b,c) is a set of three positive integers for which $T(a) + T(b) = T(c)$. Determine the smallest triangular number of the form $a + b + c$ where (a,b,c) is a triangular triple. (*Optional investigations:* Are there infinitely many such triangular numbers $a + b + c$? Is it possible for the three numbers of a triangular triple to each be triangular?)

Solution 1. [F. Barekat] For each nonnegative integer k, the triple

$$(a,b,c) = (3k+2, 4k+2, 5k+3)$$

satisfies $T(a) + T(b) = T(c)$. Indeed,

$$(3k+2)(3k+3)+(4k+2)(4k+3) = (9k^2+15k+6)+(16k^2+20k+6) = 25k^2+35k+12 = (5k+3)(5k+4) \ .$$

We have that $a+b+c = 12k+7$, so we need to determine whether there are triangular numbers congruent to 7 modulo 12. Suppose that $T(x)$ is such. Then $x(x+1)$ must be congruent to 14 modulo 24. Now, modulo 24,

$$x^2 + x - 14 \equiv x^2 + x - 110 = (x-10)(x+11) \equiv (x-10)(x-13)$$

so $T(x)$ leaves a remainder 7 upon division by 12 if and only if $x = 10 + 24m$ or $x = 13 + 24n$ for some nonnegative integers m and n. This yields

$$k = 4 + 21m + 24m^2$$

and

$$k = 7 + 27n + 24n^2$$

for nonnegative integers m and n. The smallest triples according to these formula are $(a,b,c) = (14,18,23)$ and $(a,b,c) = (23,30,38)$, with the respective values of $a+b+c$ equal to $55 = T(10)$ and $91 = T(13)$. However, it may be that there are others that do not come under this set of formulae.

The equation $a(a+1) + b(b+1) = c(c+1)$ is equivalent to $(2a+1)^2 + (2b+1)^2 = (2c+1)^2 + 1$. It is straightforward to check whether numbers of the form $n^2 + 1$ with n odd is the sum of two odd squares. We get the following triples with sums not exceeding $T(10) = 55$:

$$(2,2,3), (3,5,6), (5,6,8), (4,9,10), (6,9,11), (8,10,13), (5,14,15),$$

$$(9,13,16),(11,14,18),(6,20,21),(12,17,21),(9,21,23),(11,20,23),(14,18,23)\ .$$

The entries of none except the last of these sum to a triangular number.

Solution 2. [Y. Zhao] Let (a,b,c) be a triangular triple and let $n = a+b+c$. Now

$$\begin{aligned}2T(a)+2T(b)-2T(n-a-b) &= a^2+a+b^2+b-(n-a-b)^2-(n-a-b)\\ &= (n+1)(n+2)-2(n+1-a)(n+1-b)\ .\end{aligned}$$

Thus, $(a,b,n-a-b)$ is a triangular triple if and only if $(n+1)(n+2) = 2(n+1-a)(n+1-b)$. In this case, neither $n+1$ nor $n+2$ can be prime, as each factor on the right side is strictly less than either of them. When 1 and 2 are added to each of the first nine triangular numbers, 1, 3, 6, 10, 15, 21, 28, 36, 45, we get at least one prime. Hence $n \geq 55$. It can be checked that $(a,b,c) = (14,18,23)$ is a triangular triple.

We claim that, for every nonnegative integer k, $T(24k+10) = a+b+c$ for some triangular triple (a,b,c). Observe that. with $n = T(24k+10) = 288k^2+252k+55$,

$$\begin{aligned}(n+1)(n+2) &= (T(24k+10)+1)(T(24k+10)+2)\\ &= (288k^2+252k+56)(288k^2+252k+57)\\ &= 12(72k^2+63k+14)(96k^2+84k+19)\ .\end{aligned}$$

Select a and b so that

$$n+1-a = 3(72k^2+63k+14) = 216k^2+189k+42$$

and

$$n+1-b = 2(96k^2+84k+19) = 192k^2+168k+38\ .$$

Let $c = n-a-b$. Thus,

$$(a,b,c) = (72k^2+63k+14, 96k^2+84k+18, 120k^2+105k+23)\ .$$

From the first part of the solution, we see that (a,b,c) is a triangular triple.

We observe that $(a,b,c) = (T(59),T(77),T(83)) = (1770,3003,3486)$ is a triangular triple.

Check:

$$\begin{aligned}1770\times 1771+3003\times 3004 &= (2\times 3\times 7\times 11)(295\times 23+13\times 1502)\\ &= (2\times 3\times 7\times 11)(26311) = 2\times 3\times 7\times 11\times 83\times 317\\ &= (2\times 3\times 7\times 83)(11\times 317) = 3486\times 3487\ .\end{aligned}$$

However, the sum of the numbers is $1770+3003+3486 = 8259$, which exceeds $8256 = T(128)$ by only 3.

Comment. David Rhee observed by drawing diagrams that for any triangular triple (a,b,c), $T(a+b-c) = (c-b)(c-a)$. This can be verified directly. Checking increasing values of $a+b-c$ and factoring $T(a+b-c)$ led to the smallest triangular triple.

339. Let a,b,c be integers with $abc \neq 0$, and u,v,w be integers, not all zero, for which

$$au^2+bv^2+cw^2 = 0\ .$$

Let r be any rational number. Prove that the equation

$$ax^2+by^2+cz^2 = r$$

is solvable for rational values of x, y, z.

Solution 1. Suppose, wolog, that $u \neq 0$. Try a solution of the form

$$(x,y,z) = (u(1+t), vt, wt)\ .$$

Then $au^2 + 2au^2t + au^2t^2 + b^2v^2t^2 + cw^2t^2 = r$ implies that $2au^2t = r - au^2$, from which we find the value $t = (r - au^2)/((2au^2)$. Since a, b, c, u, v, w, r, t are all rational, so is the trial solution.

Solution 2. [R. Peng] Suppose that $r = p/q$. Then the equation with r on the right is satisfied by

$$(x, y, z) = \left(\frac{p}{2} + \frac{1}{2qa}, \left(\frac{p}{2} - \frac{1}{2qa}\right)\left(\frac{v}{u}\right), \left(\frac{p}{2} - \frac{1}{2qa}\right)\left(\frac{w}{u}\right)\right) .$$

Comment. If not all a, b, c are zero, then it is trivial to prove that the equation with r on the right has a solution; the only rôle that the equation with 0 on the right plays is to ensure that a, b, c do not all have the same sign. However, some made quite heavy weather of this. The hypothesis that integers are involved should alert you to the fact that some special character of the solution is needed. It is unreasonable to ask that the solution be in integers, but one could seek out rational solutions.

