OLYMON

Produced by the **Canadian Mathematical Society** and the **Department of Mathematics of the University of Toronto**.

Issue 8:2

March, 2007

Please send your solution to

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no later than April 30, 2007. It is important that your complete mailing address and your email address appear on the front page. If you do not write your family name last, please underline it.

486. Determine all quintuplets (a, b, c, d, u) of nonzero integers for which

$$\frac{a}{b} = \frac{c}{d} = \frac{ab+u}{cd+u} \ .$$

487. ABC is an isosceles triangle with $\angle A = 100^\circ$ and $AB = AC$. The bisector of angle B meets AC in D. Show that $BD + AD = BC$.

488. A host is expecting a number of children, which is either 7 or 11. She has 77 marbles as gifts, and distributes them into n bags in such a way that whether 7 or 11 children come, each will receive a number of bags so that all 77 marbles will be shared equally among the children. What is the minimum value of n?

489. Suppose n is a positive integer not less than 2 and that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq 0$,

$$\sum_{i=1}^n x_i \leq 400 \qquad \text{and} \qquad \sum_{i=1}^n x_i^2 \geq 10^4 \ .$$

Prove that $\sqrt{x_1} + \sqrt{x_2} \geq 10$. is it possible to have equality throughout? [*Bonus:* Formulate and prove a generalization.]

490. (a) Let a, b, c be real numbers. Prove that

$$\min\ [(a-b)^2, (b-c)^2, (c-a)^2] \leq \frac{1}{2}[a^2+b^2+c^2] \ .$$

(b) Does there exist a number k for which

$$\min\ [(a-b)^2, (a-c)^2, (a-d)^2, (b-c)^2, (b-d)^2, (c-d)^2] \leq k[a^2+b^2+c^2+d^2]$$

for any real numbers a, b, c, d? If so, determine the smallest such k.
[*Bonus:* Determine if there is a generalization.]

491. Given that x and y are positive real numbers for which $x+y=1$ and that m and n are positive integers exceeding 1, prove that
$$(1-x^m)^n + (1-y^n)^m > 1 \ .$$

492. The faces of a tetrahedron are formed by four congruent triangles. if α is the angle between a pair of opposite edges of the tetrahedron, show that
$$\cos\alpha = \frac{\sin(B-C)}{\sin(B+C)}$$
where B and C are the angles adjacent to one of these edges in a face of the tetrahedron.

Solutions to Problems 465-485.

465. For what positive real numbers a is
$$\sqrt[3]{2+\sqrt{a}} + \sqrt[3]{2-\sqrt{a}}$$
an integer?

Solution 1. Let $x = \sqrt[3]{2+\sqrt{a}}$, $y = \sqrt[3]{2-\sqrt{a}}$ and $z = x+y$. Then
$$z^3 = (x+y)^3 = x^3 + y^3 + 3(4-a)^{1/3}z = 4 + 3(4-a)^{1/3}z \ .$$
Hence $27(4-a)z^3 = (z^3-4)^3$, whence
$$a = 4 - \frac{(z^3-4)^3}{27z^3} = \frac{108z^3 - (z^3-4)^3}{27z^3} \ .$$
Since $a \geq 0$, z must be either (1) a positive integer for which $108z^3 \geq (z^3-4)^3$, or (2) a negative integer for which $108z^3 \leq (z^3-4)^3$.

Condition (1) forces $108 \geq (z^2-(4/z))^3 \geq (z^2-4)^3$, so that $z=1,2$. Condition (2) forces $108 \geq (z^2-(4/z))^3 \geq z^6$, which is satisfied by no negative integer value of z. Hence, we must have that $(z,a) = (1,5), (2,100/27)$. Since $z = x+y$ is equivalent to $z^3 = 4+3(4-a)^{1/3}z$, it is straightforward to check that both these answers are correct. Hence $a=5$ or $a = 100/27$.

Solution 2. [Yifan Wang] With x and y defined as in the first solution, note that $x > y$ and that $x^3+y^3 = (x+y)(x^2-xy+y^2)$. Since $x^2+y^2 > (x+y)^2/2$ and $-xy > -(x+y)^2/4$, we have that $4 > (x+y)^3/4$, whence $x+y \leq 2$. Since $x^3 > -y^3$, $x > -y$, so that $x+y > 0$. Hence $x+y=1$ or $x+y=2$.

When $x+y=1$, $x^2-xy+y^2=4$ and so $xy=-1$, and $x = \frac{1}{2}(1+\sqrt{5})$, $y = \frac{1}{2}(1-\sqrt{5})$. Therefore $4-a = x^3y^3 = -1$ so that $a=5$.

When $x+y=2$, then $x^2-xy+y^2=2$, so that $xy = 2/3$. Therefore $x = \frac{1}{3}(3+\sqrt{3})$, $y = \frac{1}{3}(3-\sqrt{3})$ and $4-a = 8/27$. Thus, $a = 100/27$. These solutions check out.

Solution 3. [A. Tavakoli] Denote the left side of the equation by $f(a)$. When $a \geq 4$,
$$0 \leq f(a) = (\sqrt{a}+2)^{1/3} - (\sqrt{a}-2)^{1/3} = \frac{4}{(\sqrt{a}+2)^{2/3} + (a-4)^{1/3} + (\sqrt{a}-2)^{2/3}} \leq 4^{1/3} < 3 \ .$$
Let $0 \leq a \leq 4$; again $f(a) > 0$. Observe that
$$\left(\frac{1}{2}(u+v)\right)^{\frac{1}{3}} \geq \frac{1}{2}u^{\frac{1}{3}} + \frac{1}{2}v^{\frac{1}{3}}$$

for all nonnegative values of u and v. (This can be seen by using the concavity of the function $t^{1/3}$, or from the power-mean inequality $(1/2)(s+t) \leq [(1/2)(s^3+t^3)]^{1/3}$.) Setting $u = \sqrt[3]{2+\sqrt{a}}$ and $v = \sqrt[3]{2-\sqrt{a}}$, we find that $3 > 2 \times 2^{1/3} \geq f(a) > 0$ with equality if and only if $a = 0$. Hence the only possible integer values of $f(a)$ are 0 and 1.

Let $x = \sqrt[3]{2-\sqrt{a}}$, so that $2 + \sqrt{a} = 4 - x^3$. Then

$$\begin{aligned} f(a) = 1 &\Longleftrightarrow x + (4-x^3)^{1/3} = 1 \\ &\Longleftrightarrow 4 - x^3 = 1 - 3x + 3x^2 - x^3 \\ &\Longleftrightarrow x^2 - x - 1 = 0 \Longleftrightarrow x = (1 \pm \sqrt{5})/2 \\ &\Longleftrightarrow x^3 = 2 \pm \sqrt{5}\ . \end{aligned}$$

The larger root of the quadratic leads to $x^3 > 2$ and so is extraneous. Hence $x^3 = 2 - \sqrt{5}$, and so $\sqrt{a} = \sqrt{5}$, $a = 5$.

$$\begin{aligned} f(a) = 2 &\Longleftrightarrow x + (4-x^3)^{1/3} = 2 \\ &\Longleftrightarrow 4 - x^3 = (2-x)^3 = 8 - 12x + 6x^2 - x^3 \\ &\Longleftrightarrow 3x^2 - 6x + 2 = 0 \Longleftrightarrow x = \frac{3 \pm \sqrt{3}}{3}\ . \ . \end{aligned}$$

Now,

$$\left(\frac{3 \pm \sqrt{3}}{3}\right)^3 = 2 \pm \frac{10\sqrt{3}}{9}\ .$$

The larger value of x leads to $x^3 > 2$, and so is inadmissible. The smaller value of x leads to $x^3 = 2-(10\sqrt{3}/9)$ and $\sqrt{a} = (10\sqrt{3}/9)$, $a = 100/27$. Both values of a check out.

