Department of Education, Ontario

Annual Examinations, 1948

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Obtain the factors of the expression

$$a^4(b-c)+b^4(c-a)+c^4(a-b)\ .$$

2. If x, y, z are not all equal and

$$x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}=k\ ,$$

prove that $x^2y^2z^2=1$ and $k^2=1$.

3. An infinite series $a_0+a_1x+a_2x^2+a_3x^3+\cdots$ is said to be a recurring series if a relation $a_n+pa_{n-1}+qa_{n-2}=0$, $n=2,3,4,\cdots$, is satisfied by any three consecutive coefficients. Assuming that the series whose first four terms are $2+x+5x^2+7x^3+\cdots$ is a recurring series, calculate the constants p, q, and deduce the sum to infinity of the series, assuming x to be sufficiently small numerically.

4. If a, b, c are positive and not all equal, show that

$$\frac{a^8+b^8+c^8}{a^3b^3c^3}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ .$$

5. Prove that the extremities of the latera recta of all ellipses having the same major axis lie on two parabolas. Find the vertices and foci of these parabolas.

6. Prove that, if two of the straight lines represented by the equation $ax^3+bx^2y+cxy^2+dy^3=0$, $a\neq 0$, $d\neq 0$, are normal to each other, then $a^2+ac+bd+d^2=0$.

7. Show that in the parabola $y^2=4ax$ a variable chord which subtends a right angle at the focus touches the ellipse

$$(x-3a)^2+2y^2=8a^2\ .$$

8. A right-angled triangle has its vertices on a rectangular hyperbola. Show that the tangent to the hyperbola at the right angle is perpendicular to the hypotenuse.

9. The horizontal base of a triangular pyramid is an equilateral triangle with sides each of length 10 units. If the lengths of the three edges of the pyramid are respectively 10, 10, and 6 units, find

   (a) the perpendicular height of the pyramid;

   (b) the angle of inclination of each of the three edges.

10. If the angles of a parallelogram are equal to the angles between its diagonals, show that, if the sides are a and b, the angles satisfy the equation

$$4a^2b^2\cos^4\theta - (a^2+b^2)^2\cos^2\theta + (a^2-b^2)^2 = 0 \; .$$

11. Show that $x = \sin\frac{\pi}{14}$ is a root of the equation

$$8x^3 - 4x^2 - 4x + 1 = 0 \; .$$

12. A body weighting 40 grams is placed in a smooth hemispherical bowl. A string is attached to the body, passed over the edge of the bowl, and tied at its free end to a body weighing 30 grams. If the edge of the bowl is a horizontal circle of radius 20 cm, find the position of equilibrium of the body.