340. The lock on a safe consists of three wheels, each of which may be set in eight different positions. Because of a defect in the safe mechanism, the door will open if any two of the three wheels is in the correct position. What is the smallest number of combinations which must be tried by someone not knowing the correct combination to guarantee opening the safe?

Solution. The smallest number of combinations that will guarantee success is 32. Denote the eight positions of each whell by the digits 0, 1, 2, 3, 4, 5, 6, 7, so that each combination can be represented by an ordered triple (a, b, c) of three digits. We show that a suitably selected set of 32 combinations will do the job. Let $A = \{(a, b, c) : 0 \leq a, b, c \leq 3 \text{ and } a + b + c \equiv 0 (\text{mod })4\}$ and $B = \{(u, v, w) : 4 \leq u, v, w \leq 7 \text{ and } u + v + w \equiv 0 (\text{mod })4\}$. Any entry in the triples of A and B is uniquely determined by the other two, and any ordered pair is a possibility for these two. Thus, each of A and B contains exactly 16 members. If (p, q, r) is any combination, then either two of p, q, r belong to the set $\{0, 1, 2, 3\}$ and agree with corresponding entries in a combination in A, or two belong to $\{4, 5, 6, 7\}$ and agree with corresponding entries in a combination in B.

We now show that at least 32 combinations are needed. Suppose, if possible, that a set S of combinations has three members whose first entry is 0: $(0, a, b), (0, c, d), (0, e, f)$. There will be twenty-five combinations of the form $(0, y, z)$ with $y \neq a, c, e$, $z \neq b, d, f$, that will not match in two entries any of these three. To cover such combinations, we will need at least 25 distinct combinations of the form (x, y, z) with $1 \leq x \leq 7$. None of the 28 combinations identified so far match the $7 \times 6 = 42$ combinations of the form (u, v, w), where $v \in \{a, c, e\}$, $w \in \{b, d, f\}$, $(v, w) \neq (a, b), (c, d), (e, f)$. Any combination of the form (u, v, t) or (u, t, w) can cover at most three of these and of the form (t, v, w) at most 7. Thus, S will need at least $37 = 3 + 25 + (42/7)$ members to cover all the combinations. A similar argument obtains if there are only three members in S with any other given entry. If there are only one or two members in S with a given entry, say first entry 0, then at least 36 combinations would be needed to cover all the entries with first entry 0 and the other entries differing from the entries of these two elements of S.

Thus, a set of combinations will work only if there are at least four combinations with a specific digit in each entry, in particular at least four whose first entry is k for each of $k = 0, \cdots, 7$. Thus, at least 32 entries are needed.

Comment. Some solvers formulated the problem in terms of the minimum number of rooks (castles) required to occupy or threaten every cell of a solid $8 \times 8 \times 8$ chessboard.

341. Let s, r, R respectively specify the semiperimeter, inradius and circumradius of a triangle ABC.

(a) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in arithmetic progression.

(b) Determine a necessary and sufficient condition on s, r, R that the sides a, b, c of the triangle are in geometric progression.

Comment. In the solutions, we will use the following facts, the establishment of which is left up to the reader:

$$a + b + c = 2s$$

$$ab + bc + ca = s^2 + 4Rr + r^2$$

$$abc = 4Rrs$$

An efficient way to get the second of these is to note that the square of the area is given by $r^2s^2 = s(s-a)(s-b)(s-c)$ from which

$$r^2s = s^3 - (a+b+c)s^2 + (ab+bc+ca)s - abc = s^3 - 2s^3 + (ab+bc+ca)s - 4Rrs \ .$$

Solution 1. (a) a, b, c are in arithmetic progression if and only if

$$\begin{aligned} 0 &= (2a-b-c)(2b-c-a)(2c-a-b) \\ &= (2s-3a)(2s-3b)(2s-3c) \\ &= 8s^3 - 12s^2(a+b+c) + 18s(ab+bc+ca) - 27abc \\ &= 2s^3 - 36Rrs + 18r^2s \ . \end{aligned}$$

Since $s \neq 0$, the necessary and sufficient condition that the three sides be in arithmetic progression is that $s^2 + 9r^2 = 18Rr$.

(b) First, note that

$$\begin{aligned} a^3 + b^3 + c^3 &= (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc \\ &= 2s^3 - 12Rrs - 6r^2s \ , \end{aligned}$$

and

$$\begin{aligned} a^3b^3 + b^3c^3 + c^3a^3 &= (ab+bc+ca)^3 - 3abc(a+b+c)(ab+bc+ca) + 3(abc)^2 \\ &= (s^2 + 4Rr + r^2)^3 - 24Rrs^4 - 48R^2r^2s^2 - 24Rr^3s^2 \ . \end{aligned}$$

a, b, c are in geometric progression if and only if

$$\begin{aligned} 0 &= (a^2 - bc)(b^2 - ca)(c^2 - ab) \\ &= abc(a^3 + b^3 + c^3) - (a^3b^3 + b^3c^3 + c^3a^3) \\ &= 32Rrs^4 - (s^2 + 4Rr + r^2)^3 \ , \end{aligned}$$

The necessary and sufficient condition is that

$$(s^2 + 4Rr + r^2)^3 = 32Rrs^4 \ .$$

Solution 2. The three sides of the triangle are the three real roots of the cubic equation

$$x^3 - 2sx^2 + (s^2 + r^2 + 4Rr)x - 4Rrs = 0 \ .$$

The three sides are in arithmetic progression if and only if one of them is equal to $2s/r$ and are in geometric progression if and only if one of them is equal to their geometric mean $\sqrt[3]{4Rrs}$.

(a) The condition is that $2s/3$ satisfies the cubic equation:

$$0 = 8s^3 - 6s(4s^2) + 9(s^2 + r^2 + 4Rr)(2s) - 108Rrs = 2s(s^2 + 9r^2 - 18Rr) \ .$$

(b) The condition is that $\sqrt[3]{4Rrs}$ satisfies the cubic equation: $2s(4Rrs)^{1/3} = s^2 + 4Rr + r^2$ or $32Rrs^3 = (s^2 + 4Rr + r^2)^3$.

Solution 3. [B.H. Deng] Assume that b lies between a and c, inclusive. (a) The three sides are in arithmetic progression if and only if $b = \frac{2}{3}s$ or $a + c = 2b$. Since $4Rrs = abc$, this is equivalent to $6Rr = ac$, which in turn is equivalent to

$$r^2 + s^2 + 4Rr = (a + c)b + ac = 2b^2 + ac = (8/9)s^2 + 6Rr$$

or $s^2 + 9r^2 = 18Rr$.

(b) The three sides are in geometric progression if and only if $b^3 = abc = 4Rrs$ and $ac = b^2$. This holds if and only if

$$r^2 + s^2 + 4Rr = (a + c)b + ac = (2s - b)b + ac = 2s\sqrt[3]{4Rrs} - b^2 + ac = 2s\sqrt[3]{4Rrs}$$

or $(r^2 + s^2 + 4Rr)^3 = 32Rrs^4$.