466. For a positive integer m, let $\overline{m}$ denote the sum of the digits of m. Find all pairs of positive integers (m, n) with $m < n$ for which $(\overline{m})^2 = n$ and $(\overline{n})^2 = m$.

Solution. Let $m = m_k \cdots m_1 m_0$ where $0 \leq m_i \leq 9$ are the digits of m. Then

$$10^k \leq m < n = (m_k + \cdots + m_0)^2 \leq [(k+1)10]^2\ ,$$

whence $10^{k-2} \leq (k+1)^2$ and $0 \leq k \leq 3$.

Hence $m < n = (m_3 + m_2 + m_1 + m_0)^2 \leq (4 \times 9)^2 = 36^2$. Since m and n are both perfect squares, we need only consider $m = r^2$, where $1 \leq r \leq 36$.

In the case that $k = 3$, $\overline{m} < 1 + 9 + 9 + 9 = 28$. Since $28^2 < 1000 < m < n$, there are no examples. In the case that $k = 2$, $\overline{m} < 6 + 9 + 9 = 24$ and so $n^2 \leq 24^2$. The only possibility is $(m, n) = (169, 256)$. There are no possibilities when $k = 0$ or $k = 1$.

Hence, the only number pair is $(m, n) = (169, 256)$.

Comment. This is problem **621** from *The College Mathematics Journal*.

467. For which positive integers n does there exist a set of n distinct positive integers such that

(a) each member of the set divides the sum of all members of the set, *and*

(b) none of its proper subsets with two or more elements satisfies the condition in (a)?

Solution. When $n = 1$, condition (b) is satisfied vacuously, and any singleton will do. When $n = 2$, such a set cannot be found. If a and b are any two positive integers, then condition (b) entails that both a and b divide $a + b$, and so must divide each other. This cannot happen when a and b are distinct.

When $n \geq 3$, a set of the required type can be found. For example, let

$$S_n = \{1, 2, 2 \times 3, 2 \times 3^2, \cdots 2 \times 3^{n-3}, 3^{n-2} \ .$$

The sum of the elements in S_n is $2 \times 3^{n-2}$, which is divisible by each member of S_n.

Consider any proper subset R of S_n with at least three numbers. If 3^{n-2} belongs to R, then the sum of the elements of R must be strictly between 3^{n-2} and $2 \times 3^{n-2}$, and so not divisible by 3^{n-2}. If R does not contain 3^{n-2}, then its largest entry has the form 2×3^k with $1 \leq k \leq n-3$. Then the sum of R is greater than 2×3^k and does not exceed $1 + 2(1 + 3 + \cdots + 3^k) = 3^{k+1} < 2(2 \times 3^k)$. Hence this sum is not divisible by 2×3^k. As we have seen, no doubleton satisfies the condition. Hence (b) is satisfied for all subsets of S_n.

Comment. This is problem **1504** in the October, 1996 issue of *Mathematics Magazine*.

468. Let a and b be positive real numbers satisfying $a + b \geq (a - b)^2$. Prove that

$$x^a(1-x)^b + x^b(1-x)^a \leq \frac{1}{2^{a+b-1}}$$

for $0 \leq x \leq 1$, with equality if and only if $x = \frac{1}{2}$.

Comment. Denote the left side by $f(x)$. When $a = b$, $f(x) = 2x^a(1-x)^a$, which is maximized when $x = 1/2$, its maximum value being 2×4^{-a}. In the general case, the solution can be obtained by calculus. Since $f(0) = f(1) = 0$ and the function possesses a derivative everywhere, the maximum occurs when $f'(x) = 0$ and $0 < x < 1$. Wolog, assume that $a < b$. We have that

$$f'(x) = x^{a-1}(1-x)^{a-1}[(a-(a+b)x)(1-x)^{b-a} + (b-(a+b)x)x^{b-a}] \ .$$

This solution can be found in *Mathematics Magazine* 70:4 (October, 1997), 301-302 (Problem **1505**), and is fairly technical. It would be nice to have a more transparent argument. Is there a solution that avoids calculus, at least for rational a and b?

A second solution, employs the substitution $2x = 1 - y$ to get the equivalent inequality

$$(1-y)^a(1+y)^b + (1-y)^b(1+y)^a \leq 2$$

for $|y| \leq 1$. Wolog, we can let $a = b + c$ with $c \geq 0$. Then the condition becomes $2b \geq c^2 - c$. Then the inequality is equivalent to

$$(1-y^2)^b[(1-y)^c + (1+y)^c] \leq 2 \ ,$$

for $|y| \leq 1$.

Let $0 \leq c \leq 1$. Then, for $t > 0$, the function t^c is concave, so that, for $u, v > 0$,

$$\left(\frac{u+v}{2}\right)^c \geq \frac{u^c + v^c}{2} \ .$$

Setting $(u, v) = (1 - y, 1 + y)$, we find that $(1-y)^c + (1+y)^c \leq 2$ for $|y| \leq 1$. Hence the inequality holds, with equality occurring when $y = 0$ $(x = 1/2)$.

When $c > 1$, I do not have a clean solution. First, it suffices to consider the inequality when b is replaced by $\frac{1}{2}(c^2 - c)$. Thus, we need to establish that

$$(1-y^2)^{(1/2)(c^2-c)}[(1-y)^c + (1-y)^c] \leq 2 \qquad (*)$$

for $|y| \leq 1$. The derivative of the natural logarithm of the left side is a positive multiple of

$$g(y) = (1+y)^c(1-cy) - (1-y)^c(1+cy) \ .$$

If this can be shown to be nonpositive, then the result will follow. An equivalent inequality is

$$\left(1 - \frac{2y}{1+y}\right)^2 = \left(\frac{1-y}{1+y}\right)^c \geq \left(\frac{1-cy}{1+cy}\right) = \left(1 - \frac{2cy}{1+cy}\right) ,$$

for $c > 1$ and $|y| \leq 1$.

469. Solve for t in terms of a, b in the equation

$$\sqrt{\frac{t^3 + a^3}{t+a}} + \sqrt{\frac{t^3 + b^3}{t+b}} = \sqrt{\frac{a^3 - b^3}{a-b}}$$

where $0 < a < b$.

