Department of Education, Ontario

Annual Examinations, 1949

GRADE XIII

PROBLEMS

(To be taken only by candidates writing for certain University Scholarships involving Mathematics)

Ten questions constitute a full paper.

1. Let a, b, c be different real numbers. Show that the only real solution to the system of equations
$$x + y + z = 0 \ ,$$
$$ax + by + cz = 0 \ ,$$
$$x^3 + y^3 + z^3 = 3(b-c)(c-a)(a-b) \ ,$$
is $x = b - c$, $y = c - a$, $z = a - b$.

2. Let $f(n)$ denote the number of regions into which a plane is divided by n straight lines lying in it, no two of these lines being parallel and no three of them being concurrent. Show that
$$f(n) = \frac{1}{2}(n^2 + n + 2) \ .$$

3. Show that, if $(1 + x + x^2)^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n} x^{2n}$, then
$$c_0^2 - c_1^2 + c_2^2 - \cdots + c_{2n}^2 = c_n \ .$$

4. It is given that the roots of the equation
$$17x^4 + 36x^3 - 14x^2 - 4x + 1 = 0$$
are in harmonic progression. Find these roots.

5. For the circle $x^2 + y^2 = r^2$, find the equation of the locus of the middle points of chords which subtend a right angle at the point (c, o).

6. A normal to a parabola makes an angle θ with the axis of the parabola. Show that this normal cuts the curve again at an angle whose tangent is $\frac{1}{2}\tan\theta$.

7. The tangent at a point P of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the axes of coordinates in A and B, and N is the foot of the perpendicular from the centre O on the tangent. Prove that

$$AB \cdot PN = a^2 - b^2 \ .$$

8. The normal to a hyperbola at a point P other than the vertex meets the transverse axis produced at N. From N a perpendicular is drawn to an asymptote, meeting it at L, Show that LP is parallel to the conjugate axis.

9. A regular polygon of seven sides is inscribed in a circle of unit radius. Prove that the length of a side of the polygon is a root of the equation

$$x^6 - 7x^4 + 14x^2 - 7 = 0 \ ,$$

and state the geometrical significance of the other roots.

10. The lengths of consecutive sides of a quadrilateral are a, b, c, d, respectively, and each side is produced both ways. Four circles are drawn, each touching a side and the two adjeacent sides produced, and the lengths of their radii are r_a, r_b, r_c, r_d, respectively. Prove that

$$\frac{a}{r_a} + \frac{c}{r_c} = \frac{b}{r_b} + \frac{d}{r_d} \ .$$

11. Tangents are drawn to the inscribed circle of a triange parallel to the three sides of the triangle. Given that a, b, c are the lengths of the sides and p, q, r, respectively, are the lengths of the parts of the tangents within the triangle, prove that

$$\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 \ .$$

12. By means of a rope a weight is being pulled up a plane which is inclined at an angle α to the horizontal. Given that the coefficient of friction between the weight and the plane is μ, find what angle the rope must make with the plane so that the force required shall be a minimum.