342. Prove that there are infinitely many solutions in positive integers, whose greatest common divisor is equal to 1, of the system

$$a + b + c = x + y$$
$$a^3 + b^3 + c^3 = x^3 + y^3 \ .$$

Solution 1. Suppose that a, b, c are in arithmetic progression, so that $c = 2b - a$ and $x + y = 3b$. Then

$$x^2 - xy + y^2 = \frac{a^3 + b^3 + c^3}{a + b + c} = 3b^2 - 4ab + 2a^2$$

so that

$$3xy = (x + y)^2 - (x^2 - xy + y^2) = 6b^2 + 4ab - 2a^2$$

and

$$xy = 2b^2 + \frac{2a(2b - a)}{3} \ .$$

Therefore

$$(y - x)^2 = (x + y)^2 - 4xy = b^2 - \frac{8a(2b - a)}{3} = \frac{(3b - 8a)^2 - 40a^2}{9} \ .$$

Let $p = 3b - 8a$, $q = 2a$. We can get solutions by solving $p^2 - 10q^2 = 9$. Three solutions are $(p, q) = (3, 0), (7, 2), (13, 4)$. The fundamental solution of $u^2 - 10v^2 = 1$ is $(u, v) = (19, 6)$. So from any solution $(p, q) = (r, s)$ of $p^2 - 10q^2 = 9$, we get another $(p, q) = (19r + 60s, 6r + 19s)$. For these to yield solutions $(a, b, c; x, y)$ of the original system, we require q to be even and $p + 4q$ to be divisible by 3. Since $19r + 60s \equiv r \pmod 3$ and $6r + 19s \equiv s \pmod 2$, if $(p, q) = (r, s)$ has these properties, then so also does $(p, q) = (19r+60s, 6r+19s)$. Starting with (r, s), we can define integers p and q, and then solve the equations $x + y = 3b$, $y - x = 1$. Since p and so b are odd, these equations have integer solutions. Here are some examples:

$$(p, q; a, b, c; x, y) = (3, 0; 0, 1, 2; 1, 2), (57, 18; 9, 43, 77; 64, 65),$$
$$(7, 2; 1, 5, 9; 7, 8), (253, 80; 40, 191, 342; 286, 287) \ .$$

Solution 2. [D. Dziabenko] Let $a = 3d$. $c = 2b - 3d$, so that $x + y = 3b$ and a, b, c are in arithmetic progression. Then

$$a^3 + b^3 + c^3 = 27d^3 + b^3 + 8b^3 - 36b^2d + 54bd^2 - 27d^3$$
$$= 9b^3 - 36b^2d + 54bd^2 = 9b(b^2 - 4b^2d + 6d^2) \ ,$$

whence $x^2 - xy + y^2 = 3b^2 - 12bd + 18d^2$. Therefore

$$3xy = (x + y)^2 - (a^2 - xy + y^2) = 6b^2 + 12bd - 18d^2$$

so that $xy = 2b^2 + 4bd - 6d^2$ and

$$(x-y)^2 = (x+y)^2 - 4xy = b^2 - 16bd + 24d^2 = (b-8d)^2 - 40d^2 \ .$$

Let $b - 8d = p^2 + 10q^2$ and $d = pq$. Then

$$x - y = \sqrt{p^4 - 20p^2q^2 + 100q^4} = p^2 - 10q^2 \ .$$

Solving this, we find that

$$(a, b, c; x, y) = (3pq, p^2 + 8pq + 10q^2, 2p^2 + 13pq + 20q^2; 2p^2 + 12pq + 10q^2, p^2 + 12pq + 20q^2) \ .$$

Some numerical examples are

$$(a, b, c; p, q) = (3, 19, 35; 24, 33), (6, 30, 54; 42, 48), (6, 57, 108; 66, 105) \ .$$

Any common divisor of a and b must divide $3pq$ and $p^2 + 10q^2$, and so must divide both p and q. [Justify this; you need to be a little careful.] We can get the solutions we want by arranging that the p and q are coprime.

Solution 3. [F. Barekat] Let $m = a + b + c = x + y$ and $n = a^3 + b^3 + c^3 = x^3 + y^3$. Then

$$3xy = m^2 - \frac{n}{m} = \frac{m^3 - n}{m}$$

and

$$\begin{aligned}(x-y)^2 &= \frac{x^3 + y^3}{x + y} - xy = \frac{4n - m^3}{3n} \\ &= \frac{4(a^3 + b^3 + c^3) - (a + b + c)^3}{3(a + b + c)} \\ &= (c - a - b)^2 - \frac{4ab(a + b)}{a + b + c} \ .\end{aligned}$$

Select a, b, c so that

$$\frac{4ab(a + b)}{a + b + c} = 2(c - a - b) - 1 \ .$$

so that $x - y = c - a - b - 1$. Then we can solve for rational values of x and y. If we can do this $x + y = a + b + c$ and $x - y = c - a - b - 1$. Note that, these two numbers have different parity, so we will obtain fractional values of x and y, whose denominators are 2. However, the equations to be solved are homogeneous, so we can get integral solutions by doubling: $(2a, 2b, 2c; 2x, 2y)$.

Let $c = u(a + b)$. Then

$$4ab = 2(u - 1)(u + 1)(a + b) - (u + 1) \ .$$

Let $u = 4v + 3$, Then we get

$$ab = 4(v + 1)(2v + 1)(a + b) - 4(v + 1) \ ,$$

from which

$$[a - 4(v + 1)(2v + 1)][b - 4(v + 1)(2v + 1)] = (v + 1)[16(v + 1)(2v + 1)^2 - 1] \ .$$

We use various factorizations of the right side and this equation to determine integer values of a and b, from which the remaining variables c, x and y can be determined.

For example, $v = 0$ yields the equation $(a - 4)(b - 4) = 15$ from which we get the possibilities

$$(a, b, c) = (5, 19, 72), (7, 9, 48) \ .$$

Doubling to clear fractions, yields the solutions

$$(a, b, c; x, y) = (10, 38, 144; 49, 143), (14, 18, 96; 33, 95) .$$

Additional solutions come from $v = 1$:

$$(a, b, c; x, y) = (76, 130, 1442; 207, 1441), (50, 1196, 8722; 1247, 8721) .$$

Solution 4. [D. Rhee] An infinite set of solutions is given by the formula

$$\begin{aligned}(a, b, c; x, y) &= (2, n^2 + 3n, n^2 + 5n + 4; n^2 + 4n + 2, n^2 + 4n + 4) \\ &= (2, n(n+3), (n+1)(n+4); (n+2)^2 - 2, (n+2)^2) .\end{aligned}$$

Examples are $(a, b, c; x, y) = (2, 4, 10; 7, 9), (2, 10, 18; 14, 16), (2, 18, 28; 23, 25)$.