Solution 1. The equation is equivalent to

$$\sqrt{t^2 - at + a^2} + \sqrt{t^2 - bt + b^2} = \sqrt{a^2 + ab + b^2} .$$

Square both sides of the equation, collect the nonradical terms on one side and the radical on the other and square again. Once the polynomials are expanded and like terms collected, we obtain the equation

$$0 = t^2(a+b)^2 - 2ab(a+b)t + a^2b^2 = [t(a+b) - ab]^2 ,$$

whence $t = ab/(a+b)$. This can be checked by substituting it into the equation.

Solution 2. [Y. Wang] As in solution 1, we can find an equivalent equation, which can then be manipulated to

$$\sqrt{(t-(a/2))^2 + (\sqrt{3}a/2)^2} + \sqrt{(t-(b/2))^2 + (-\sqrt{3}b/2)^2} = \sqrt{(a/2 - b/2)^2 + (\sqrt{3}a/2 + \sqrt{3}b/2)} .$$

If we consider the points $A \sim (a/2, \sqrt{3}a/2)$, $B \sim (b/2, -\sqrt{3}b/2)$ and $T \sim (t, 0)$, then we can interpret this equation as stating that $AT + BT = AB$. By the triangle inequality, we see that T must lie on AB, so that the slopes of AT and BT are equal. Thus

$$\frac{\sqrt{3}a}{a - 2t} = \frac{\sqrt{3}b}{2t - b} ,$$

whence $t = ab/(a+b)$.

470. Let ABC, ACP and BCQ be nonoverlapping triangles in the plane with angles CAP and CBQ right. Let M be the foot of the perpendicular from C to AB. Prove that lines AQ, BP and CM are concurrent if and only if $\angle BCQ = \angle ACP$.

Solution 1. [A. Tavakoli] Let BP and AQ intersect at K. Let $\angle BCQ = \alpha$, $\angle ACP = \beta$ and $\angle BCA = \gamma$. By the trigonometric form of Ceva's theorem, CM, AP and BQ are concurrent if and only if

$$\frac{\sin \angle BCM}{\sin \angle ACM} \cdot \frac{\sin \angle KAC}{\sin \angle KAB} \cdot \frac{\sin \angle KBA}{\sin \angle KBC} = 1 . \qquad (1)$$

This holds whether K lies inside or outside of the triangle.

We have that $\sin \angle BCM = \cos \angle CBA$, $\sin \angle ACM = \cos \angle CAB$, and, by the Law of Sines applied to triangles ACQ and ABQ,

$$\sin \angle KAC = \sin \angle QAC = (\sin \angle ACQ)(|QC|)/(|AQ|) ,$$

and

$$\sin \angle KAB = \sin \angle QAB = (\sin \angle ABQ)(|QB|)/(|AQ|) \ .$$

Therefore

$$\frac{\sin \angle KAC}{\sin \angle KAB} = \left(\frac{\sin \angle ACQ}{\sin \angle ABQ}\right) \cdot \left(\frac{|QC|}{|QB|}\right) = \left(\frac{\sin(\gamma+\alpha)}{\sin(\angle ABC + 90^\circ)}\right) \cdot \left(\frac{1}{\sin \alpha}\right) = \frac{-\sin(\gamma+\alpha)}{(\cos \angle CBA)\sin \alpha} \ .$$

Similarly,

$$\sin \angle KBA = \sin \angle BAP(|AP|/|BP|)$$

$$\sin \angle KBC = \sin \angle BCP(|PC|/|BP|)$$

and so

$$\frac{\sin \angle KBA}{\sin \angle KBC} = \frac{\sin(\angle BAC + 90^\circ)}{\sin(\beta+\gamma)} \cdot \frac{|AP|}{|PC|} = \frac{-\cos(\angle BAC)\sin \beta}{\sin(\beta+\gamma)} \ .$$

Hence the condition for concurrency becomes

$$\frac{\sin(\gamma+\alpha)}{\sin \alpha} \cdot \frac{\sin \beta}{\sin(\gamma+\beta)} = 1$$

$$\Longleftrightarrow \sin \gamma \cot \alpha + \cos \gamma = \sin \gamma \cot \beta + \cos \gamma$$

$$\Longleftrightarrow \cot \alpha = \cot \beta \Longleftrightarrow \angle BCQ = \alpha = \beta = \angle ACP \ .$$

This is the required result.

Solution 2. We do some preliminary work. Suppose that PB and AQ intersect at O, and that X and Y are the respective feet of the perpendiculars from C to PB and AQ. Since $\angle CXP = \angle CAP = 90^\circ$, $CAXP$ is concyclic and so $\angle ACP = \angle AXP$. Similarly $CQBY$ is concyclic and so $\angle BCQ = \angle BYQ$. Since $\angle CXO = \angle CYO = 90^\circ$, X and Y lie on the circle with diameter CO. Hence $\angle YCO = \angle YXO = \angle YXB$.

Now suppose that $\angle BCQ = \angle ACP$. Let CO produced meet AB at N. Since $\angle AXP = \angle ACP = \angle BCQ = \angle BYQ$, it follows that $\angle AXB = \angle AYB$ so that $BYXA$ is concyclic and so $\angle YXB = \angle YAB$. Therefore

$$\angle YCN = \angle YCO = \angle YXB = \angle YAB = \angle YAN$$

and $ANYC$ is concyclic/ Hence $\angle CNA = \angle CYA = 90^\circ$ and N must coincide with M.

On the other hand, let CM pass through O. Since $\angle CYA = \angle CMA = 90^\circ$, $AMYC$ is concyclic so that

$$\angle YAB = \angle YAM = \angle YCM = \angle YCO = \angle YXB \ .$$

Therefore $BAXY$ is concyclic and $\angle BXA = \angle BYA \Rightarrow \angle AXP = \angle BYQ$. Since $CAXP$ and $CYBQ$ are concyclic, $\angle ACP = \angle AXP = \angle BYQ = \angle BCQ$.

471. Let I and O denote the incentre and the circumcentre, respectively, of triangle ABC. Assume that triangle ABC is not equilateral. Prove that $\angle AIO \leq 90^\circ$ if and only if $2BC \leq AB + CA$, with equality holding only simultaneously.

Solution 1. Wolog, let $AB \geq AC$. Suppose that the circumcircle of triangle ABC intersects AI in D. Construct the circle Γ with centre D that passes through B and C. By the symmetry of AB and AC in the angle bisector AD, this circle intersects segment AB in a point F such that $AF = AC$. Let Γ intersect AD at P. Then chords CP and FP have the same length. If $AB > AC$, this implies that P is on the angle bisector of angle ABC. If $AB = AC$, then $\angle ABC = \angle ADC = \angle PDC = 2\angle PBC$. In either case, $P = I$.

Let E be on the ray BA produced such that $AE = AC$. Since $\angle DAC = \frac{1}{2}\angle BAC = \angle AEC$ and $\angle ADC = \angle ABC = \angle EBC$, triangles ADC and EBC are similar, and so

$$ID : AD = CD : AD = BC : BE = BC : (AB + AC) \ .$$

But $\angle AIO \leq 90°$ if and only if $ID/AD \leq 1/2$, and so is equivalent to $2BC \leq AB + AC$, with equality holding only simultaneously. (Solution due to Wu Wei Chao in China.)

Solution 2. We have that $\angle AIO \leq 90°$ if and only if $\cos \angle AIO \geq 0$, if and only if $|AO|^2 \leq |OI|^2 + |IA|^2$. Let a, b, c be the respective sidelengths of BC, CA, AB; let R be the circumradius and let r be the inradius of triangle ABC. Since, by Euler's formula, $|OI|^2 = R^2 - 2Rr$, and $r = |IA|\sin(A/2)$, the foregoing inequality is equivalent to

$$2R \leq \frac{r}{\sin^2(A/2)} = \frac{2r}{1-\cos A} \ .$$

Applying $R = a/(2\sin A)$, $r = bc\sin A/(a+b+c)$ and $2bc\cos A = b^2 + c^2 - a^2$, we find that

$$\begin{aligned}
r - R(1-\cos A) &= \frac{bc\sin A}{a+b+c} - \frac{a(1-\cos A)}{2\sin A} \\
&= \sin A\left[\frac{bc}{a+b+c} - \frac{a(1-\cos A)}{2\sin^2 A}\right] \\
&\frac{\sin A}{2(1+\cos A)(a+b+c)}[2bc + 2bc\cos A - a(a+b+c)] \\
&\frac{\sin A}{2(1+\cos A)(a+b+c)}[2bc + b^2 + c^2 - a^2 - a(a+b+c)] \\
&\frac{\sin A}{2(1+\cos A)(a+b+c)}[(b+c)^2 - 2a^2 - a(b+c)] \\
&\frac{\sin A}{2(1+\cos A)(a+b+c)}[(b+c+a)(b+c-2a)] \ .
\end{aligned}$$

Hence the inequality $R(1-\cos A) \leq r$ is equivalent to $2a \leq b + c$. The desired result follows. (Solution due to Can A. Minh, USA)

Solution 3. [Y. Wang] Let AI intersect the circumcircle of triangle ABC at D. Since AI bisects the angle BAC and the arc BC, we have that $BD = BC$. Also,

$$\angle DIC = \angle CAD + \angle ACI = \angle BCD + \angle BCI = \angle DCI \ ,$$

whence $DC = DI = DB$. Using Ptolemy's Theorem, we have that

$$AB \times CD + BD \times AC = AD \times BC \ ,$$

so that

$$AB \times DI + DI \times AC = (AI + ID) \times BC \ .$$

Hence

$$k \equiv \frac{AB + AC}{BC} = 1 + \frac{AI}{ID} \ .$$

If $AB = AC$, then A, O, I are collinear. Let $k < 2$; then $AI < ID$ and I lies between A and O and $\angle AIO = 180°$. Let $k > 2$; then $AI > ID$, O lies between A and I and $\angle AIO = 0°$. [If $k = 2$, then $AI = ID$, the incentre and circumcentre coincide and the triangle is equilateral – the excluded case.]

Wolog, suppose that $AB > AC$. Then the circumcentre O lies within the triangle ABD. Let P be the foot of the perpendicular from O to AD. Then P is the midpoint of AD and the angle AIO is greater than, equal to or less than $90°$ according as I is in the segment AP, coincides with P or is in the segment PD. These correspond to $k < 2$, $k = 2$ and $k > 2$, and the result follows.

472. Find all integers x for which

$$(4-x)^{4-x} + (5-x)^{5-x} + 10 = 4^x + 5^x \ .$$

Solution. If $x < 0$, then the left side is an integer, but the right side is positive and less than $\frac{1}{4} + \frac{1}{5} < 1$. If $x > 5$, then the left side is less than $\frac{1}{4}$, while the right side is a positive integer. Therefore, the only candidates for solution are the integers between 0 and 5 inclusive. Checking, we find that the only solution is $x = 2$.

473. Let $ABCD$ be a quadrilateral; let M and N be the respective midpoint of AB and BC; let P be the point of interesection of AN and BD, and Q be the point of intersection of DM amd AC. Suppose the $3BP = BD$ and $3AQ = AC$. Prove that $ABCD$ is a parallelogram.

Solution. Let $\overrightarrow{AB} = \mathbf{x}$, $\overrightarrow{BC} = \mathbf{y}$ and $\overrightarrow{CD} = a\mathbf{x} + b\mathbf{y}$, where a and b are real numbers. Then

$$\overrightarrow{AD} = (a+1)\mathbf{x} + (b+1)\mathbf{y}$$

and

$$\overrightarrow{AN} = \mathbf{x} + \frac{1}{2}\mathbf{y} \ .$$

But $\overrightarrow{BD} = 3\overrightarrow{BP}$, so that

$$\overrightarrow{AP} = \frac{2\overrightarrow{AB} + \overrightarrow{AD}}{3} = \frac{a+3}{3}\mathbf{x} + \frac{b+1}{3}\mathbf{y} \ .$$

Since the vectors $\overrightarrow{AP}$ and $\overrightarrow{AN}$ are collinear, $a + 3 : 1 = b + 1 : \frac{1}{2}$, whence $a - 2b + 1 = 0$. Also

$$\overrightarrow{DM} = \overrightarrow{AM} - \overrightarrow{AD} = \left(\frac{1}{2} - a - 1\right)\mathbf{x} - (b+1)\mathbf{y} = -\left(a + \frac{1}{2}\right)\mathbf{x} - (b+1)\mathbf{y}$$

and

$$\overrightarrow{DQ} = \overrightarrow{AQ} - \overrightarrow{AD} = \frac{1}{3}(\mathbf{x} + \mathbf{y}) - (a+1)\mathbf{x} - (b+1)\mathbf{y} = -\frac{1}{3}[(3a+2)\mathbf{x} + (3b+2)\mathbf{y}] \ .$$

Since the vectors $\overrightarrow{DQ}$ and $\overrightarrow{DM}$ are collinear, we must have $(3a+2) : (a + \frac{1}{2}) = (3b+2) : (b+1)$, whence $2a + b + 2 = 0$. Therefore $(a, b) = (-1, 0)$, $\overrightarrow{CD} = -\mathbf{x} = \overrightarrow{BA}$ and $\overrightarrow{AD} = \mathbf{y} = \overrightarrow{BC}$. Hence $ABCD$ is a parallelogram.

474. Solve the equation for positive real x:

$$(2^{\log_5 x} + 3)^{\log_5 2} = x - 3 \ .$$

Solution. Recall the identity $u^{\log_b v} = v^{\log_b u}$ for positive u, v and positive base $b \neq 1$. (Take logarithms to base b.) Then, for all real t, $(2^t + 3)^{\log_5 2} = 2^{\log_5 (2^t + 3)}$. This is true in particular when $t = \log_5 x$.

Let $f(x) = 2^{\log_5 x} + 3$ for $x > 0$. Then $f(x) = x^{\log_5 2} + 3$ and the equation to be solved is $f(f(x)) = x$. The function $f(x)$ is an increasing function of the positive variable x. If $f(x) < x$, then $f(f(x)) < f(x)$; if $f(x) > x$, then $f(f(x)) > f(x)$. Hence, for $f(f(x)) = x$ to be true, we must have $f(x) = x$. With $t = \log_5 x$, the equation becomes $2^t + 3 = 5^t$, or equivalently, $(2/5)^t + 3(1/5)^t = 1$. The left side is a stricly decreasing function of t, and so equals the right side only when $t = 1$. Hence the unique solution of the equation is $x = 5$.