Comment. M. Fatehi gave the solution

$$(a, b, c; x, y) = (5, 6, 22; 12, 21) .$$

343. A sequence $\{a_n\}$ of integers is defined by

$$a_0 = 0 , \quad a_1 = 1 , \quad a_n = 2a_{n-1} + a_{n-2}$$

for $n > 1$. Prove that, for each nonnegative integer k, 2^k divides a_n if and only if 2^k divides n.

Solution 1. Let m and n be two nonnegative integers. Then $a_{m+n} = a_m a_{n+1} + a_{m-1}a_n = a_{m+1}a_n + a_m a_{n-1}$. This can be checked for small values of m and n and established by induction. The induction step is

$$\begin{aligned}a_{m+n+1} &= 2a_{m+n} + a_{m+n-1} = 2(a_m a_{n+1} + a_{m-1}a_n) + (a_m a_n + a_{m-1}a_{n-1} \\ &= a_m(2a_{n+1} + a_n) + a_{m-1}(2a_n + a_{n-1}) = a_m a_{n+2} + a_{m-1}a_{n+1} .\end{aligned}$$

In particular, for each integer n,

$$a_{2n} = a_n(a_{n-1} + a_{n+1}) .$$

It is straightforward to show by induction from the recursion that a_n is odd whenever n is odd and even whenever n is even. Suppose now that n is even. Then $a_{n+1} = 2a_n + a_{n-1} \equiv a_{n-1} \equiv a_1 = 1 \pmod 4$, so that $a_{n-1} + a_{n+1} = 2b_n$ for some odd number b_n. Hence $a_{2n} = 2a_n b_n$. For $k = 0$, we have that $2^k | a_n$ if and only if $2^k | n$. Suppose that this has been established for $k = r$.

Suppose that $n = 2^{r+1}m$ for some integer m. Then $n/2$ is divisible by 2^r, and therefore so is $a_{n/2}$. Hence $a_n = 2a_{n/2}b_{n/2}$ is divisible by 2^{r+1}. On the other hand, suppose that n is not divisible by 2^{r+1}. If n is not divisible by 2^r, then a_n is not so divisible by the induction hypothesis, and so not divisible by 2^{r+1}. On the other hand, if $n = 2^r c$, with c odd, then a_n is divisible by 2^r. But $n/2 = 2^{r-1}c$, so $a_{n/2}$ is not divisible by 2^r. Hence $a_n = 2a_{n/2}b_{n/2}$ is not divisible by 2^{r+1}. The result follows.

Solution 2. For convenience, imagine that the sequence is continued backwards using the recursion $a_{n-2} = a_n - 2a_{n-1}$ for all integer values of the index n. We have for every integer n, $a_{n+1} = 2a_n + a_{n-1} \Rightarrow 2a_{n+1} = 4a_n + 2a_{n-1} \Rightarrow a_{n+2} - a_n = 4a_n + a_n - a_{n-2} \Rightarrow a_{n+2} = 6a_n - a_{n-2}$. Suppose, for some positive integer r, we have established that, for every integer n,

$$a_{n+2^r} = b_r a_n - a_{n-2^r}$$

where $b^r \equiv 2 \pmod 4$. This is true for $r = 1$ with $b_1 = 6$. Then

$$b_r a_{n+2^r} = b_r^2 a_n - b_r a_{n-2^r}$$

$$\implies a_{n+2^{r+1}} + a_n = b_r^2 a_n - (a_n + a_{n-2^{r+1}})$$

$$\implies a_{n+2^r} = b^{r+1} a_n - a_{n-2^{r+1}} \ ,$$

where $b_{r+1} = b_2^2 - 2 \equiv 2 \pmod 4$.

Observe that, since $a_{n+1} \equiv a_{n-1} \pmod 2$ and $a_0 = 0$, $a_1 = 1$, a_n is even if and only if n is even. When n is even, then $a_{n+2} \equiv a_{n-2} \pmod 4$, so that a_n is divisible by 4 if and only if n is.

Let $m \geq 2$ be a positive integer. Suppose that it has been established for $1 \leq s \leq m$, that 2^s divides a_n if and only if 2^s divides n. Then 2^{s+1} will divide a_n only if $n = 2^s p$ for some integer p. Now

$$a_{2^s} = b_{s-1} a_{2^{s-1}} - a_0 = b_{s-1} a_{2^{s-1}} \ ;$$

since $2 \| b^{s-1}$ and $2^{s-1} \| a^{2^{s-1}}$, it follows that $2^s \| a_{2^s}$. (The notation $2^k \| q$ means that 2^k is the highest power of 2 that divides q.) Thus 2^{s+1} does not divide 2^s.

Suppose that it has been established for $1 \leq i \leq p$ that when $n = 2^s i$, $2^{s+1} | n$ if and only if p is even. We have that

$$a_{2^s(p+1)} = b_s 2^s p - a_{2^s(p-1)} \ .$$

If p is even, then $b^s a_{2^s p} \equiv 0 \pmod{2^{s+1}}$, so that $a_{2^s(p+1)} \equiv a_2^s(p-1) \equiv 2^s \pmod{2^{s+1}}$, and $a_{2^s(p+1)}$ is not a multiple of 2^{s+1}. If p is odd, then each term on the right side of the foregoing equation is a multiple of 2^{s+1}, and therefore so is $a^{2^s(p+1)}$. The desired result follows by induction.

Solution 3. The characteristic equation for the recursion is $t^2 - 2t - 1 = 0$, with roots $t = 1 \pm \sqrt{2}$. Solving the recursion, we find that

$$\begin{aligned} a_n &= \frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n - (1-\sqrt{2})^n] \\ &= \frac{1}{\sqrt{2}}\left[\sum_{k=0}^{\infty} \binom{n}{2k+1} 2^k \sqrt{2}\right] \\ &= \sum_{k=0}^{\infty} \binom{n}{2k+1} 2^k = n + \sum_{k=1}^{\infty} \binom{n}{2k+1} 2^k \\ &= n + \sum_{k=1}^{\infty} \frac{n}{2k+1} \binom{n-1}{2k} 2^k \ . \end{aligned}$$

(We use the convention that $\binom{i}{j} = 0$ when $i < j$. Suppose that $n = 2^r s$ where r is a nonnegative integer and s is odd. Since the odd number $2k+1$ divides $n\binom{n-1}{2k} = 2^r s \binom{n-1}{2k}$, $2k+1$ must divide $s\binom{n-1}{2k}$, so that 2^s must divide $n\binom{n-1}{2k} = \binom{n}{2k+1}$. Therefore, 2^{s+1} must divide each term $\binom{n}{2k+1}$ for $k \geq 1$. Therefore $a_n \equiv n \pmod{2^s}$ and the desired result follows.