475. Let z_1, z_2, z_3, z_4 be distinct complex numbers for which $|z_1| = |z_2| = |z_3| = |z_4|$. Suppose that there is a real number $t \neq 1$ for which

$$|tz_1 + z_2 + z_3 + z_4| = |z_1 + tz_2 + z_3 + z_4| = |z_1 + z_2 + tz_3 + z_4| \ .$$

Show that, in the complex plane, z_1, z_2, z_3, z_4 lie at the vertices of a rectangle.

Solution. Let $s = z_1 + z_2 + z_3 + z_4$. Then

$$|s - (1-t)z_1| = |s - (1-t)z_2| = |s - (1-t)z_3| \ .$$

Therefore, s is equidistant from the three distinct points $(1-t)z_1$, $(1-t)z_2$ and $(1-t)z_3$; but these three points are on the circle with centre 0 and radius $(1-t)z_1$. Therefore $s = 0$.

Since $z_1 - (-z_2) = z_1 + z_2 = -z_3 - z_4 = (-z_4) - z_3$ and $z_2 - (-z_3) = z_2 + z_3 = -z_4 - z_1 = (-z_4) - z_1$, z_1, $-z_2$, z_3 and $-z_4$ are the vertices of a parallelogram inscribed in a circle centered at 0, and hence of a rectangle whose diagonals intersect at 0. Therefore, $-z_2$ is the opposite of one of z_1, z_3 and $-z_4$. Since z_2 is unequal to z_1 and z_3, we must have that $-z_2 = z_4$. Also $z_1 = -z_3$. Hence z_1, z_2, z_3 and z_4 are the vertices of a rectangle.

476. Let p be a positive real number and let $|x_0| \leq 2p$. For $n \geq 1$, define

$$x_n = 3x_{n-1} - \frac{1}{p^2}x_{n-1}^3 \ .$$

Determine x_n as a function of n and x_0.

Solution. Let $x_n = 2py_n$ for each nonnegative integer n. Then $|y_0| \leq 1$ and $y_n = 3y_{n-1} - 4y_{n-1}^3$. Recall that

$$\sin 3\theta = \sin 2\theta \cos\theta + \sin\theta\cos 2\theta = 2\sin\theta(1-\sin^2\theta) + \sin\theta(1-2\sin^2\theta) = 3\sin\theta - 4\sin^3\theta \ .$$

Select $\theta \in [-\pi/2, \pi/2]$. Then, by induction, we determine that $y_n = \sin 3^n\theta$ and $x_n = 2p\sin 3^n\theta$, for each nonnegative integer n, where $\theta = \arcsin(x_0/2p)$.

477. Let S consist of all real numbers of the form $a + b\sqrt{2}$, where a and b are integers. Find all functions that map S into the set $\mathbf{R}$ of reals such that (1) f is increasing, and (2) $f(x+y) = f(x) + f(y)$ for all x, y in S.

Solution. Since $f(0) = f(0) + f(0)$, $f(0) = 0$ and $f(x) \geq 0$ for $x \geq 0$. Let $f(1) = u$ and $f(\sqrt{2}) = v$; u and v are both nonnegative. Since $f(0) = f(x) + f(-x)$, $f(-x) = -f(x)$ for all x. Since, by induction, it can be shown that $f(nx) = nf(x)$ for every positive integer n, it follows that

$$f(a + b\sqrt{2}) = au + bv \ ,$$

for every pair (a, b) of integers.

Since f is increasing, for every positive integer n, we have that

$$f(\lfloor n\sqrt{2}\rfloor) \leq f(n\sqrt{2}) \leq f(\lfloor n\sqrt{2}\rfloor + 1) \ ,$$

so that

$$\lfloor n\sqrt{2}\rfloor u \leq nv \leq (\lfloor n\sqrt{2}\rfloor + 1)u \ .$$

Therefore,

$$\left(\sqrt{2} - \frac{1}{n}\right)u \leq \left(\frac{\lfloor n\sqrt{2}\rfloor}{n}\right)u \leq v \leq \frac{1}{n}(\lfloor n\sqrt{2}\rfloor + 1)u \leq \left(\sqrt{2} + \frac{1}{n}\right)u \ ,$$

for every positive integer n. It follows that $v = u\sqrt{2}$, so that $f(x) = ux$ for every $x \in S$. It is readily checked that this equation satisfies the conditions for all nonnegative u.

478. Solve the equation

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + x}}} + \sqrt{3}\sqrt{2 - \sqrt{2 + \sqrt{2 + x}}} = 2x$$

for $x \geq 0$

Solution. Since $2 - \sqrt{2 + \sqrt{2 + x}} \geq 0$, we must have $0 \leq x \leq 2$. Therefore, there exists a number $t \in [0, \frac{1}{2}\pi]$ for which $\cos t = \frac{1}{2}x$. Now we have that,

$$\begin{aligned}\sqrt{2+\sqrt{2+\sqrt{2+x}}} &= \sqrt{2+\sqrt{2+\sqrt{2+2\cos t}}} \\ &= \sqrt{2+\sqrt{2+\sqrt{4\cos^2(t/2)}}} = \sqrt{2+\sqrt{2+2\cos(t/2)}} \\ &= \sqrt{2+2\cos(t/4)} = 2\cos(t/8)\ .\end{aligned}$$

Similarly, $\sqrt{2-\sqrt{2+\sqrt{2+x}}} = 2\sin(t/8)$. Hence the equation becomes

$$2\cos\frac{t}{8} + 2\sqrt{3}\sin\frac{t}{8} = 4\cos t$$

or

$$\frac{1}{2}\cos\frac{t}{8} + \frac{\sqrt{3}}{2}\sin\frac{t}{8} = \cot t\ .$$

Thus,

$$\cos\left(\frac{\pi}{3} - \frac{t}{8}\right) = \cos t\ .$$

Since the argument of the cosine on the left side lies between 0 and $\pi/3$, we must have that $(\pi/3) - (t/8) = t$, or $t = 8\pi/27$.

479. Let x, y, z be positive integer for which

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

and the greatest common divisor of x and z is 1. Prove that $x + y$, $x - z$ and $y - z$ are all perfect squares. Give two examples of triples (x, y, z) that satisfy these conditions.

Solution 1. [G. Ghosn] Since $(1/y) = (x - z)/(xz)$ and gcd $(x, x - z) =$ gcd $(z, x - z) = 1$, the fractions on both sides of the equation are in lowest terms, and so $x - z = 1$ and $xz = y$. Hence $x + y = x(1 + z) = x^2$ and $y - z = z(x - 1) = z^2$.

Solution 2. Since $z(x + y) = xy$ and the greatest common divisor of x and z is 1, x, being a divisor of $z(x + y)$ must be a divisor of $x + y$ and so of y. Let $y = ux$ for some positive integer u. Then $z(1 + u) = ux$. Since u and $1 + u$ have greatest common divisor 1, u must divide z and $1 + u$ must divide x, Hence $z = uv$ and $x = (1 + u)w$, for some positive integers v and w. Therefore $uv(1 + u) = u(1 + u)w$, whence $v = w$.

Therefore $(x, y, z) = ((1 + u)v, u(1 + u)v, uv)$. Since x and z have greatest common divisor 1, $v = 1$ and $(x, y, z) = (1 + u, u(1 + u), u)$. This satisfies the given equation as well as $x + y = (1 + u)^2 = x^2$, $x - z = 1$ and $y - z = u^2 = z^2$. Particular examples are $(x, y, z) = (2, 2, 1), (3, 6, 2), (4, 12, 3), (5, 20, 4)$.

Solution 3. We have that $z(x + y) = xy$ and $x(y - z) = yz$. Since gcd $(x, z) = 1$, z and x both must divide y, so that $y = vz = wx$ for some positive integers v and w. Since $z(1 + w)x = xvz$, $1 + w = v$ and gcd $(v, w) = 1$. Since $wx = vz$,,we must have that $x = v$ and $z = w$ and $y = vw$. This satisfies the equation as well as $x + y = v^2$, $x - z = 1$ and $y - z = w^2$.

Solution 4. [K. Huynh] Observe that $x > y$ and $z > y$. From the equation, we obtain that $xz + yz = xy$ whence $(x - z)(y - z) = z^2$. Since gcd $(x, z) = 1$, there is no prime that divides $x - z$ and z^2, so that gcd $(x - z, z^2) = 1$. Therefore $x - z = 1$, $y - z = z^2$, $y = z^2 + z$ and $x + y = (z + 1)^2$.

480. Let a and b be positive real numbers for which $60^a = 3$ and $60^b = 5$. Without the use of a calculator or of logarithms, determine the value of

$$12^{\frac{1-a-b}{2(1-b)}} .$$

Solution 1. [V. Zhou]

$$\begin{aligned} 12^{\frac{1-a-b}{2(1-b)}} &= \left(\frac{60}{5}\right)^{\frac{1-a-b}{2(1-b)}} = 60^{(1-b)\cdot(\frac{1-a-b}{2(1-b)})} \\ &= \left(\frac{60}{60^{a+b}}\right)^{\frac{1}{2}} = \left(\frac{60}{60^a \cdot 60^b}\right)^{\frac{1}{2}} \\ &= \left(\frac{60}{3\times 5}\right)^{\frac{1}{2}} = 2 . \end{aligned}$$

Solution 2. Since $60^b = 5$, $12^b = 5^{1-b}$ and $5 = 12^{b/(1-b)}$. Since $60^a = 3$, $2^2 5^a 12^a = 12$. Therefore

$$2^2 = 12^{1-a}5^{-a} = 12^{1-a}12^{-ab/(1-b)} = 12^{(1-a-b+ab-ab)/(1-b)} = 12^{(1-a-b)/(1-b)} .$$

Therefore $2 = 12^{(1-a-b)/2(1-b)}$.

Solution 3. [A. Guo; D. Shi] Since $a = \log_{60} 3$ and $b = \log_{60} 5$,

$$1 - (a+b) = 1 - \log_{60}(15) = \log_{60}(60/15) = \log_{60} 4 .$$

Also, $1 - b = 1 - \log_{60} 5 = \log_{60} 12$, so that

$$\frac{1-a-b}{1-b} = \frac{\log_{60} 4}{\log_{60} 12} = \log_{12} 4 = 2\log_{12} 2 .$$

Therefore

$$12^{\frac{1-a-b}{2(1-b)}} = 12^{\log_{12} 2} = 2 .$$

481. In a certain town of population $2n+1$, one knows those to whom one is known. For any set A of n citizens, there is some person among the other $n+1$ who knows everyone in A. Show that some citizen of the town knows all the others.

Solution 1. [K. Huynh] We prove that there is a set of $n+1$ people in the town, each of whom knows (and is known by) each of the rest. First, observe that for any set of k people, with $k \leq n$, there is a person not among them who knows them all. This follows by augmenting the set to n people and applying the condition of the problem.

Let p_1 be any person. There is a person, say p_2 who knows p_1. A person p_3 can be found who knows both p_1 and p_2, so that $\{p_1, p_2, p_3\}$ is a triplet each of whom knows the other two. Suppose, as an induction hypothesis, that $3 \leq k \leq n$, and $\{p_1, p_2, \cdots, p_k\}$ is a set of k people any pair of whom know each other. By the foregoing observation, there is another person p_{k+1} who knows them all. By induction, we can find a set $\{p_1, p_2, \cdots, p_{n+1}\}$, each pair of whom know each other.

Consider the remaining n people. There must be one among the p_i who knows all of these remaining people. This person p_i therefore knows everyone.

Solution 2. Let us suppose that the persons are numbered from 0 to $2n$ inclusive. The notation $(a : a_1, a_2, \cdots, a_k)$ will mean that a is knows and is known by each of $a_1, a_2, \cdots, a_k$. Begin with the set $\{1, 2, \cdots, n\}$; some person, say 0, knows everyone in this set, so that

$$(0 : 1, 2, 3, \cdots, n) .$$

If person 0, knows everyone else, then we are done. Otherwise, there is a person, say, $n+1$, not known to 0, so that everyone in the set $\{n+1, n+2, \cdots, 2n\}$, is known by a person in the first set, say 1, so that

$$(1 : 0, n+1, n+2, \cdots, 2n) \ .$$

Consider the set $\{0, 2, 3, \cdots, n\}$. If 1 knows everyone in this set, then 1 knows everyone and we are done. If 1 does not know everyone in this set, then there is someone else, say $n+1$, who does, so that

$$(n+1 : 0, 1, \cdots, n) \quad \text{and} \quad (0 : 1, 2, \cdots, n+1) \ .$$

If 0 knows everyone in the set $\{1, n+2, \cdots, 2n\}$, then 0 knows everyone; if $n+1$ knows everyone in this set, then $n+1$ knows everyone, and we are done. If not, then there is a person 2, say, who knows everyone in the set:

$$(2 : 0, 1, n+1, n+2, \cdots, 2n) \ .$$

Consider the set $\{0, 3, \cdots, n, n+1\}$. If 1 or 2 knows everyone in this set, then 1 or 2 knows everybody and we are done. Otherwise, there is a person, say $n+2$ who knows everyone in the set, so that

$$(n+2 : 0, 1, 2, \cdots, n+1) \quad \text{and} \quad (0 : 1, 2, \cdots, n+1, n+2) \ .$$

We can continue on in this way either until we find someone that knows everyone, or until we reach the ith stage for which

$$(i : 0, 1, 2, \cdots, i-1, n+1, \cdots, 2n) \quad \text{and} \quad (n+i : 0, 1, 2, \cdots, n, n+1, \cdots, n+i-1) \ .$$

If we get to the nth stage, then n and $2n$ each know everyone.