Comment. Y. Zhao obtained by induction that

$$\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$$

from which the matrix equation

$$\begin{pmatrix} a_{2n+1} & a_{2n} \\ a_{2n} & a_{2n-1} \end{pmatrix} = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{2n-1} \end{pmatrix}^2$$

yields the equation $a_{2n} = a_n(a_{n-1} + a_{n+1}$.

344. A function f defined on the positive integers is given by

$$f(1) = 1 \ , \quad f(3) = 3 \ , \quad f(2n) = f(n) \ ,$$

$$f(4n + 1) = 2f(2n + 1) - f(n)$$
$$f(4n + 3) = 3f(2n + 1) - 2f(n) \ ,$$

for each positive integer n. Determine, with proof, the number of positive integers no exceeding 2004 for which $f(n) = n$.

Solution. Let $g(n)$ be defined for positive integer n by writing n to base 2 and reversing the digits. Specifically, if $n = \sum_{k=0}^{r} a_k 2^r$ with each a_k equal to 0 or 1 and $a_r = 1$, then $g(n) = \sum_{k=0}^{n} a_{r-k} 2^k$. We prove that $g(n)$ has the properties ascribed to $f(n)$. It is checked that $g(1) = g(2) = g(3) = 1$. Let $n = a_r 2^r + a_{r-1} 2^{r-1} + \cdots + a_1 2 + a_0$. Then $2n = a_r 2^{r+1} + \cdots + a_0 2 + 0$ and $g(2n) = 0 \cdot 2^{r+1} + a_0 2^r + \cdots + a_{r-1} 2 + a_r = g(n)$.

Since $4n + 1 = a_r 2^{r+2} + a_{r-1} 2^{r+1} + \cdots + a_1 2^3 + a_0 2 + 0 \cdot 2 + 1$,

$$\begin{aligned} g(n) + g(4n+1) &= (a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1} 2 + a_r) + (2^{r+2} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1} 2 + a_r) \\ &= 2^{r+2} + a_0 2^{r+1} + a_1 r^r + \cdots + a_{r-1} 2^2 + a_r 2 = 2(2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1} 2 + a_r \\ &= 2g(a_2 2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1} 2 + a_r) = 2g(2n+1) \ . \end{aligned}$$

(This uses the fact that $a2^i + a2^i = a2^{i+1}$.)

Since $4n + 3 = a_r 2^{r+2} + a_{r-1} 2^{r+1} + \cdots + a_1 2^3 + a_0 2 + 1 \cdot 2 + 1$,

$$\begin{aligned} &2g(n) + g(4n+3) \\ &\quad = (a_0 2^{r+1} + a_1 2^r + a_2 2^{r-1} + \cdots + a_{r-1} 2^2 + a_r 2) \\ &\qquad + (2^{r+2} + 2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_{r-1} 2 + a^r) \\ &\quad = (2^{r+2} + a_0 2^{r+1} + a_1 2^r + \cdots + a_r 2) + (2^{r+1} + a_0 2^r + a_1 2^{r-1} + \cdots + a_r) \\ &\quad = 2g(2n+1) + g(2n+1) = 3g(2n+1) \ . \end{aligned}$$

We show by induction that $f(n) = g(n)$ for every positive integer n. This is true for $1 \leq n \leq 4$. Suppose it holds for $1 \leq n \leq 4m$. Then

$$f(4m + 1) = 2f(2m + 1) - f(m) = 2g(2m + 1) - g(m) = g(4m + 1) \ ;$$

$$f(4m + 2) = f(2m + 1) = g(2m + 1) = g(4m + 2) \ ;$$

$$f(4m + 3) = 3f(2m + 1) - 2f(m) = 3g(2m + 1) - 2g(m) = g(4m + 3) \ ;$$

$$f(4m + 4) = f(2m + 2) = g(2m + 2) = f(4m + 4) \ .$$

Thus we have a description of $f(n)$.

For $f(n) = n$, it is necessary and sufficient that n is a palindrome when written to base 2. We need to find the number of palindromes between 1 and $2004 = (11111010100)_2$ inclusive. The number of $(2r - 1)-$ and $2r-$digit palindromes is each 2^{r-1} as the first and last digits must be 1 and there are $r - 1$ other matching pairs of digits or central digits that can be set to either 0 or 1. The number of palindromes up to $2^{11} - 1 = 2047$ is $2(1 + 2 + 4 + 8 + 16) + 32 = 94$. The only palindromes between 2004 and 2048 are $(11111011111)_2$ and $(11111111111)_2$, and these should not be counted. Therefore, there are exactly 92 palindromes, and therefor 92 solutions of $f(n) = n$ between 1 and 2004, inclusive.

345. Let $\mathfrak{C}$ be a cube with edges of length 2. Construct a solid figure with fourteen faces by cutting off all eight corners of $\mathfrak{C}$, keeping the new faces perpendicular to the diagonals of the cuhe and keeping the

newly formed faces identical. If the faces so formed all have the same area, determine the common area of the faces.

Solution 1. In the situation where the cuts pass through the midpoints of the edges, yielding a cube-octahedron with six square and eight equilateral-triangular sides, we find that the square faces have area 2 and the triangular faces have area $(\sqrt{3}/4)(\sqrt{2}) = \sqrt{6}/4 < 2$. Moving the cuts closer to the vertices yields triangular faces of area less than 2 and octahedral faces of area greater than 2. Thus, for equal areas of the corner and face figures, the cuts must be made a a distance exceeding 1 from each vertex.

The corner faces of the final solid are hexagons formed by large equilateral triangles with smaller equilateral triangles clipped off each vertex; the other faces are squares (diamonds) in the middle of the faces of the cube. Let the square faces have side length x. The vertices of this face are distant $1-(x/\sqrt{2})$ from the edge of the cube, so that smaller equilateral triangles of side $\sqrt{2}(1-(x/\sqrt{2})) = \sqrt{2}-x$ are clipped off from a larger equilateral triangle of side $2(\sqrt{2}-x)+x = 2\sqrt{2}-x$. The areas of the hexagonal faces of the solid figure are each

$$\frac{\sqrt{3}}{4}[(2\sqrt{2}-x)^2 - 3(\sqrt{2}-x)^2] = \frac{\sqrt{3}}{2} + \frac{x\sqrt{6}}{2} - \frac{x^2\sqrt{3}}{2} \ .$$

For equality, we need

$$x^2 = \frac{\sqrt{3}}{2}[1 + x\sqrt{2} - x^2] \ ,$$

or

$$(2+\sqrt{3})x^2 - x\sqrt{6} - \sqrt{3} = 0 \ .$$

Hence

$$x = \frac{\sqrt{6} + \sqrt{8\sqrt{3}+18}}{2(2+\sqrt{3})}$$

and the common area is

$$x^2 = \frac{6+2\sqrt{3}+\sqrt{27+12\sqrt{3}}}{7+4\sqrt{3}} = (6+2\sqrt{3}+\sqrt{27+12\sqrt{3}})(7-4\sqrt{3}) \ .$$

Solution 2. Let the cut be made distant u from a vartex. As in Solution 1, we argue that $1 < u < 2$. Then the edge of the square face of the final solid is distant $u/\sqrt{2}$ from the vertex of the cube and $\sqrt{2}(1-(u/2))$ from the centre of the face. Thus, the square face has side length $\sqrt{2}(2-u)$ and area $8-8u+2u^2$.

The hexagonal face of the solid consists of an equilateral triangle of side $\sqrt{2}u$ with three equilateral triangles of side $\sqrt{2}(u-1)$ clipped off. Its area is $(\sqrt{3}/2)[-2u^2+6u-3$. For equality of area of all the faces, we require that

$$2(8-8u+2u^2) = \sqrt{3}(-2u^2+6u-3)$$

or

$$2(2+\sqrt{3})u^2 - 2(8+3\sqrt{3})u + (16+3\sqrt{3}) = 0 \ .$$

Solving this equation and taking the root less than 2 yields that

$$u = \frac{(8+3\sqrt{3}) - \sqrt{9+4\sqrt{3}}}{2(2+\sqrt{3})} \ ,$$

whence

$$2-u = \frac{\sqrt{3}+\sqrt{9+4\sqrt{3}}}{2(2+\sqrt{3})} \ .$$

Thus, the common area is

$$2(2-u)^2 = \frac{6+2\sqrt{3}+\sqrt{27+12\sqrt{3}}}{7+4\sqrt{3}} = (6+2\sqrt{3}+\sqrt{27+12\sqrt{3}})(7-4\sqrt{3}) \ .$$

346. Let n be a positive integer. Determine the set of all integers that can be written in the form

$$\sum_{k=1}^{n} \frac{k}{a_k}$$

where $a_1, a_2, \cdots, a_n$ are all positive integers.

Solution 1. The sum cannot exceed $\sum_{k=1}^n k = \frac{1}{2}n(n+1)$. We prove by induction that the set of integers that can be written in the required form consists of the integers from 1 to $\frac{1}{2}n(n+1)$ inclusive. Observe that 1 is representable for any n (for example, by making each a_k equal to kn). Also, $\frac{1}{2}n(n+1)$ is representable, by taking $a_1 = a_2 = \cdots = a_n$. Thus, the result holds for $n = 1$ and $n = 2$.

Suppose that it holds for $n = m \geq 2$. Then, by taking $a_{m+1} = m+1$ and appending $(m+1)/(m+1)$ to each integer representable by an $m-$term sum, we find that each integer between 2 and $\frac{1}{2}m(m+1) + 1 = (m^2+m+2)/2$ inclusive can be represented with a $(m+1)-$ term sum. By taking $a_{m+1} = 1$ and appending $m+1$ to each integer representable by an $m-$term sum, we find that each integer between $m+2$ and $\frac{1}{2}m(m+1) + (m+1) = \frac{1}{2}(m+1)(m+2)$ can be represented with an $(m+1)-$term sum. Since $[(m^2+m+2)/2] - (m+2) = \frac{1}{2}(m^2-m-2) = \frac{1}{2}(m-2)(m+1) \geq 0$ for $m \geq 2$, $\frac{1}{2}(m^2+m+2) \geq m+2$. Thus, we can represent all numbers between 1 and $\frac{1}{2}(n+1)(n+2)$ inclusive. The result follows.

Solution 2. [Y. Zhao] `Lemma.` For any integer k with $1 \leq k \leq \frac{1}{2}n(n+1)$, there is a subset T_k of $\{1, 2, \cdots, n\}$ for which the sum of the numbers in T_k is k.

`Proof.` T_k is the entire set when $k = \frac{1}{2}n(n+1)$. For the other values of k, we give a proof by induction on k. A singleton suffices for $1 \leq k \leq n$. Suppose that the result holds for $k = m - 1 < \frac{1}{2}n(n+1)$. Then T_{m-1} must lack at least one number. If $1 \notin T_{m-1}$, let $T_m = T_{m-1} \cup \{1\}$. If $1 \in T_{m-1}$, let $i > 1$ be the least number not in T_{m-1}. Then let $T_m = T_{m-1} \cup \{i+1\} \backslash \{i\}$. ♠

Now to the result. Let $2 \leq k \leq \frac{1}{2}n(n+1)$ be given and determine T_{k-1}. Define $a_i = 1$ when $i \in T_{k-1}$ and $a_i = \frac{1}{2}n(n+1) - (k-1)$ when $i \notin T_{k-1}$. Then

$$\begin{aligned}\sum \left\{ \frac{i}{a_i} : i \notin T_{k-1} \right\} &= \left[\binom{n+1}{2} - (k-1) \right]^{-1} \sum \{ i : i \notin T_{k-1} \} \\ &= \left[\binom{n+1}{2} - (k-1) \right]^{-1} \left[\binom{n+1}{2} - \sum \{ i : i \in T_{k-1} \right] = 1 \ .\end{aligned}$$

Since one can give a representation for 1 and since no number exceeding $\binom{n+1}{2}$ can be represented, it follows that the set of representable integers consists of those between 1 and $\binom{n+1}{2}$ inclusive.

347. Let n be a positive integer and $\{a_1, a_2, \cdots, a_n\}$ a finite sequence of real numbers which contains at least one positive term. Let S be the set of indices k for which at least one of the numbers

$$a_k, a_k + a_{k+1}, a_k + a_{k+1} + a_{k+2}, \cdots, a_k + a_{k+1} + \cdots + a_n$$

is positive. Prove that

$$\sum \{ a_k : k \in S \} > 0 \ .$$

Solution. We prove the result by induction on n. When $n = 1$, the result is obvious. Let $m \geq 2$ and suppose that the result holds for all $n \leq m-1$. Suppose a suitable sequence $\{a_1, a_2, \cdots, a_m\}$ is given. If $a_1 \notin S$, then $\sum \{a_k : k \in S\} > 0$, by the induction hypothesis applied to the $(m-1)-$element set $\{a_2, \cdots, a_m\}$. Suppose that $a_1 \in S$ and that r is the smallest index for which $a_1 + a_2 + \cdots + a_r > 0$. Then, for $1 \leq i \leq r-1$, $a_1 + \cdots + a_i \leq 0$ and so

$$(a_{i+1} + \cdots + a_r) = (a_1 + \cdots + a_r) - (a_1 + \cdots + a_i) > 0 \ ,$$

i.e., $a_2, a_3, \cdots, a_r \in S$. Hence $\sum\{a_k : 2 \leq k \leq r\} > 0$. If there are no elements of S that exceed r. then the desired conclusion follows. Otherwise, by the induction hypothesis applied to the $(m-r)-$ element set $\{a_{r+1}, \cdots, a_m\}$, we have that $\sum\{a_k : k \in S, r+1 \leq k \leq m\} > 0$. The desired conclusion follows.