482. A trapezoid whose parallel sides have the lengths a and b is partitioned into two trapezoids of equal area by a line segment of length c parallel to these sides. Determine c as a function of a and b.

Solution. Let u be the distance between the segment of length a and that of length c, and v the distance between the segment of length c and that of length b. Then

$$\frac{u+v}{u} = \frac{b-a}{c-a} \ .$$

From the area condition, we have that

$$2\left(\frac{c+a}{2}\right)u = \left(\frac{b+a}{2}\right)(u+v) = \left(\frac{b^2-a^2}{2(c-a)}\right)u \ ,$$

whence $2(c^2 - a^2) = b^2 - a^2$ and $c^2 = \frac{1}{2}(a^2 + b^2)$. Therefore

$$c = \sqrt{\frac{a^2+b^2}{2}} \ .$$

483. Let A and B be two points on the circumference of a circle, and E be the midpoint of arc AB (either arc will do). Let P be any point on the minor arc EB and N the foot of the perpendicular from E to AP. Prove that $AN = NP + PB$.

Solution 1. Produce ANP to M so that $AN = NM$. Then $EM = AE = EB$. Hence $\angle EBM = \angle EMB$, so that

$$\angle PBM = \angle EBM - \angle EBP = \angle EMB - \angle EAP = \angle EMB - \angle EMA = \angle PMB \ .$$

Therefore $PB = PM$, so that

$$AN = NM = NP + PM = NP + PB \ .$$

Solution 2. [V. Zhou] Determine Q on AN so that $AQ = BP$. Then, also, $\angle EAQ = \angle EAP = \angle EPB$ and $AE = EB$, so that triangles AEQ and BEP are congruent. Hence $EQ = EP$ and so $QN = NP$. Therefore $AN = QN + AQ = NP + PB$.

Solution 3. [Y. Wang] Let O be the centre and r the radius of the circle. Let F and G be the respective midpoints of AP and AB. Then $FG \| BP$ and, since $\angle AFO = \angle AGO = 90°$, the quadrilateral $AFGO$ is concyclic.

Let $\alpha = \angle AOF = \angle AGF$ and $\beta = \angle AOE = \angle BOE$. Then

$$\angle PAB = \angle FAG = \angle FOG = \angle FOE = \angle NEO = \beta - \alpha \ .$$

Also, $|FN| = |OE| \sin(\beta - \alpha) = r \sin(\beta - \alpha)$ and $|AF| = r \sin \alpha$. By the Law of Sines applied to triangle AFG,

$$\frac{|FG|}{\sin(\beta - \alpha)} = \frac{|AF|}{\sin \alpha} = r,$$

whence $|FG| = r \sin(\beta - \alpha) = |FN|$. Hence $AN = PF + FN = PN + 2FN = PN + 2FG = NP + PB$.

484. ABC is a triangle with $\angle A = 40°$ and $\angle B = 60°$. Let D and E be respective points of AB and AC for which $\angle DCB = 70°$ and $\angle EBC = 40°$. Furthermore, let F be the point of intersection of DC and EB. Prove that $AF \perp BC$.

Solution 1. [J. Schneider] Let AH be the altitude from A to BC. We apply the converse of Ceva's Theorem in the trigonometric form to show that the cevians AH, BE and CD concur.

$$\frac{\sin 30° \sin 40° \sin 10°}{\sin 10° \sin 20° \sin 70°} = \frac{\sin 30°(2 \sin 20° \cos 20°)}{\sin 20° \cos 20°} = 2 \sin 30° = 1 \ .$$

Hence AH, BE and CD concur, so that AH passes through F and the result follows.

Solution 2. [A. Siddhour] In triangle BCF, since $\angle CBF = 40°$ and $\angle CBF = 40°$, it follows that $\angle BFC = 70° = \angle CBF$ and $BF = BC$. Hence $|BF| = a$ (using the standard convention for lengths of the sides of the triangle ABC). Assign coordinates:

$$B \sim (0,0), \quad C \sim (a,0), \quad A \sim (c \cos 60°, c \sin 60°), \quad F \sim (a \cos 40°, a \sin 40° \ .$$

By the Law of sines, we have that $c \sin 40° = a \sin 80°$, whence $c = 2a \cos 40°$.

We have that

$$\begin{aligned} \overrightarrow{FA} \cdot \overrightarrow{BC} &= (c \cos 60° - a \cos 40°, c \sin 60° - a \sin 60°) \cdot (a, 0) \\ &= a(2a \cos 40° \cos 60° - a \cos 40° = a \cos 40° - a \cos 40° = 0 \ , \end{aligned}$$

from which it follows that $AF \perp BC$.

Solution 3. [Y. Wang] The result will follow if one can show that $\angle FAC = 10°$. Since $\angle FCA = \angle BCA - \angle DCB = 80° - 70° = 10°$, it is enough to show that the perpendicular from F to AC bisects AC, *i.e.*, $2|CF| \cos \angle FCA = |AC|$.

Since $\angle FBC = 40°$ and $\angle BCF = 70°$, it follows that $\angle BFC = 70°$ so that $|CF| = 2|BC| \cos 70°$. Since $BC : AC = \sin \angle BAC : \sin \angle ABC = \sin 40° : \sin 60°$,

$$2|CF| \cos \angle FCA = 4|BC| \cos 70° \cos 10° = 4|AC| \sin 40° \sin 20° \sin 80° / \sin 60° \ .$$

For each angle θ,

$$\begin{aligned} 4\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta) &= 2\sin\theta[\cos 2\theta - \cos 120^\circ] \\ &= 2\sin\theta\cos 2\theta + 2\sin\theta\sin 30^\circ \\ &= \sin 3\theta - \sin\theta + \sin\theta = \sin 3\theta \ . \end{aligned}$$

When $\theta = 20^\circ$, this becomes $4\sin 20^\circ\sin 40^\circ\sin 80^\circ = \sin 60^\circ$. so that $2|CF|\cos\angle FCA = |AC|$, as desired.

Solution 4. Since $\angle BFC = 70^\circ = \angle BCD$, $BF = BC$. Let $|BF| = |BC| = 1$, $|AF| = u$ and $|CF| = v$. Let $\angle BAF = \theta$, so that $\angle CAF = 40^\circ - \theta$. By the Sine law applied to triangles BFC and AFC,

$$\frac{\sin 40^\circ}{\sin 70^\circ} = v = \frac{u\sin(40^\circ-\theta)}{\sin 10^\circ} \ .$$

By the Sine Law applied to triangle ABF, $u = \sin 20^\circ/\sin\theta$. Hence

$$\frac{\sin 40^\circ}{\sin 70^\circ} = \frac{\sin 20^\circ\sin(40^\circ-\theta)}{\sin 10^\circ\sin\theta} \ ,$$

so that

$$\sin 10^\circ\sin 40^\circ\sin\theta = \sin 20^\circ\cos 20^\circ\sin(40^\circ-\theta) \ ,$$

whence

$$2\sin 10^\circ\sin\theta = \sin(40^\circ-\theta) = \sin 40^\circ\cos\theta - \cos 40^\circ\sin\theta$$

and

$$\sin\theta(2\sin 10^\circ + \cos 40^\circ) = \cos\theta\sin 40^\circ \ .$$

Now

$$\begin{aligned} 2\sin 10^\circ + \cos 40^\circ &= \sin 10^\circ + (\sin 10^\circ + \sin 50^\circ) \\ &= \sin 10^\circ + 2\sin 30^\circ\cos 20^\circ = \sin 10^\circ + \sin 70^\circ \\ &= 2\sin 40^\circ\cos 30^\circ = \sqrt{3}\sin 40^\circ \ . \end{aligned}$$

Hence $\sqrt{3}\sin\theta = \cos\theta$, so that $\cot\theta = \sqrt{3}$. Hence $\theta = 30^\circ$ and the result follows.