Comment. Most solvers had much more elaborate solutions which essentially used this idea. No one recognized that the proliferation of cases could be sidestepped by the technical use of an induction argument.

348. Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that $f(x) - x^3$ is an increasing function. Must $f(x) - x - x^2$ also be increasing?

(b) Suppose that $f(x)$ is a real-valued function defined for real values of x. Suppose that both $f(x) - 3x$ and $f(x) - x^3$ are increasing functions. Must $f(x) - x - x^2$ also be increasing on all of the real numbers, or on at least the positive reals?

Solution. (a) The answer is *no*. Consider $f(x) = x^3 + x$. Then $x = f(x) - x^3$ is increasing, but $g(x) = f(x) - x - x^2 = x^3 - x^2$ is not. Indeed, $g(0) = 0$ while $g(\frac{1}{2}) = -\frac{1}{8} < 0$.

Solution 1. (b) Let $u \geq v$. Suppose that $u + v \leq 2$. Then, since $f(x) - 3x$ is increasing, $f(u) - 3u \geq f(v) - 3v$, whence

$$f(u) - f(v) \geq 3(u - v) \geq (u + v + 1)(u - v) = u^2 - v^2 + u - v \Longrightarrow f(u) - u - u^2 \geq f(v) - v - v^2 \ .$$

Suppose that $u + v \geq 2$. Then, since $f(x) - x^3$ is increasing,

$$f(u) - u^3 \geq f(v) - v^3 \Longrightarrow f(u) - f(v) \geq u^3 - v^3 = (u - v)(u^2 + uv + v^2) \ .$$

Now

$$2[(u^2 + uv + v^2) - (u + v + 1)] = (u + v)^2 + (u - 1)^2 + (v - 1)^2 - 4 \geq 0 \ ,$$

so that $u^2 + uv + v^2 \geq u + v + 1$ and

$$f(u) - f(v) \geq (u - v)(u + v + 1) = u^2 - v^2 + u - v \Longrightarrow f(u) - u - u^2 \geq f(v) - v - v^2 \ .$$

Hence $f(u) - u - u^2 \geq f(v) - v - v^2$ whenever $u \geq v$, so that $f(x) - x - x^2$ is increasing. ♠

Solution 2. [F. Barekat] (b) Let $u \geq v$. Then, as in Solution 1, we find that $f(u) - f(v) \geq 3(u - v)$ and $f(u) - f(v) \geq u^3 - v^3 = (u - v)(u^2 + uv + v^2)$. If $1 \geq u \geq v$, then $3 \geq u + v + 1$, so that

$$f(u) - f(v) \geq 3(u - v) \geq (u + v + 1)(u - v) \ .$$

If $u \geq v \geq 1$, then $u^2 \geq u$, $v^2 \geq v$ and

$$f(u) - f(v) \geq (u - v)(u^2 + uv + v^2) \geq (u - v)(u + 1 + v) \ .$$

In either case, we have that $f(u) - u - u^2 \geq f(v) - v - v^2$. Finally, if $u \geq 1 \geq v$, then

$$f(u) - u - u^2 \geq f(1) - 2 \geq f(v) - v - v^2 \ .$$

The result follows. ♠

Comment. D. Dziabenko assumed that f was differentiable on $\mathbf{R}$, so that $f'(x) \geq 3$ and $f'(x) \geq 3x^2$ everywhere. Hence, for all x, $3f'(x) \geq 3x^2 + 6$, so that $f'(x) \geq x^2 + 2 \geq 2x + 1$. Hence, the derivative of $f(x) - x - x^2$ is always nonnegative, so that $f(x) - x - x^2$ is increasing. However, there is nothing in the hypothesis that forces f to be differentiable, so this is only a partial solution and its solver would have to settle for a grade of 2 out of 7. A little knowledge is a dangerous thing. If calculus is used, you need to make sure that everything is in place, all assumptions made identified and justified. Often, a more efficient and transparent solution exists without recourse to calculus.

349. Let s be the semiperimeter of triangle ABC. Suppose that L and N are points on AB and CB produced (*i.e.*, B lies on segments AL and CN) with $|AL| = |CN| = s$. Let K be the point symmetric to B with respect to the centre of the circumcircle of triangle ABC. Prove that the perpendicular from K to the line NL passes through the incentre of triangle ABC.

Let the incentre of the triangle be I.

Solution 1. Let P be the foot of the perpendicular from I to AK, and Q the foot of the perpendicular from I to CK. Since BK is a diameter of the circumcircle of triangle ABC, $\angle BAK = \angle BCK = 90°$ and $IP \| BA$, $IQ \| BC$. Now $|IP| = s - a = |BN|$, $|IQ| = s - c = |BL|$ and $\angle PIQ = \angle ABC = \angle NBL$, so that $\Delta IPQ \equiv \Delta BNL$ (SAS). Select R on IP and S on IQ (possibly produced) so that $IR = IQ$, $IS = IP$. Thus, $\Delta ISR \equiv \Delta BNL$ and $RS \| NL$ (why?). Since $IPKQ$ is concyclic, $\angle KIP + \angle IRS = \angle KIP + \angle IQP = \angle KIP + \angle IKP = 90°$. Therefore IK is perpendicular to RS, and so to NL.

Solution 2. `Lemma.` Let W, X, Y, Z be four points in the plane. Then $WX \perp YZ$ if and only ig $|WY|^2 - |WZ|^2 = |XY|^2 - |XZ|^2$.

`Proof.` Note that

$$2(\overrightarrow{W} - \overrightarrow{X}) \cdot (\overrightarrow{Z} - \overrightarrow{Y}) = (\overrightarrow{Y} - \overrightarrow{W})^2 - (\overrightarrow{Z} - \overrightarrow{W})^2 - (\overrightarrow{Y} - \overrightarrow{X})^2 + (\overrightarrow{Z} - \overrightarrow{X})^2 \; . \spadesuit \; .$$

Since BK is a diameter of the circumcircle, $\angle LAK = \angle NCK = 90°$. We have that

$$\begin{aligned} |KL|^2 - |KN|^2 &= (|KA|^2 + |AL|^2) - (|KC|^2 + |CN|^2) = |KA^2| - |KC|^2 \\ &= (|BK|^2 - |AB|^2) - (|BK|^2 - |BC|^2) = |BC|^2 - |AB|^2 \; . \end{aligned}$$

Let U and V be the respective feet of the perpendiculars from I to BA and BC. Observe that $|AU| = |BN| = s - a$, $|CV| = |BL| = s - c$ and $|BU| = |BV| = s - b$, so that $|UL| = |BC| = a$, $|VN| = |AB| = c$. Then

$$|IL|^2 - |IN|^2 = (|IU|^2 + |UL|^2) - (|IV|^2 + |VN|^2) = |UL|^2 - |VN|^2 = |BC|^2 - |AB|^2 \; ,$$

which, along with the lemma, implies the result.

350. Let $ABCDE$ be a pentagon inscribed in a circle with centre O. Suppose that its angles are given by $\angle B = \angle C = 120°$, $\angle D = 130°$, $\angle E = 100°$. Prove that BD, CE and AO are concurrent.

Solution 1. [P. Shi; Y. Zhao] The vertices $ABCDE$ are the vertices A_1, A_5, A_7, A_{11}, A_{12} of a regular $18-$gon. (Since A, B, C, D, E lie on a circle, the position of the remaining vertices are determined by that of A are the angle sizes. Alternatively, look at the angles subtended at the centre by the sides of the pentagon.) Since the sum of the angles of a pentagon is $540°$, $\angle A = 70°$. Since $\angle AED > 90°$, D and E lie on the same side of the diameter through A, and B and C lie on the other side. Thus, the line AO produced intersects the segment CD. Consider the triangle ACD for which AO produced, BD and CE are cevians. We apply the trigonometric version of Ceva's theorem.

We have that $\angle CAO = 30°$, $\angle ADB = 40°$, $\angle DCE = 10°$, $\angle OAD = 10°$, $\angle BDC = 20°$ and $\angle ECA = 70°$. Hence

$$\frac{\sin \angle CAO \cdot \sin \angle ADB \cdot \sin \angle DCE}{\sin \angle OAD \cdot \sin \angle BDC \cdot \sin \angle ECA} = \frac{(\frac{1}{2}) \sin 40° \sin 10°}{\sin 10° \sin 20° \sin 70°} = \frac{\cos 20°}{\sin 70°} = 1 \; .$$

Hence the three lines AO, BD and CE are concurrent as desired.

Solution 2. [C. Sun] Let BD and CE intersect at P. We can compute the following angles: $\angle EAB = \angle EBA = 70°$, $\angle AEB = 40°$, $\angle BEP = \angle PDC = 20°$, $\angle EBP = \angle PCD = 10°$, $\angle PBC = \angle PED = 40°$, $\angle BCP = \angle EDP = 110°$ and $\angle BPC = \angle EPD = 30°$. Since triangle ABE is acute, the circumcentre O of it (and the pentagon) lie in its interior, and $\angle AOB = \frac{1}{2}\angle AEB = 80°$. Since triangle ABE is isosceles, $\angle BAO = \angle ABO = 50°$.

Let Q be the foot of the perpendicular from E to AB and R the foot of the perpendicular from B to EP produced. Since $\Delta EQB \equiv \Delta ERB$ (ASA), $QB = RB$. Since $\angle BPR = 30°$, $\angle PBR = 60°$ and $PB = 2RB = 2QB = AB$. Hence ABP is isosceles with apex $\angle ABP = 80°$. Thus, $\angle BAP = \angle BPA = 50°$. Hence $\angle BAO = \angle BAP = 50°$, so that AO must pass through P and the result follows.

351. Let $\{a_n\}$ be a sequence of real numbers for which $a_1 = 1/2$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1} \ .$$

Prove that, for all n, $a_1 + a_2 + \cdots + a_n < 1$.

Solution. Let $b_n = 1/a_n$ for $n \geq 1$, whence $b_1 = 2$ and $b_{n+1} = b_n^2 - b^n + 1 = b_n(b_n - 1) + 1$ for $n \geq 1$. Then $\{b_n\}$ is an increasing sequence of integers and

$$\frac{1}{b_n} = \frac{1}{b_n - 1} - \frac{1}{b_{n+1} - 1}$$

for $n \geq 1$. Hence

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} \\ &= \left[\frac{1}{b_1 - 1} - \frac{1}{b_2 - 1}\right] + \left[\frac{1}{b_2 - 1} - \frac{1}{b_3 - 1}\right] + \cdots + \left[\frac{1}{b_n - 1} - \frac{1}{b_{n+1} - 1}\right] \\ &= \frac{1}{b_1 - 1} - \frac{1}{b_{n+1} - 1} = 1 - \frac{1}{b_{n+1} - 1} < 1 \ . \end{aligned}$$

352. Let $ABCD$ be a unit square with points M and N in its interior. Suppose, further, that MN produced does not pass through any vertex of the square. Find the smallest value of k for which, given any position of M and N, at least one of the twenty triangles with vertices chosen from the set $\{A, B, C, D, M, N\}$ has area not exceeding k.

Solution 1. Wolog, suppose that M lies in the interior of triangle ABN. Then

$$[ABM] + [AMN] + [BMN] + [CND] = [ABN] + [CND] = \frac{1}{2} \ ,$$

so that at least one of the four triangles on the left has area not exceeding $1/8$. Hence $k \leq 1/8$. We give a configuration for which each of the twenty triangles has area not less than $1/8$, so that $k = 1/8$.

Suppose that M and N are both located on the line joining the midpoints of AD and BC with M distant $1/4$ from the side AD and N distant $1/4$ from the side BC. Then

$$\frac{1}{4} = [ABM] = [CDM] = [ABN] = [CDN]$$

$$\frac{3}{8} = [BCM] = [DAN]$$

$$\frac{1}{2} = [ABC] = [BCD] = [CDA] = [DAB]$$

$$\frac{1}{8} = [DAM] = [BCN] = [AMN] = [BMN] = [CMN] = [DMN] = [AMC] = [ANC] = [BMD] = [BND] \ .$$

Solution 2. [L. Fei] Suppose that all triangle have area exceed k. Then M and N must be in the interior of the square distant more than $2k$ from each edge to ensure that areas of triangles like ABM exceed k. Similarly, M and N must be distant more than $k\sqrt{2}$ from the diagonals of the square. For points M and N to be available that satisfy both conditions, we need to find a point that is distant at least $2k$ from the edges and $k\sqrt{2}$ from the diagonal; such a point would lie on a midline of the square. The condition is that $2k + \sqrt{2}(k\sqrt{2}) < \frac{1}{2}$ or $k < \frac{1}{8}$. On the other hand, we can give a configuration in which each area is at least equal to k and some areas are exactly $\frac{1}{8}$. This would have M and N on the same midline, each equidistant from an edge and the centre of the square.