Solution 5. [K. Huynh] Let a, b, c be the sides of triangle ABC according to convention. Since $\angle BFC = \angle FCB = 70^\circ$, $|BF| = |BC| = a$. Let the respective feet of the perpendiculars from A and F to BC be P and Q. Then $|BP| = c\cos 60^\circ = c/2$ and $|BQ| = a\cos 40^\circ$. From the Law of Sines, $a\sin 80^\circ = c\sin 40^\circ$, so that $c = 2a\cos 40^\circ$. Hence $BP = BQ$, and the result follows.

Solution 6. [G. Ghosn] Applying the Law of Sines to triangles BCE and BEA using their common side BE, we obtain that

$$\frac{|EC|}{|EA|} = \left(\frac{\sin 40^\circ}{\sin 80^\circ}\right)\left(\frac{\sin 40^\circ}{\sin 20^\circ}\right) = \frac{\sin^2 40^\circ}{\sin 20^\circ\sin 80^\circ} = \frac{2\cos 20^\circ\sin 40^\circ}{\sin 80^\circ} \ .$$

Similarly,

$$\frac{|DA|}{|DB|} = \frac{\sin 10^\circ\sin 60^\circ}{\sin 40^\circ\sin 70^\circ} \ .$$

By Ceva's therem

$$\begin{aligned} 1 &= \frac{|EC|}{|EA|}\frac{|DA|}{|DB|}\frac{|MB|}{|MC|} \\ &= \frac{2\cos 20^\circ\sin 40^\circ\sin 10^\circ\sin 60^\circ}{\sin 80^\circ\sin 40^\circ\sin 70^\circ}\frac{|MB|}{|MC|} \\ &= \frac{2\cos 80^\circ\sin 60^\circ}{\sin 80^\circ}\frac{|MB|}{|MC|} \ , \end{aligned}$$

whence we find that $|MB| : |MC| = \tan 80° : \tan 60°$.

Let AN be an altitude of triangle ABC, so that $|AN| = |NB| \tan 60° = |CN| \tan 80°$. Then $MB : MC = NB : NC$, so that $M = N$ and the desired result follows.

485. From the foot of each altitude of the triangle, perpendiculars are dropped to the other two sides. Prove that the six feet of these perpendiculars lie on a circle.

Solution 1. Let ABC be the triangle with altitudes AP, BQ and CR; let H be the orthocentre. Let $PU \perp AB$, $QV \perp BC$, $RW \perp CA$, $PX \perp CA$, $QY \perp AB$ and $RZ \perp BC$, where $U, Y \in AB$; $V, Z \in BC$; and $W, X \in CA$.

Consider triangles AQR and ABC. Since $ARHQ$ is concyclic (right angles at Q and R),

$$\angle ARQ = \angle AHQ = \angle BHP = 90° - \angle HBP = 90° - \angle QBC = \angle ACB \ .$$

Similarly, $\angle AQR = \angle ABC$. Thus, triangles AQR and ABC are similar, the similarity being implemented by a dilatation of centre A followed by a reflection about the bisector of angle BAC. Since QY and RW are altitudes of triangle AQR, triangle AYW is formed from triangle AQR as triangle AQR is formed from triangle ABC. Hence triangles AYW and AQR are similar by the combination of a dilatation with centre A and a reflection about the bisector of angle BAC.

Therefore, triangle AYW and ABC are directly similar and $YW \| BC$. Similarly triangles BZU and BCA as well as triangles CXV and CAB are similar and $ZU \| CA$ and $XV \| AB$. (We note that this means that $XWYUZV$ is a hexagon with opposite sides parallel, although this is not needed here.)

Since $PX \| HQ$ and $PU \| HR$, $AU : AR = AP : AH = AX : AQ$, so that there is a dilatation taking $U \rightarrow R$, $P \rightarrow H$ and $X \rightarrow Q$. Therefore $UX \| RQ$ and triangle AXU is similar to triangle AQR and to triangle ABC.

Consider quadrilateral $UZVX$.

$$\begin{aligned}\angle UZV + \angle UXV &= (180° - \angle BZU) + (180° - \angle AXU - \angle CXV) \\ &= (180° - \angle ACB) + (180° - \angle ABC - \angle BAC) = 180° \ .\end{aligned}$$

Hence $UZVX$ is concyclic. Similarly, $VXWY$ and $WYUZ$ are concyclic.

Since triangles AYW and AXU are similar with $\angle AWY = \angle AUX$ and $\angle AYW = \angle AXU$, $XWYU$ is concyclic. Similarly, $YUZV$ and $ZVXW$ are concylclic. Hence $XWYUZV$ is a hexagon, any consecutive four vertices of which are concylcic, and so is itself concyclic.

Solution 2. [K. Huynh] Let a, b, c be the lengths of the sides and A, B, C the angles of the triangle ABC according to convention. Use the notation of Solution 1. We have that $|BU| = |BP| \cos B = (c \cos B) \cos B = c \cos^2 B$. Similarly, $|BZ| = a \cos^2 B$, $|AY| = c \cos^2 A$ and $|CV| = a \cos^2 C$. Therefore, $|BY| = c(1 - \cos^2 A) = c \sin^2 A$ and $|CV| = a(1 - \cos^2 C) = a \sin^2 C$.

Since $a \sin C = c \sin A$,

$$\begin{aligned}|BU||BY| &= (c \cos^2 B)(a \sin^2 A) = \cos^2 B (c \sin A)^2 \\ &= \cos^2 B (a \sin C)^2 = (a \cos^2 B)(a \sin^2 C) = |BZ||BV| \ .\end{aligned}$$

from which, by a power-of-the-point argument [give details!], we deduce that $YUZV$ is concyclic. Similarly, $ZVXW$ and $XWYU$ are concyclic.

Suppose that the circumcircle of $YUZV$ intersects AZ at L and the circumcircle of $ZVXW$ intersects AZ at M. Since $XWYU$ is concyclic, $|AY||AU| = |AW||AX|$. Therefore,

$$|AL||AZ| = |AY||AU| = |AW||AX| = |AM||AZ| \ .$$

Hence $L = M$. Thus, the circumcircles of $YUZV$ and $ZVXW$ share three noncollinear points, Z, V and $L = M$, and so must coincide. Similarly, each coincides with the circumcircle of $XWYU$ and the result follows.